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STRONG LAWS OF LARGE NUMBERS FOR THE SEQUENCE OF THE MAXIMUM OF PARTIAL SUMS OF I.I.D. RANDOM VARIABLES

BY

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Abstract. Let $0 < p \leqslant 2$, let $\{X_n;\ n \geqslant 1\}$ be a sequence of independent copies of a real-valued random variable X, and set $S_n = X_1 + \ldots + X_n$, $n \geqslant 1$. Motivated by a theorem of Mikosch (1984), this note is devoted to establishing a strong law of large numbers for the sequence $\{\max_{1 \leqslant k \leqslant n} |S_k|\ ;\ n \geqslant 1\}$. More specifically, necessary and sufficient conditions are given for

$$\lim_{n\to\infty} \big(\max_{1 \leqslant k \leqslant n} |S_k|\big)^{(\log n)^{-1}} = e^{1/p} \text{ a.s.},$$

where $\log x = \log_e \max\{e, x\}$, $x \geqslant 0$.

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1. A THEOREM OF MIKOSCH

Throughout this note, let $\{X, X_n;\ n \geqslant 1\}$ be a sequence of independent and identically distributed (i.i.d.) real-valued random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. As usual, let $S_n = \sum_{k=1}^{n} X_k,\ n \geqslant 1$, denote their partial sums. For $a, b \in \mathbb{R} = (-\infty, \infty)$, we denote $\max\{a, b\}$ by $a \vee b$ and $\min\{a, b\}$ by $a \wedge b$. Write $\log x = \log_e(e \vee x)$, $x \geqslant 0$. If $0 < p < 2$, then

$$\lim_{n\to\infty} \frac{S_n}{n^{1/p}} = 0 \text{ almost surely (a.s.)}$$

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if and only if

$$\mathbb{E}|X|^p < \infty, \quad \text{where } \mathbb{E}X = 0 \text{ whenever } p \geqslant 1.$$

This is the celebrated Kolmogorov–Marcinkiewicz–Zygmund strong law of large numbers (SLLN); see Kolmogoroff [7] for $p = 1$ and Marcinkiewicz and Zygmund [8] for $p \neq 1$. The origin of the current investigation is the following strong limit theorem established by Mikosch [9] (see Addendum 7.5.16 of Petrov [10], p. 258) which is related to the Kolmogorov–Marcinkiewicz–Zygmund SLLN.

THEOREM 1.1 (Mikosch [9]). *Suppose that* $\mathbb{E}|X|^\beta < \infty$ *for some* $\beta > 0$ *and* $\mathbb{E}X = 0$ *if* $\beta \geqslant 1$. *Let*

$$\beta_0 = \sup\{\beta > 0 : \ \mathbb{E}|X|^\beta < \infty\} \quad \textit{and} \quad p = \beta_0 \wedge 2.$$

Then

$$\limsup_{n\to\infty} \left|\frac{S_n}{n^{1/p}}\right|^{(\log n)^{-1}} = 1 \ \textit{a.s.} \tag{1.1}$$

REMARK 1.1. *Since, for* $n \geqslant 3$,

$$e^{-1/p}\,|S_n|^{(\log n)^{-1}} = \left|\frac{S_n}{n^{1/p}}\right|^{(\log n)^{-1}},$$

(1.1) *is equivalent to*

$$\limsup_{n\to\infty} |S_n|^{(\log n)^{-1}} = e^{1/p} \ \textit{a.s.} \tag{1.2}$$

REMARK 1.2. *Recently Zou and Liu* [11] *proved for* $0 < p \leqslant 2$ *that* (1.2) *holds if and only if*

$$\begin{cases} \mathbb{P}(X = 0) < 1, \ \ \mathbb{E}X = 0, & \\ \textit{and } \sup\{\beta \geqslant 0 : \ \mathbb{E}|X|^\beta < \infty\} \geqslant 2 & \textit{if } p = 2, \\ \mathbb{E}X = 0 \textit{ and } \sup\{\beta \geqslant 0 : \ \mathbb{E}|X|^\beta < \infty\} = p & \textit{if } 1 < p < 2, \\ \textit{either } \sup\{\beta \geqslant 0 : \ \mathbb{E}|X|^\beta < \infty\} = 1 & \\ \textit{or } \mathbb{E}|X| < \infty \textit{ and } \mathbb{E}X \neq 0 & \textit{if } p = 1, \\ \sup\{\beta \geqslant 0 : \ \mathbb{E}|X|^\beta < \infty\} = p & \textit{if } 0 < p < 1. \end{cases} \tag{1.3}$$

We now look at the following two examples.

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EXAMPLE 1.1. Assume that $\{X,\ X_n;\ n \geqslant 1\}$ is a Rademacher sequence; that is, $\{X, X_n;\ n \geqslant 1\}$ is a sequence of i.i.d. random variables with $\mathbb{P}(X = 1) = \mathbb{P}(X = -1) = 1/2$. Then

$$\mathbb{E}X = 0 \quad \text{and} \quad \mathbb{E}X^2 = 1,$$

and hence, by Theorem 1.1,

$$\limsup_{n\to\infty} |S_n|^{(\log n)^{-1}} = e^{1/2} \ \text{ a.s.}$$

On the other hand, for this example, it is well known that

$$\mathbb{P}\left(S_n = 0 \text{ infinitely often (i.o.)}\right) = 1,$$

and hence

$$\liminf_{n\to\infty} |S_n|^{(\log n)^{-1}} = 0 \ \text{ a.s.}$$

Thus, for this example,

$$\lim_{n\to\infty} |S_n|^{(\log n)^{-1}} \quad \text{does not exist a.s.}$$

EXAMPLE 1.2. Let $\{X,\ X_n;\ n \geqslant 1\}$ be a sequence of i.i.d. real-valued random variables with a symmetric distribution given by

$$\mathbb{P}(X = k) = \mathbb{P}(X = -k) = \frac{c_1(\log k)^2}{k^2}, \qquad k = 1, 2, 3, \ldots,$$

where $c_1 = \left(2\sum_{k=1}^{\infty}(\log k)^2/k^2\right)^{-1}$. Then

$$\sup\{\beta \geqslant 0 :\ \mathbb{E}|X|^{\beta} < \infty\} = 1,$$

and hence, by Remark 1.2,

$$\limsup_{n\to\infty} |S_n|^{(\log n)^{-1}} = e \ \text{ a.s.}$$

On the other hand, for this example, Kesten [6], pp. 1182–1183, showed that

$$\liminf_{n\to\infty} \frac{|S_n|}{n^{\alpha}} = 0 \ \text{ a.s. } \forall\, \alpha > 0.$$

Thus

$$\liminf_{n\to\infty} |S_n|^{(\log n)^{-1}} = \liminf_{n\to\infty} e^{\alpha} \left|\frac{S_n}{n^{\alpha}}\right|^{(\log n)^{-1}} \leqslant e^{\alpha} \ \text{ a.s. } \forall\, \alpha > 0,$$

and hence

$$\liminf_{n\to\infty} |S_n|^{(\log n)^{-1}} \leqslant 1 \ \text{ a.s.}$$

Thus, for this example,

$$\lim_{n\to\infty} |S_n|^{(\log n)^{-1}} \quad \text{does not exist a.s.}$$

Motivated by the theorem of Mikosch [9] and two examples above, this note is devoted to establishing an SLLN for the sequence $\{\max_{1\leqslant k\leqslant n}|S_k|\,;\ n\geqslant 1\}$ of the maximum of partial sums of i.i.d. real-valued random variables. Necessary and sufficient conditions are given for

$$\lim_{n\to\infty}\big(\max_{1\leqslant k\leqslant n}|S_k|\big)^{(\log n)^{-1}} = e^{1/p} \text{ a.s.,}\quad \text{where } 0<p\leqslant 2.$$

Our main results are Theorems 2.1–2.4 stated in Section 2. In regard to Example 1.1 above, it follows from Theorem 2.1 that

$$\lim_{n\to\infty}\big(\max_{1\leqslant k\leqslant n}|S_k|\big)^{(\log n)^{-1}} = e^{1/2} \text{ a.s.,}$$

and in regard to Example 1.2 above, it follows from Theorem 2.3 that

$$\lim_{n\to\infty}\big(\max_{1\leqslant k\leqslant n}|S_k|\big)^{(\log n)^{-1}} = e \text{ a.s.}$$

2. THE SLLN FOR THE SEQUENCE OF THE MAXIMUM OF PARTIAL SUMS

We start with some notation. Let X be a given real-valued random variable. Write

$$\rho_1 = \sup\{r\geqslant 0:\ \lim_{x\to\infty} x^r\mathbb{P}(|X|>x)=0\},$$
$$\rho_2 = \sup\{r\geqslant 0:\ \liminf_{x\to\infty} x^r\mathbb{P}(|X|>x)=0\}.$$

Clearly, ρ_1 and ρ_2 are two parameters of the distribution of the random variable X and satisfy

$$0\leqslant\rho_1\leqslant\rho_2\leqslant\infty.$$

We say that X is a *symmetric* random variable if

$$\mathbb{P}(X\leqslant x)=\mathbb{P}(X\geqslant -x)\ \ \forall\, x\in\mathbb{R}.$$

Let $\{X_n;\ n\geqslant 1\}$ be a sequence of independent copies of a real-valued random variable X. Let $0<p\leqslant 2$. In this section, the SLLN for $\{\max_{1\leqslant k\leqslant n}|S_k|\,;\ n\geqslant 1\}$ is presented by the following Theorems 2.1–2.4.

THEOREM 2.1. *The following three statements are equivalent*:

$$\lim_{n\to\infty}\big(\max_{1\leqslant k\leqslant n}|S_k|\big)^{(\log n)^{-1}} = e^{1/2} \text{ a.s.,}\tag{2.1}$$

$$0<\limsup_{n\to\infty}\big(\max_{1\leqslant k\leqslant n}|S_k|\big)^{(\log n)^{-1}} \leqslant e^{1/2} \text{ a.s.,}\tag{2.2}$$

$$\mathbb{P}(X=0)<1,\quad \mathbb{E}X=0,\quad \textit{and}\quad \rho_1\geqslant 2.\tag{2.3}$$

REMARK 2.1. *It is easy to see that* (2.1) *holds for Example* 1.1. *Furthermore*, (2.1) *holds for any real-valued random variable* X *satisfying*

$$\mathbb{E}X = 0 \quad \textit{and} \quad 0 < \mathbb{E}X^2 < \infty.$$

However, *the converse is not true. Thus*, *we see that Theorem* 2.1 *is a kind of supplement to the classical Hartman and Wintner* [4] *law of the iterated logarithm* (*LIL*) *for the partial sums of i.i.d. random variables.*

THEOREM 2.2. *Let* $1 < p < 2$. *Then*

$$\lim_{n\to\infty} \big(\max_{1\leqslant k\leqslant n} |S_k|\big)^{(\log n)^{-1}} = e^{1/p} \;\; \textit{a.s. if and only if} \;\; \mathbb{E}X = 0 \;\; \textit{and} \;\; \rho_1 = \rho_2 = p.$$

THEOREM 2.3. (i) *Let* X *be a real-valued random variable such that*

$$\textit{either} \;\; \mathbb{E}|X| < \infty \; \textit{and} \; \mathbb{E}X \neq 0 \;\; \textit{or} \;\; \rho_1 = \rho_2 = 1. \tag{2.4}$$

Then

$$\lim_{n\to\infty} \big(\max_{1\leqslant k\leqslant n} |S_k|\big)^{(\log n)^{-1}} = e \;\; \textit{a.s.} \tag{2.5}$$

(ii) *If* X *is a real-valued symmetric random variable*, *then*

(2.5) *holds if and only if* $\rho_1 = \rho_2 = 1$.

REMARK 2.2. *We now reconsider Example* 1.2. *Clearly*, X *is symmetric. Since*

$$\sum_{k=n}^{\infty} \frac{c_1(\log k)^2}{k^2} \sim \frac{c_1(\log n)^2}{n} \quad \textit{as } n \to \infty,$$

we see that

$$\rho_1 = \rho_2 = 1,$$

and hence, *by Theorem* 2.3, (2.5) *holds.*

THEOREM 2.4. *Let* $0 < p < 1$. *Then*

$$\lim_{n\to\infty} \big(\max_{1\leqslant k\leqslant n} |S_k|\big)^{(\log n)^{-1}} = e^{1/p} \;\; \textit{a.s. if and only if} \;\; \rho_1 = \rho_2 = p.$$

To prove Theorems 2.1–2.4, some preliminary lemmas will first be established in Section 3. These lemmas may be of independent interest. Our main results will then be proved in Section 4. We refer the reader to Chow and Teicher [1] for any basic results in probability theory that are used in this note.

3. PRELIMINARY LEMMAS

To prove the SLLN for the sequence of the maximum of partial sums, we use the following preliminary lemmas.

LEMMA 3.1. *Let $\{a_n;\ n \geqslant 1\}$ be a nondecreasing sequence of positive real numbers such that*

$$\lim_{n\to\infty} a_n = \infty.$$

Then, for any sequence $\{b_n;\ n \geqslant 1\}$ of real numbers such that $\sup_{n\geqslant 1}|b_n| > 0$, *we have*

$$\limsup_{n\to\infty}\big(\max_{1\leqslant k\leqslant n}|b_k|\big)^{1/a_n} = 1 \vee \limsup_{n\to\infty}|b_n|^{1/a_n}. \tag{3.1}$$

Proof. Set $\gamma = \limsup_{n\to\infty}|b_n|^{1/a_n}$. Since $\sup_{n\geqslant 1}|b_n| > 0$, there exists $n_0 \geqslant 1$ such that $|b_{n_0}| > 0$. Note that

$$\big(\max_{1\leqslant k\leqslant n}|b_k|\big)^{1/a_n} \geqslant |b_n|^{1/a_n} \quad \text{and} \quad \big(\max_{1\leqslant k\leqslant n}|b_k|\big)^{1/a_n} \geqslant |b_{n_0}|^{1/a_n}, \quad n \geqslant n_0.$$

We thus see that

$$\limsup_{n\to\infty}\big(\max_{1\leqslant k\leqslant n}|b_k|\big)^{1/a_n} \geqslant \limsup_{n\to\infty}|b_n|^{1/a_n} = \gamma,$$

and it follows from $\lim_{n\to\infty} a_n = \infty$ that

$$\limsup_{n\to\infty}\big(\max_{1\leqslant k\leqslant n}|b_k|\big)^{1/a_n} \geqslant \limsup_{n\to\infty}|b_{n_0}|^{1/a_n} = 1,$$

and hence

$$\limsup_{n\to\infty}\big(\max_{1\leqslant k\leqslant n}|b_k|\big)^{1/a_n} \geqslant 1 \vee \gamma. \tag{3.2}$$

We now show that

$$\limsup_{n\to\infty}\big(\max_{1\leqslant k\leqslant n}|b_k|\big)^{1/a_n} \leqslant 1 \vee \gamma. \tag{3.3}$$

Clearly, (3.3) holds if $\gamma = \infty$.

We now turn our attention to the case $0 \leqslant \gamma < \infty$. For given $\epsilon > 0$, there exists a positive integer $n_\epsilon \geqslant n_0$ such that

$$|b_n|^{1/a_n} \leqslant \gamma + \epsilon \ \ \forall\, n \geqslant n_\epsilon,$$

and hence

$$|b_n| \leqslant (\gamma+\epsilon)^{a_n} \leqslant \big((1\vee\gamma)+\epsilon\big)^{a_n} \ \ \forall\, n \geqslant n_\epsilon.$$

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Since $\{a_n;\ n \geqslant 1\}$ is a nondecreasing sequence of positive real numbers and $(1 \vee \gamma) + \epsilon > 1$, we see that

$$|b_k| \leqslant \big((1 \vee \gamma) + \epsilon\big)^{a_n} \quad \forall\, n_\epsilon \leqslant k \leqslant n.$$

Thus we have

$$\begin{aligned}(\max_{1\leqslant k\leqslant n} |b_k|)^{1/a_n} &\leqslant (\max_{1\leqslant k\leqslant n_\epsilon} |b_k|)^{1/a_n} \vee (\max_{n_\epsilon\leqslant k\leqslant n} |b_k|)^{1/a_n} \\ &\leqslant (\max_{1\leqslant k\leqslant n_\epsilon} |b_k|)^{1/a_n} \vee \big((1 \vee \gamma) + \epsilon\big) \quad \forall\, n \geqslant n_\epsilon.\end{aligned}$$

Since $\lim_{n\to\infty} a_n = \infty$, we get

$$\limsup_{n\to\infty} (\max_{1\leqslant k\leqslant n} |b_k|)^{1/a_n} \leqslant 1 \vee \big((1 \vee \gamma) + \epsilon\big) = (1 \vee \gamma) + \epsilon.$$

Letting $\epsilon \searrow 0$, we obtain (3.3), which together with (3.2) yields the conclusion (3.1). The proof of Lemma 3.1 is now complete. ■

Note that, for $r > 0$,

$$\text{if} \ \lim_{x\to\infty} x^r \mathbb{P}(|X| > x) = 0, \ \text{then} \ \mathbb{E}|X|^{r_1} < \infty \ \forall\, 0 \leqslant r_1 < r,$$

and

$$\text{if} \ \mathbb{E}|X|^r < \infty, \ \text{then} \ \lim_{x\to\infty} x^{r_1} \mathbb{P}(|X| > x) = 0 \ \forall\, 0 \leqslant r_1 \leqslant r.$$

We thus infer that

$$\rho_1 = \sup\{r \geqslant 0 :\ \lim_{x\to\infty} x^r \mathbb{P}(|X| > x) = 0\} = \sup\{r \geqslant 0 :\ \mathbb{E}|X|^r < \infty\}.$$

Thus, by Lemma 3.1 and Remark 1.2, we have the following strong limit theorem for the sequence of the maximum of partial sums of i.i.d. real-valued random variables. This theorem will be used in the proofs of Theorems 2.1–2.4.

THEOREM 3.1. *Let $0 < p \leqslant 2$. Let $\{X_n;\ n \geqslant 1\}$ be a sequence of independent copies of a real-valued random variable X. Then*

$$\limsup_{n\to\infty} (\max_{1\leqslant k\leqslant n} |S_k|)^{(\log n)^{-1}} = e^{1/p} \ \textit{a.s.}$$

if and only if

$$\begin{cases} \mathbb{P}(X = 0) < 1,\ \mathbb{E}X = 0, \textit{ and } \rho_1 \geqslant 2 & \textit{if } p = 2, \\ \mathbb{E}X = 0 \textit{ and } \rho_1 = p & \textit{if } 1 < p < 2, \\ \textit{either } \rho_1 = 1 \textit{ or } \mathbb{E}|X| < \infty \textit{ and } \mathbb{E}X \neq 0 & \textit{if } p = 1, \\ \rho_1 = p & \textit{if } 0 < p < 1. \end{cases}$$

The following lemma will be used in the proof of Theorem 2.1.

LEMMA 3.2. *Let* $\{X_n;\ n \geqslant 1\}$ *be a sequence of independent copies of a real-valued nondegenerate random variable* X. *Then*

$$\liminf_{n\to\infty}\big(\max_{1\leqslant k\leqslant n}|S_k|\big)^{(\log n)^{-1}} \geqslant e^{1/2} \quad a.s. \tag{3.4}$$

P r o o f. If $0 < \mathbb{E}X^2 < \infty$ and $\mathbb{E}X = 0$, then it follows from the so-called other LIL due to Chung [2] and Jain and Pruitt [5] that

$$\liminf_{n\to\infty}\frac{\max_{1\leqslant k\leqslant n}|S_k|}{\sqrt{n/\log\log n}} = \frac{\pi}{\sqrt{8}}(\mathbb{E}X^2)^{1/2} \ \text{ a.s.} \tag{3.5}$$

If $0 < \mathbb{E}X^2 < \infty$ and $\mathbb{E}X \neq 0$, then it follows from the Kolmogorov SLLN that

$$\liminf_{n\to\infty}\frac{\max_{1\leqslant k\leqslant n}|S_k|}{\sqrt{n/\log\log n}} \geqslant \lim_{n\to\infty}\frac{n\,|S_n/n|}{\sqrt{n/\log\log n}} = \infty \ \text{ a.s.} \tag{3.6}$$

If $\mathbb{E}X^2 = \infty$, then it follows from Theorem 3.2 of Csáki [3] (see Addendum 7.5.19 of Petrov [10], p. 258) that

$$\lim_{n\to\infty}\frac{\max_{1\leqslant k\leqslant n}|S_k|}{\sqrt{n/\log\log n}} = \infty \ \text{ a.s.} \tag{3.7}$$

Thus, from (3.5)–(3.7) we obtain

$$\liminf_{n\to\infty}\frac{\max_{1\leqslant k\leqslant n}|S_k|}{\sqrt{n/\log\log n}} > 0 \ \text{ a.s.},$$

which ensures that

$$\begin{aligned}
&\liminf_{n\to\infty}\big(\max_{1\leqslant k\leqslant n}|S_k|\big)^{(\log n)^{-1}}\\
&\quad = \liminf_{n\to\infty}\left(\big(\sqrt{n/\log\log n}\big)^{(\log n)^{-1}}\left(\frac{\max_{1\leqslant k\leqslant n}|S_k|}{\sqrt{n/\log\log n}}\right)^{(\log n)^{-1}}\right)\\
&\quad \geqslant e^{1/2} \ \text{ a.s.},
\end{aligned}$$

i.e., (3.4) holds. ■

The following lemma will be used in the proofs of Theorems 2.2–2.4.

LEMMA 3.3. *Let* $\{X_n;\ n \geqslant 1\}$ *be a sequence of independent copies of a real-valued random variable* X *such that* $0 < \rho_2 \leqslant 2$. *Then*

$$\liminf_{n\to\infty}\big(\max_{1\leqslant k\leqslant n}|S_k|\big)^{(\log n)^{-1}} \geqslant e^{1/\rho_2} \quad a.s. \tag{3.8}$$

P r o o f. For given $\rho_2 < r < \infty$, let $r_1 = (r + \rho_2)/2$ and $\tau = 1 - (r_1/r)$. Then $\rho_2 < r_1 < r < \infty$ and $\tau > 0$. By the definition of ρ_2, we have

$$\lim_{x\to\infty} x^{r_1}\mathbb{P}(|X| > x) = \infty,$$

and hence for all sufficiently large x,

$$\mathbb{P}(|X| > x) \geqslant x^{-r_1}.$$

Thus, for all sufficiently large n,

$$n\mathbb{P}(|X| > n^{1/r}) \geqslant n(n^{1/r})^{-r_1} = n^{1-(r_1/r)} = n^{\tau},$$

and hence

$$\mathbb{P}(\max_{1\leqslant k\leqslant n} |X_k| \leqslant n^{1/r}) = \big(1 - \mathbb{P}(|X| > n^{1/r})\big)^n \leqslant e^{-n\mathbb{P}(|X|>n^{1/r})} \leqslant e^{-n^{\tau}}.$$

Since

$$\sum_{n=1}^{\infty} e^{-n^{\tau}} < \infty,$$

by the Borel–Cantelli lemma, we have

$$\mathbb{P}\big((\max_{1\leqslant k\leqslant n} |X_k|)^{(\log n)^{-1}} \leqslant e^{1/r} \text{ i.o.}\big) = \mathbb{P}(\max_{1\leqslant k\leqslant n} |X_k| \leqslant n^{1/r} \text{ i.o.}) = 0,$$

which implies

$$\liminf_{n\to\infty}(\max_{1\leqslant k\leqslant n} |X_k|)^{(\log n)^{-1}} \geqslant e^{1/r} \quad \text{a.s.} \tag{3.9}$$

Letting $r \searrow \rho_2$, from (3.9) we obtain

$$\liminf_{n\to\infty}(\max_{1\leqslant k\leqslant n} |X_k|)^{(\log n)^{-1}} \geqslant e^{1/\rho_2} \quad \text{a.s.} \tag{3.10}$$

Note that, for each $n \geqslant 1$,

$$|X_k| = |S_k - S_{k-1}| \leqslant |S_k| + |S_{k-1}| \quad \forall\, 1 \leqslant k \leqslant n.$$

Thus

$$\max_{1\leqslant k\leqslant n} |X_k| \leqslant 2 \max_{1\leqslant k\leqslant n} |S_k| \quad \forall\, n \geqslant 1.$$

It thus follows from (3.10) that

$$\begin{aligned}\liminf_{n\to\infty}&(\max_{1\leqslant k\leqslant n} |S_k|)^{(\log n)^{-1}} \\ &\geqslant \liminf_{n\to\infty}\left(\frac{1}{2}\right)^{(\log n)^{-1}} \liminf_{n\to\infty}(\max_{1\leqslant k\leqslant n} |X_k|)^{(\log n)^{-1}} \geqslant e^{1/\rho_2} \quad \text{a.s.,}\end{aligned}$$

i.e., (3.8) holds. ■

The following lemma will be used in the proof of Theorem 2.2.

LEMMA 3.4. *Let $\{X_n;\ n \geqslant 1\}$ be a sequence of independent copies of a real-valued random variable X such that*

$$\mathbb{E}X = 0 \quad \textit{and} \quad 1 < \rho_1 < 2 \wedge \rho_2.$$

Then

$$\liminf_{n\to\infty} \big(\max_{1\leqslant k\leqslant n} |S_k|\big)^{(\log n)^{-1}} < e^{1/\rho_1} \quad \textit{a.s.} \tag{3.11}$$

Proof. Let $h = \big((2 \wedge \rho_2) - \rho_1\big)/4$. Since $1 < \rho_1 < 2 \wedge \rho_2$, we infer that $h > 0$, $\rho_1 < \rho_1 + 3h < 2 \wedge \rho_2$, and by the definition of ρ_2,

$$\liminf_{x\to\infty} x^{\rho_1+3h}\mathbb{P}(|X| > x) = 0.$$

Hence, by letting $x = t^{1/(\rho_1+2h)}$, we obtain

$$\liminf_{t\to\infty} t^{1+\eta}\mathbb{P}(|X| > t^{1/b}) = 0,$$

where

$$\eta = \frac{h}{\rho_1 + 2h} > 0 \quad \text{and} \quad b = \rho_1 + 2h \in (\rho_1, 2 \wedge \rho_2).$$

Then, proceeding inductively, we can choose an increasing sequence $\{n_m;\ m \geqslant 1\}$ of positive integers such that $n_1 = 1$ and

$$n_m = \min\left\{k \geqslant 2^m \vee (n_{m-1} + 1) :\ \mathbb{P}(|X| > k^{1/b}) \leqslant \frac{1}{k^{1+\eta}}\right\}, \quad m > 1.$$

Write, for $(x, y) \in (0, 2) \times (0, 2)$,

$$\varphi_1(x, y) = 1 + \frac{2}{b} - \frac{x}{b} - \frac{2}{y} \quad \text{and} \quad \varphi_2(x, y) = -\eta + \frac{1+\eta}{x} - \frac{1}{y}.$$

Since

$$\begin{aligned}\lim_{\substack{x\nearrow\rho_1\\ y\searrow\rho_1}} \varphi_1(x, y) &= 1 + \frac{2}{b} - \frac{\rho_1}{b} - \frac{2}{\rho_1}\\ &= \frac{\rho_1 b + 2\rho_1 - \rho_1^2 - 2b}{\rho_1 b} = \frac{(\rho_1 - 2)(b - \rho_1)}{\rho_1 b} < 0\end{aligned}$$

and

$$\lim_{\substack{x\nearrow\rho_1\\ y\searrow\rho_1}} \varphi_2(x, y) = -\eta + \frac{1+\eta}{\rho_1} - \frac{1}{\rho_1} = \frac{\eta(1 - \rho_1)}{\rho_1} < 0,$$

we can choose r and q such that $1 < r < \rho_1,\ \rho_1 < q < b$, and

$$\varphi_1(r,q) = 1 + \frac{2}{b} - \frac{r}{b} - \frac{2}{q} < 0, \quad \text{and} \quad \varphi_2(r,q) = -\eta + \frac{1+\eta}{r} - \frac{1}{q} < 0,$$

and hence

$$\mathbb{E}|X|^r < \infty, \quad \frac{1}{b} < \frac{1}{q} < \frac{1}{\rho_1}, \quad \text{and} \quad \mathbb{P}(|X| > n_m^{1/q}) \leqslant \frac{1}{n_m^{1+\eta}},\ m \geqslant 1.$$

Write, for $1 \leqslant i \leqslant n_m,\ m \geqslant 1$,

$$\mu_m = \mathbb{E}\big(XI(|X| \leqslant n_m^{1/q})\big), \quad X_{m,i} = X_i I(|X_i| \leqslant n_m^{1/q}) - \mu_m.$$

Note that

$$S_k = k\mu_m + \sum_{i=1}^{k} X_{m,i} + \sum_{i=1}^{k} X_i I(|X_i| > n_m^{1/q}), \quad 1 \leqslant k \leqslant n_m,\ m \geqslant 1.$$

We thus have

(3.12)

$$\max_{1\leqslant k\leqslant n_m} |S_k| \leqslant n_m\,|\mu_m| + \max_{1\leqslant k\leqslant n_m} \Big|\sum_{i=1}^{k} X_{m,i}\Big| + \sum_{i=1}^{n_m} |X_i|\, I(|X_i| > n_m^{1/q}),\ m \geqslant 1.$$

We now show that

$$\lim_{m\to\infty} \frac{n_m\,|\mu_m|}{n_m^{1/q}} = 0, \tag{3.13}$$

$$\lim_{m\to\infty} \frac{\max_{1\leqslant k\leqslant n_m} \big|\sum_{i=1}^{k} X_{m,i}\big|}{n_m^{1/q}} = 0 \ \text{ a.s.,} \tag{3.14}$$

and

$$\lim_{m\to\infty} \frac{\sum_{i=1}^{n_m} |X_i|\, I(|X_i| > n_m^{1/q})}{n_m^{1/q}} = 0 \ \text{ a.s.} \tag{3.15}$$

To verify (3.13), let $s = r/(r-1)$. It follows from $r > 1$ that

$$s > 1 \quad \text{and} \quad \frac{1}{r} + \frac{1}{s} = 1.$$

Since $\mathbb{E}X = 0$ and $\mathbb{E}|X|^r < \infty$, using Hölder's inequality, we have

$$\begin{aligned}
\frac{n_m\,|\mu_m|}{n_m^{1/q}} &= \frac{n_m\big|\mathbb{E}\big(XI(|X| > n_m^{1/q})\big)\big|}{n_m^{1/q}} \\
&\leqslant \frac{n_m(\mathbb{E}|X|^r)^{1/r}\big(\mathbb{E}\big(I(|X| > n_m^{1/q})\big)^s\big)^{1/s}}{n_m^{1/q}} \\
&= \frac{n_m(\mathbb{E}|X|^r)^{1/r}\big(\mathbb{P}(|X| > n_m^{1/q})\big)^{1-1/r}}{n_m^{1/q}} \\
&\leqslant \frac{n_m(\mathbb{E}|X|^r)^{1/r}(n_m^{-1-\eta})^{1-1/r}}{n_m^{1/q}} = (\mathbb{E}|X|^r)^{1/r} n_m^{\varphi_2(r,q)}, \qquad m \geqslant 1,
\end{aligned}$$

and hence, by recalling that $\varphi_2(r,q) < 0$, (3.13) follows.

We now verify (3.14). Note that for each $m \geqslant 1$, $X_{m,i}$, $1 \leqslant i \leqslant n_m$, are i.i.d. real-valued random variables such that $\mathbb{E}X_{m,i} = 0$, $1 \leqslant i \leqslant n_m$. Thus, using Kolmogrov's inequality, we see that, for all $\epsilon > 0$ and $m \geqslant 1$,

$$\begin{aligned}
&\mathbb{P}\Big(\max_{1\leqslant k\leqslant n_m}\Big|\sum_{i=1}^{n_m} X_{m,i}\Big| > \epsilon n_m^{1/q}\Big) \\
&\leqslant \frac{\mathrm{Var}\big(\sum_{i=1}^{n_m} X_{m,i}\big)}{\epsilon^2 n_m^{2/q}} \leqslant \frac{n_m\mathbb{E}\big(X^2 I(|X| \leqslant n_m^{1/q})\big)}{\epsilon^2 n_m^{2/q}} \\
&= \frac{n_m\mathbb{E}\big(|X|^r|X|^{2-r}I(|X| \leqslant n_m^{1/b})\big)}{\epsilon^2 n_m^{2/q}} + \frac{n_m\mathbb{E}\big(X^2 I(n_m^{1/b} < |X| \leqslant n_m^{1/q})\big)}{\epsilon^2 n_m^{2/q}} \\
&\leqslant (\mathbb{E}|X|^r)\frac{n_m n_m^{(2/b)-(r/b)}}{\epsilon^2 n_m^{2/q}} + \frac{n_m n_m^{2/q}\mathbb{P}(n_m^{1/b} < |X| \leqslant n_m^{1/q})}{\epsilon^2 n_m^{2/q}} \\
&\leqslant (\mathbb{E}|X|^r/\epsilon^2) n_m^{\varphi_1(r,q)} + (1/\epsilon^2)\frac{1}{n_m^{\eta}} \\
&\leqslant (\mathbb{E}|X|^r/\epsilon^2)\,(2^m)^{\varphi_1(r,q)} + (1/\epsilon^2)\frac{1}{2^{\eta m}} = (\mathbb{E}|X|^r/\epsilon^2)\lambda^m + (1/\epsilon^2)\frac{1}{2^{\eta m}},
\end{aligned}$$

where we let $\lambda = 2^{\varphi_1(r,q)}$. Then $0 < \lambda < 1$ since $\varphi_1(r,q) < 0$. Hence

$$\begin{aligned}
&\sum_{m=1}^{\infty} \mathbb{P}\Big(\max_{1\leqslant k\leqslant n_m}\Big|\sum_{i=1}^{k} X_{m,i}\Big| > \epsilon n_m^{1/q}\Big) \\
&\qquad\qquad \leqslant \frac{\mathbb{E}|X|^r}{\epsilon^2}\sum_{m=1}^{\infty}\lambda^m + \frac{1}{\epsilon^2}\sum_{m=1}^{\infty}\frac{1}{2^{\eta m}} < \infty \ \ \forall\, \epsilon > 0,
\end{aligned}$$

which, by applying the Borel–Cantelli lemma, yields (3.14).

To verify (3.15), note that

$$\sum_{m=1}^{\infty} \mathbb{P}\big(\sum_{i=1}^{n_m} |X_i|\, I(|X_i| > n_m^{1/q}) \neq 0\big) = \sum_{m=1}^{\infty} \mathbb{P}\big(\max_{1\leqslant i\leqslant n_m} |X_i| > n_m^{1/q}\big)$$
$$\leqslant \sum_{m=1}^{\infty} n_m \mathbb{P}(|X| > n_m^{1/q}) \leqslant \sum_{m=1}^{\infty} \frac{1}{n_m^{\eta}} \leqslant \sum_{m=1}^{\infty} \frac{1}{2^{\eta m}} < \infty.$$

Thus, applying the Borel–Cantelli lemma, we have

$$\mathbb{P}\big(\sum_{i=1}^{n_m} |X_i|\, I(|X_i| > n_m^{1/q}) \neq 0 \text{ i.o.}\big) = 0,$$

which ensures (3.15).

It thus follows from (3.12)–(3.15) that

$$\lim_{m\to\infty} \frac{\max_{1\leqslant k\leqslant n_m} |S_k|}{n_m^{1/q}} = 0 \ \text{ a.s.},$$

and hence

$$\liminf_{n\to\infty} \frac{\max_{1\leqslant k\leqslant n} |S_k|}{n^{1/q}} = 0 \ \text{ a.s.},$$

which ensures that

$$\liminf_{n\to\infty} \big(\max_{1\leqslant k\leqslant n} |S_k|\big)^{(\log n)^{-1}} = \liminf_{n\to\infty} e^{1/q} \left(\frac{\max_{1\leqslant k\leqslant n} |S_k|}{n^{1/q}}\right)^{(\log n)^{-1}}$$
$$\leqslant e^{1/q} < e^{1/\rho_1} \ \text{ a.s.},$$

where the first inequality follows from the observation that if $0 < a_{n_j} \to 0$ and $0 < b_j \to 0$, then $a_{n_j}^{b_j} \leqslant 1$ for all large j. Thus (3.11) holds. ■

The following Lemmas 3.5 and 3.6 will be used in the proofs of Theorems 2.3 and 2.4, respectively. Their proofs are similar to that of Lemma 3.4.

LEMMA 3.5. *Let $\{X_n;\ n \geqslant 1\}$ be a sequence of independent copies of a real-valued symmetric random variable X such that $\rho_1 = 1 < \rho_2$. Then*

$$\liminf_{n\to\infty} \big(\max_{1\leqslant k\leqslant n} |S_k|\big)^{(\log n)^{-1}} < e \ \text{ a.s.} \tag{3.16}$$

Proof. Let $h = \big((2 \wedge \rho_2) - 1\big)/4$, $\eta = h/(1 + 2h)$, and $b = 1 + 2h$. Since $1 < \rho_2$, we have $h > 0$, $\eta > 0$, $b \in (1, 2 \wedge \rho_2)$, and by the definition of ρ_2,

$$\liminf_{t\to\infty} t^{1+\eta} \mathbb{P}(|X| > t^{1/b}) = 0.$$

Then, proceeding inductively, we can choose an increasing sequence $\{n_m;\ m \geqslant 1\}$ of positive integers such that $n_1 = 1$ and

$$n_m = \min\left\{k \geqslant 2^m \vee (n_{m-1}+1):\ \mathbb{P}(|X| > k^{1/b}) \leqslant \frac{1}{k^{1+\eta}}\right\}, \qquad m > 1.$$

Write

$$\phi(x,y) = 1 + \frac{2}{b} - \frac{x}{b} - \frac{2}{y}, \qquad (x,y) \in (0,2)\times(0,2).$$

Since $\rho_1 = 0$ and

$$\lim_{\substack{x \nearrow \rho_1 \\ y \searrow \rho_1}} \phi(x,y) = 1 + \frac{2}{b} - \frac{1}{b} - 2 = \frac{1-b}{b} < 0,$$

we can choose r and q such that

$$0 < r < 1 < q < b \quad \text{and} \quad \varphi(r,q) = 1 + \frac{2}{b} - \frac{r}{b} - \frac{2}{q} < 0,$$

and hence

$$\mathbb{E}|X|^r < \infty, \qquad \frac{1}{b} < \frac{1}{q} < 1, \qquad \text{and} \qquad \mathbb{P}(|X| > n_m^{1/q}) \leqslant \frac{1}{n_m^{1+\eta}},\ \ m \geqslant 1.$$

Since $\{X_n;\ n \geqslant 1\}$ is a sequence of independent copies of the real-valued symmetric random variable X, using Lévy's inequality and Chebyshev's inequality, we have for $m \geqslant 1$,

$$\begin{aligned}
(3.17)\quad \mathbb{P}\big(\max_{1\leqslant k\leqslant n_m} |S_k| > 2\epsilon n_m^{1/q}\big) &\leqslant 2\mathbb{P}(|S_{n_m}| > 2\epsilon n_m^{1/q}) \\
&\leqslant 2\mathbb{P}\Big(\Big|\sum_{i=1}^{n_m} X_i I(|X_i| \leqslant n_m^{1/q})\Big| > \epsilon n_m^{1/q}\Big) \\
&\quad + 2\mathbb{P}\Big(\Big|\sum_{i=1}^{n_m} X_i I(|X_i| > n_m^{1/q})\Big| > \epsilon n_m^{1/q}\Big) \\
&\leqslant \left(\frac{2}{\epsilon^2}\right) \frac{n_m \mathbb{E}\big(X^2 I(|X| \leqslant n_m^{1/q})\big)}{n_m^{2/q}} + 2n_m \mathbb{P}(|X| > n_m^{1/q})\ \ \forall\, \epsilon > 0.
\end{aligned}$$

Let $\zeta = 2^{\varphi(r,q)}$. Then $0 < \zeta < 1$ (since $\varphi(r,q) < 0$). Note that

$$
\begin{aligned}
(3.18)\quad & \frac{n_m \mathbb{E}\big(X^2 I(|X| \leqslant n_m^{1/q})\big)}{n_m^{2/q}} \\
&= \frac{n_m \mathbb{E}\big(|X|^r |X|^{2-r} I(|X| \leqslant n_m^{1/b})\big)}{n_m^{2/q}} + \frac{n_m \mathbb{E}\big(X^2 I(n_m^{1/b} < |X| \leqslant n_m^{1/q})\big)}{n_m^{2/q}} \\
&\leqslant (\mathbb{E}|X|^r)\, \frac{n_m n_m^{(2/b)-(r/b)}}{n_m^{2/q}} + \frac{n_m n_m^{2/q} \mathbb{P}(n_m^{1/b} < |X| \leqslant n_m^{1/q})}{n_m^{2/q}} \\
&\leqslant (\mathbb{E}|X|^r)\, n_m^{\varphi(r,q)} + \frac{1}{n_m^{\eta}} \\
&\leqslant (\mathbb{E}|X|^r)\, (2^m)^{\varphi(r,q)} + \frac{1}{2^{\eta m}} = (\mathbb{E}|X|^r)\, \zeta^m + \frac{1}{2^{\eta m}} \quad \forall\, m \geqslant 1
\end{aligned}
$$

and

$$
(3.19)\qquad n_m \mathbb{P}(|X| > n_m^{1/q}) \leqslant \frac{1}{n_m^{\eta}} \leqslant \frac{1}{2^{\eta m}} \quad \forall\, m \geqslant 1.
$$

It thus follows from (3.17)–(3.19) that

$$
\begin{aligned}
(3.20)\quad & \sum_{m=1}^{\infty} \mathbb{P}\big(\max_{1 \leqslant k \leqslant n_m} |S_k| > 2\epsilon n_m^{1/q}\big) \\
&\leqslant \frac{2}{\epsilon^2} \left(\mathbb{E}|X|^r \sum_{m=1}^{\infty} \zeta^m + \sum_{m=1}^{\infty} \frac{1}{2^{\eta m}} \right) + 2 \sum_{m=1}^{\infty} \frac{1}{2^{\eta m}} < \infty \quad \forall\, \epsilon > 0.
\end{aligned}
$$

Applying the Borel–Cantelli lemma, we see that (3.20) implies

$$
\lim_{m \to \infty} \frac{\max_{1 \leqslant k \leqslant n_m} |S_k|}{n_m^{1/q}} = 0 \ \text{ a.s.},
$$

which ensures that

$$
\begin{aligned}
\liminf_{n \to \infty} \big(\max_{1 \leqslant k \leqslant n} |S_k|\big)^{(\log n)^{-1}} &= \liminf_{n \to \infty} e^{1/q} \left(\frac{\max_{1 \leqslant k \leqslant n} |S_k|}{n^{1/q}} \right)^{(\log n)^{-1}} \\
&\leqslant e^{1/q} < e \ \text{ a.s.},
\end{aligned}
$$

i.e., (3.16) holds. ■

LEMMA 3.6. *Let $\{X_n;\ n \geqslant 1\}$ be a sequence of independent copies of a real-valued random variable X such that $0 < \rho_1 < 1 \wedge \rho_2$. Then*

$$
(3.21)\qquad \liminf_{n \to \infty} \big(\max_{1 \leqslant k \leqslant n} |S_k|\big)^{(\log n)^{-1}} < e^{1/\rho_1} \ \text{ a.s.}
$$

Proof. Let $h = \big((1\wedge\rho_2)-\rho_1\big)/4,\ \eta = h/(\rho_1+2h)$, and $b=\rho_1+2h$. Since $0<\rho_1<1\wedge\rho_2$, we have $h>0$, $\eta>0$, $b\in(\rho_1,1\wedge\rho_2)$, and by the definition of ρ_2,

$$\liminf_{t\to\infty} t^{1+\eta}\mathbb{P}(|X|>t^{1/b})=0.$$

Then, proceeding inductively, we can choose an increasing sequence $\{n_m;\ m\geqslant 1\}$ of positive integers such that $n_1=1$ and

$$n_m=\min\left\{k\geqslant 2^m\vee(n_{m-1}+1):\ \mathbb{P}(|X|>k^{1/b})\leqslant\frac{1}{k^{1+\eta}}\right\},\qquad m>1.$$

Write

$$\phi(x,y)=1+\frac{1}{b}-\frac{x}{b}-\frac{1}{y},\qquad (x,y)\in(0,1)\times(0,1).$$

Since

$$\lim_{\substack{x\nearrow\rho_1\\ y\searrow\rho_1}}\phi(x,y)=1+\frac{1}{b}-\frac{\rho_1}{b}-\frac{1}{\rho_1}=\frac{\rho_1 b+\rho_1-\rho_1^2-b}{\rho_1 b}=\frac{(\rho_1-1)(b-\rho_1)}{\rho_1 b}<0,$$

we can choose r and q such that

$$0<r<\rho_1<q<b\quad\text{ and }\quad\phi(r,q)=1+\frac{1}{b}-\frac{r}{b}-\frac{1}{q}<0,$$

and hence

$$\mathbb{E}|X|^r<\infty,\qquad\frac{1}{b}<\frac{1}{q}<\frac{1}{\rho_1},\qquad\text{and}\qquad\mathbb{P}(|X|>n_m^{1/q})\leqslant\frac{1}{n_m^{1+\eta}},\ \ m\geqslant 1.$$

Using Markov's inequality, we have for $m\geqslant 1$,

$$(3.22)\quad \mathbb{P}\big(\max_{1\leqslant k\leqslant n_m}|S_k|>2\epsilon n_m^{1/q}\big)\leqslant\mathbb{P}\Big(\sum_{i=1}^{n_m}|X_i|>2\epsilon n_m^{1/q}\Big)$$
$$\leqslant\mathbb{P}\Big(\sum_{i=1}^{n_m}|X_i|I(|X_i|\leqslant n_m^{1/q})>\epsilon n_m^{1/q}\Big)+\mathbb{P}\Big(\sum_{i=1}^{n_m}|X_i|I(|X_i|>n_m^{1/q})>\epsilon n_m^{1/q}\Big)$$
$$\leqslant\left(\frac{1}{\epsilon}\right)\frac{n_m\mathbb{E}\big(|X|I(|X|\leqslant n_m^{1/q})\big)}{n_m^{1/q}}+n_m\mathbb{P}(|X|>n_m^{1/q})\ \ \forall\,\epsilon>0.$$

Let $\tau = 2^{\phi(r,q)}$. Then $0 < \tau < 1$ (since $\phi(r,q) < 0$). Note that

$$\begin{aligned}
(3.23)\quad & \frac{n_m \mathbb{E}\big(|X| I(|X| \leqslant n_m^{1/q})\big)}{n_m^{1/q}} \\
&= \frac{n_m \mathbb{E}\big(|X|^r |X|^{1-r} I(|X| \leqslant n_m^{1/b})\big)}{n_m^{1/q}} + \frac{n_m \mathbb{E}\big(|X| I(n_m^{1/b} < |X| \leqslant n_m^{1/q})\big)}{n_m^{1/q}} \\
&\leqslant (\mathbb{E}|X|^r) \frac{n_m n_m^{(1/b)-(r/b)}}{n_m^{1/q}} + \frac{n_m n_m^{1/q} \mathbb{P}(n_m^{1/b} < |X| \leqslant n_m^{1/q})}{n_m^{1/q}} \\
&\leqslant (\mathbb{E}|X|^r)\, n_m^{\phi(r,q)} + \frac{1}{n_m^{\eta}} \\
&\leqslant (\mathbb{E}|X|^r)\, (2^m)^{\phi(r,q)} + \frac{1}{2^{\eta m}} = (\mathbb{E}|X|^r)\, \tau^m + \frac{1}{2^{\eta m}} \ \ \forall\, m \geqslant 1
\end{aligned}$$

and

$$(3.24)\qquad n_m \mathbb{P}(|X| > n_m^{1/q}) \leqslant n_m \mathbb{P}(|X| > n_m^{1/b}) \leqslant \frac{1}{2^{\eta m}} \ \ \forall\, m \geqslant 1.$$

It thus follows from (3.22)–(3.24) that

$$\begin{aligned}
&\sum_{m=1}^{\infty} \mathbb{P}\big(\max_{1 \leqslant k \leqslant n_m} |S_k| > 2\epsilon n_m^{1/q}\big) \\
&\qquad \leqslant \frac{1}{\epsilon}\left(\mathbb{E}|X|^r \sum_{m=1}^{\infty} \tau^m + \sum_{m=1}^{\infty} \frac{1}{2^{\eta m}}\right) + \sum_{m=1}^{\infty} \frac{1}{2^{\eta m}} < \infty \ \ \forall\, \epsilon > 0,
\end{aligned}$$

which, by applying the Borel–Cantelli lemma, yields

$$\lim_{m \to \infty} \frac{\max_{1 \leqslant k \leqslant n_m} |S_k|}{n_m^{1/q}} = 0 \ \text{ a.s.}$$

Hence

$$\liminf_{n \to \infty} \frac{\max_{1 \leqslant k \leqslant n} |S_k|}{n^{1/q}} = 0 \ \text{ a.s.,}$$

which ensures that

$$\begin{aligned}
\liminf_{n \to \infty} \big(\max_{1 \leqslant k \leqslant n} |S_k|\big)^{(\log n)^{-1}} &= \liminf_{n \to \infty} e^{1/q} \left(\frac{\max_{1 \leqslant k \leqslant n} |S_k|}{n^{1/q}}\right)^{(\log n)^{-1}} \\
&\leqslant e^{1/q} < e^{1/\rho_1} \ \text{ a.s.,}
\end{aligned}$$

i.e., (3.21) holds. ■

4. PROOFS OF THEOREMS 2.1–2.4

With the preliminary results provided in the previous sections, Theorems 2.1–2.4 may be proved.

Proof of Theorem 2.1. By Theorem 3.1 with $p = 2$ and Lemma 3.2, (2.1) and (2.3) are equivalent.

Clearly, (2.2) follows from (2.1).

It remains to show that (2.2) implies (2.1). It follows from (2.2) that X is a nondegenerate random variable. By Lemma 3.2, (3.4) holds, and hence (2.1) follows. ■

Proof of Theorem 2.2. The "if" part follows from Theorem 3.1 with $1 < p < 2$ and Lemma 3.3.

We now establish the "only if" part. Since $1 < p < 2$, by Theorem 3.1, it follows from

$$\lim_{n\to\infty}\big(\max_{1\leqslant k\leqslant n}|S_k|\big)^{(\log n)^{-1}} = e^{1/p} \text{ a.s.} \tag{4.1}$$

that $\mathbb{E}X = 0$ and $\rho_1 = p$. If $\rho_2 \neq p$, then $\rho_1 < \rho_2$ (since $\rho_1 \leqslant \rho_2$). By Lemma 3.4, we have

$$\liminf_{n\to\infty}\big(\max_{1\leqslant k\leqslant n}|S_k|\big)^{(\log n)^{-1}} < e^{1/\rho_1} = e^{1/p} \text{ a.s.,}$$

which contradicts (4.1). Thus (4.1) implies that $\mathbb{E}X = 0$ and $\rho_1 = \rho_2 = p$. ■

Proof of Theorem 2.3. (i) If $\rho_1 = \rho_2 = 1$, then (2.5) follows from Theorem 3.1 with $p = 1$ and Lemma 3.3.

If $\mathbb{E}|X| < \infty$ and $\mathbb{E}X \neq 0$, then, by the Kolmogorov SLLN,

$$\lim_{n\to\infty}\frac{\max_{1\leqslant k\leqslant n}|S_k|}{n} = |\mathbb{E}X| \text{ a.s.,}$$

and hence

$$\lim_{n\to\infty}\big(\max_{1\leqslant k\leqslant n}|S_k|\big)^{(\log n)^{-1}} = \lim_{n\to\infty} e\left(\frac{\max_{1\leqslant k\leqslant n}|S_k|}{n}\right)^{(\log n)^{-1}} = e \text{ a.s.,}$$

i.e., (2.5) holds.

(ii) From part (i), we only need to prove the "only if" part. Since X is a symmetric random variable, by Theorem 3.1 with $p = 1$ and Lemma 3.5, (2.5) implies that $\rho_1 = \rho_2 = 1$. ■

REMARK 4.1. *We now construct a counterexample to show that* (2.4) *is not necessary for* (2.5) *to hold. In fact, for given* $1 < \rho < \infty$, *let* X *be a real-valued random variable with probability distribution given by*

$$\mathbb{P}(X = d_n) = \frac{c_2}{d_n}, \quad n \geqslant 1,$$

where

$$d_n = 2^{\rho^n},\ n \geqslant 1, \quad \textit{and} \quad c_2 = \left(\sum_{n=1}^{\infty} \frac{1}{d_n}\right)^{-1} > 0.$$

Let $\{X_n;\ n \geqslant 1\}$ *be a sequence of independent copies of* X. *Then*

$$\mathbb{E}X = \infty, \quad \rho_1 = 1, \quad \rho_2 = \rho > 1,$$

and (2.5) *holds by* $S_n \geqslant nd_1$, $n \geqslant 1$, *and Theorem* 3.1 *with* $p = \rho_1 = 1$.

P r o o f o f T h e o r e m 2.4. By using Theorem 3.1 with $0 < p < 1$ and Lemma 3.3, the "if" part follows.

Since $0 < p < 1$, the "only if" part follows from Theorem 3.1 with $0 < p < 1$ and Lemma 3.6. ■

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REFERENCES

[1] Y. S. Chow and H. Teicher, *Probability Theory: Independence, Interchangeability, Martingales*, third edition, Springer, New York 1997.

[2] K. L. Chung, *On the maximum partial sums of sequences of independent random variables*, Trans. Amer. Math. Soc. 64 (1948), pp. 205–233.

[3] E. Csáki, *On the lower limits of maxima and minima of Wiener process and partial sums*, Z. Wahrsch. Verw. Gebiete 43 (1978), pp. 205–221.

[4] P. Hartman and A. Wintner, *On the law of the iterated logarithm*, Amer. J. Math. 63 (1941), pp. 169–176.

[5] N. C. Jain and W. E. Pruitt, *The other law of the iterated logarithm*, Ann. Probab. 3 (1975), pp. 1046–1049.

[6] H. Kesten, *The limit points of a normalized random walk*, Ann. Math. Statist. 41 (1970), pp. 1173–1205.

[7] A. Kolmogoroff, *Sur la loi forte des grands nombres*, C. R. Acad. Sci. Paris Sér. Math. 191 (1930), pp. 910–912.

[8] J. Marcinkiewicz and A. Zygmund, *Sur les fonctions indépendantes*, Fund. Math. 29 (1937), pp. 60–90.

[9] T. Mikosch, *The law of the iterated logarithm for independent random variables outside the domain of partial attraction of the normal law*, Vestnik Leningrad. Univ. Mat. Mekh. Astronom. no. 3 (1984), pp. 35–39 (in Russian).

[10] V. V. Petrov, *Limit Theorems of Probability Theory: Sequences of Independent Random Variables*, Clarendon, Oxford 1995.

[11] Y. Zou and X. Liu, *An extension of a theorem of Mikosch*, Statist. Probab. Lett. 120 (2017), pp. 81–86.

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