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DENSENESS OF CERTAIN SMOOTH LÉVY FUNCTIONALS IN $\mathbb{D}_{1,2}$

BY

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Abstract. The Malliavin derivative for a Lévy process (X_t) can be defined on the space $\mathbb{D}_{1,2}$ using a chaos expansion or in the case of a pure jump process also *via* an increment quotient operator. In this paper we define the Malliavin derivative operator $\boldsymbol{D}$ on the class $\mathcal{S}$ of smooth random variables $f(X_{t_1}, \ldots, X_{t_n})$, where f is a smooth function with compact support. We show that the closure of $L_2(\mathbb{P}) \supseteq \mathcal{S} \xrightarrow{\boldsymbol{D}} L_2(\mathbb{m} \otimes \mathbb{P})$ yields to the space $\mathbb{D}_{1,2}$. As an application we conclude that Lipschitz functions operate on $\mathbb{D}_{1,2}$.

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1. INTRODUCTION

In the recent years Malliavin calculus for Lévy processes has been developed using various types of chaos expansions. For example, Lee and Shih [5] applied a white noise approach, León et al. [6] worked with certain strongly orthogonal martingales, Løkka [7] and Di Nunno et al. [2] considered multiple integrals with respect to the compensated Poisson random measure and Solé et al. [11] used the chaos expansion proved by Itô [4].

This chaos representation from Itô applies to any square integrable functional of a general Lévy process. It uses multiple integrals like in the well-known Brownian motion case but with respect to an independent random measure associated with the Lévy process. Solé et al. propose in [12] a canonical space for a general Lévy process. They define for random variables on the canonical space the increment quotient operator

$$\Psi_{t,x} F(\omega) = \frac{F(\omega_{t,x}) - F(\omega)}{x}, \quad x \neq 0,$$

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in a pathwise sense, where, roughly speaking, $\omega_{t,x}$ can be interpreted as the outcome of adding at time t a jump of the size x to the path ω. They show that on the canonical Lévy space the Malliavin derivative $D_{t,x}F$ defined *via* the chaos expansion due to Itô and $\Psi_{t,x}F$ coincide a.e. on $\mathbb{R}_+ \times \mathbb{R}_0 \times \Omega$ (where $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$) whenever $F \in L_2$ and $\mathbb{E} \int_{\mathbb{R}_+ \times \mathbb{R}_0} |\Psi_{t,x}F|^2 d\mathbb{m}(t,x) < \infty$ (see Section 2 for the definition of $\mathbb{m}$). On the other hand, on the Wiener space, the Malliavin derivative is introduced as an operator D mapping smooth random variables of the form $F = f\big(W(h_1), \ldots, W(h_n)\big)$ into $L_2(\Omega; H)$, i.e.

$$DF = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f\big(W(h_1), \ldots, W(h_n)\big) h_i$$

(see, for example, [8]). Here f is a smooth function mapping from $\mathbb{R}^n$ into $\mathbb{R}$ such that all its derivatives have at most polynomial growth, and $\{W(h), h \in H\}$ is an isonormal Gaussian family associated with a Hilbert space H. The closure of the domain of the operator D is the space $\mathbb{D}_{1,2}$.

In the present paper we proceed in a similar way for a Lévy process $(X_t)_{t \geqslant 0}$. We will define a Malliavin derivative on a class of smooth random variables and determine its closure. The class of smooth random variables we consider consists of elements of the form $F = f(X_{t_1}, \ldots, X_{t_n})$, where $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth function with compact support.

Analogously to results of Solé et al. [12] about the canonical Lévy space the Malliavin derivative $DF \in L_2(\mathbb{m} \otimes \mathbb{P})$, defined *via* chaos expansion, can be expressed explicitly as a two-parameter operator $D_{t,x}$. For certain smooth random variables of the form $F = f(X_{t_1}, \ldots, X_{t_n})$ we have

$$\begin{aligned} D_{t,x} f(X_{t_1}, \ldots, X_{t_n}) = & \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(X_{t_1}, \ldots, X_{t_n}) \mathbb{1}_{[0,t_i] \times \{0\}}(t,x) \\ & + \Psi_{t,x} f(X_{t_1}, \ldots, X_{t_n}) \mathbb{1}_{\{x \neq 0\}}(x) \end{aligned}$$

for $\mathbb{m} \otimes \mathbb{P}$-a.e. (t, x, ω). Here $\Psi_{t,x}$ for $x \neq 0$ is given by

$$\begin{aligned} & \Psi_{t,x} f(X_{t_1}, \ldots, X_{t_n}) \\ & \quad := \frac{f\big(X_{t_1} + x\mathbb{1}_{[0,t_1]}(t), \ldots, X_{t_n} + x\mathbb{1}_{[0,t_n]}(t)\big) - f(X_{t_1}, \ldots, X_{t_n})}{x}. \end{aligned}$$

Our main result is that the smooth random variables $f(X_{t_1}, \ldots, X_{t_n})$ are dense in the space $\mathbb{D}_{1,2}$ defined *via* the chaos expansion. This implies that defining $\boldsymbol{D}$ as an operator on the smooth random variables as in Definition 3.2 below and taking the closure leads to the same result as defining D using Itô's chaos expansion (see Definition 2.1).

The paper is organized as follows. In Section 2 we shortly recall Itô's chaos expansion, the definition of the Malliavin derivative and some related facts. The third and fourth sections focus on the introduction of the Malliavin derivative operator on smooth random variables and the determination of its closure. Applying the denseness result from the previous section we show in Section 5 that Lipschitz functions map from $\mathbb{D}_{1,2}$ into $\mathbb{D}_{1,2}$.

2. THE MALLIAVIN DERIVATIVE VIA ITÔ'S CHAOS EXPANSION

We assume a càdlàg Lévy process $X = (X_t)_{t\geqslant 0}$ on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with Lévy triplet (γ, σ^2, ν), where $\gamma \in \mathbb{R}$, $\sigma \geqslant 0$ and ν is the Lévy measure. Then X has the Lévy–Itô decomposition

$$X_t = \gamma t + \sigma W_t + \int\limits_{[0,t]\times\{|x|\geqslant 1\}} x dN(t,x) + \int\limits_{[0,t]\times\{0<|x|<1\}} x d\tilde{N}(t,x),$$

where W denotes a standard Brownian motion, N is the Poisson random measure associated with the process X and $\tilde{N}$ the compensated Poisson random measure, $d\tilde{N}(t,x) = dN(t,x) - dtd\nu(x)$. Consider the measures μ on $\mathcal{B}(\mathbb{R})$,

$$d\mu(x) := \sigma^2 d\delta_0(x) + x^2 d\nu(x),$$

and $\mathbb{m}$ on $\mathcal{B}(\mathbb{R}_+ \times \mathbb{R})$, where $\mathbb{R}_+ := [0,\infty)$,

$$d\mathbb{m}(t,x) := dtd\mu(x).$$

For $B \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R})$ such that $\mathbb{m}(B) < \infty$ let

$$M(B) = \sigma \int\limits_{\{t\in\mathbb{R}_+:(t,0)\in B\}} dW_t + \lim_{n\to\infty} \int\limits_{\{(t,x)\in B: 1/n<|x|<n\}} x d\tilde{N}(t,x),$$

where the convergence is taken in the space $L_2(\Omega, \mathcal{F}, \mathbb{P})$. Now $\mathbb{E}M(B_1)M(B_2) = \mathbb{m}(B_1 \cap B_2)$ for all B_1, B_2 with $\mathbb{m}(B_1) < \infty$ and $\mathbb{m}(B_2) < \infty$. For $n = 1, 2, \ldots$ let us write

$$L_2^n := L_2\left((\mathbb{R}_+ \times \mathbb{R})^n, \mathcal{B}(\mathbb{R}_+ \times \mathbb{R})^{\otimes n}, \mathbb{m}^{\otimes n}\right).$$

For $f \in L_2^n$ Itô [4] defines a multiple integral $I_n(f)$ with respect to the random measure M. It follows that $I_n(f) = I_n(\tilde{f})$ a.s., where $\tilde{f}$ is the symmetrization of f,

$$\tilde{f}(z_1, \ldots, z_n) = \frac{1}{n!} \sum_{\pi\in\mathcal{S}_n} f(z_{\pi(1)}, \ldots, z_{\pi(n)})$$

for all $z_i = (t_i, x_i) \in \mathbb{R}_+ \times \mathbb{R}$, and $\mathcal{S}_n$ denotes the set of all permutations on $\{1, \ldots, n\}$.

Let $(\mathcal{F}_t^X)_{t\geqslant 0}$ be the augmented natural filtration of X. Then $(\mathcal{F}_t^X)_{t\geqslant 0}$ is right continuous ([9], Theorem I 4.31). Set $\mathcal{F}^X := \bigvee_{t\geqslant 0} \mathcal{F}_t^X$. By Theorem 2 of Itô [4] the chaos decomposition

$$L_2 := L_2(\Omega, \mathcal{F}^X, \mathbb{P}) = \bigoplus_{n=0}^{\infty} I_n(L_2^n)$$

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holds, where $I_0(L_2^0) := \mathbb{R}$ and $I_n(L_2^n) := \{I_n(f_n) : f_n \in L_2^n\}$ for $n = 1, 2, \dots$ For $F \in L_2$ the representation

$$F = \sum_{n=0}^{\infty} I_n(f_n)$$

with $I_0(f_0) = \mathbb{E}F$ a.s. is unique if the functions f_n are symmetric. Furthermore,

$$\|F\|_{L_2}^2 = \sum_{n=0}^{\infty} n! \|\tilde{f}_n\|_{L_2^n}^2.$$

DEFINITION 2.1. Let $\mathbb{D}_{1,2}$ be the space of all $F = \sum_{n=0}^{\infty} I_n(f_n) \in L_2$ such that

$$\|F\|_{\mathbb{D}_{1,2}}^2 := \sum_{n=0}^{\infty} (n+1)! \|\tilde{f}_n\|_{L_2^n}^2 < \infty.$$

Set $L_2(\mathbb{m} \otimes \mathbb{P}) := L_2\big(\mathbb{R}_+ \times \mathbb{R} \times \Omega, \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}) \otimes \mathcal{F}^X, \mathbb{m} \otimes \mathbb{P}\big)$. The *Malliavin derivative* $D : \mathbb{D}_{1,2} \to L_2(\mathbb{m} \otimes \mathbb{P})$ is defined by

$$D_{t,x} F := \sum_{n=1}^{\infty} n I_{n-1}\Big(\tilde{f}_n\big((t,x), \cdot\big)\Big), \quad (t,x,\omega) \in \mathbb{R}_+ \times \mathbb{R} \times \Omega. \tag{2.1}$$

We consider (as Solé et al. [12]) the operators $D_{\cdot,0}$ and $D_{\cdot,x}$, $x \neq 0$, and their domains $\mathbb{D}_{1,2}^0$ and $\mathbb{D}_{1,2}^J$. For $\sigma > 0$ assume that $\mathbb{D}_{1,2}^0$ consists of random variables $F = \sum_{n=0}^{\infty} I_n(f_n) \in L_2$ such that

$$\|F\|_{\mathbb{D}_{1,2}^0}^2 := \|F\|_{L_2}^2 + \sum_{n=1}^{\infty} n \cdot n! \|\tilde{f}_n \mathbb{1}_{(\mathbb{R}_+ \times \{0\}) \times (\mathbb{R}_+ \times \mathbb{R})^{n-1}}\|_{L_2^n}^2 < \infty.$$

For $\nu \neq 0$, let $\mathbb{D}_{1,2}^J$ be the set of $F \in L_2$ such that

$$\|F\|_{\mathbb{D}_{1,2}^J}^2 := \|F\|_{L_2}^2 + \sum_{n=1}^{\infty} n \cdot n! \|\tilde{f}_n \mathbb{1}_{(\mathbb{R}_+ \times \mathbb{R}_0) \times (\mathbb{R}_+ \times \mathbb{R})^{n-1}}\|_{L_2^n}^2 < \infty,$$

where $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$. If both $\sigma > 0$ and $\nu \neq 0$, then

$$\mathbb{D}_{1,2} = \mathbb{D}_{1,2}^0 \cap \mathbb{D}_{1,2}^J. \tag{2.2}$$

In case $\nu = 0$, $D_{\cdot,0}$ coincides with the classical Malliavin derivative D^W (see, for example, [8]) except for a multiplicative constant, $D_t^W F = \sigma D_{t,0} F$.

In the next lemma we formulate a denseness result which will be used to determine the closure of the Malliavin operator from Definition 3.2 below.

LEMMA 2.1. *Let $\mathcal{L} \subseteq L_2$ be the linear span of random variables of the form*

$$M(T_1 \times A_1) \dots M(T_n \times A_n), \quad n = 1, 2, \dots$$

where the A_i's are finite intervals of the form $(a_i, b_i]$ and the T_i's are finite disjoint intervals of the form $T_i = (s_i, t_i]$. Then $\mathcal{L}$ is dense in L_2, $\mathbb{D}_{1,2}$, $\mathbb{D}_{1,2}^0$ and $\mathbb{D}_{1,2}^J$.

P r o o f. 1° First we consider the class of all linear combinations of

$$M(B_1)\dots M(B_n) = I_n(\mathbb{1}_{B_1\times\dots\times B_n}),$$

$n = 1, 2, \dots$, where the sets $B_i \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R})$ are disjoint and fulfill the condition $\mathbb{m}(B_i) < \infty$. It follows from the completeness of the multiple integrals in L_2 (see [4], Theorem 2) that this class is dense in L_2. Especially, the class of all linear combinations of $\mathbb{1}_{B_1\times\dots\times B_n}$ with disjoint sets $B_1, \dots, B_n$ of finite measure $\mathbb{m}$ is dense in $L_2^n = L_2\big((\mathbb{R}_+ \times \mathbb{R})^n, \mathcal{B}(\mathbb{R}_+ \times \mathbb{R})^{\otimes n}, \mathbb{m}^{\otimes n}\big)$. Let $\mathcal{H}_n$ be the linear span of $\mathbb{1}_{(T_1\times A_1)\times\dots\times(T_n\times A_n)}$, where $A_i = (a_i, b_i]$ and $T_i = (s_i, t_i]$. One can easily see that $\mathcal{H}_n$ is dense in L_2^n as well. Indeed, because $\mathbb{m}$ is a Radon measure, there are compact sets $C_i \subseteq B_i$ such that $\mathbb{m}(B_i \setminus C_i)$ is sufficiently small to get

$$\|\mathbb{1}_{B_1\times\dots\times B_n} - \mathbb{1}_{C_1\times\dots\times C_n}\|_{L_2^n} < \varepsilon$$

for some given $\varepsilon > 0$. Since the compact sets (C_i) are disjoint, one can find disjoint bounded open sets $U_i \supseteq C_i$ such that $\|\mathbb{1}_{C_1\times\dots\times C_n} - \mathbb{1}_{U_1\times\dots\times U_n}\|_{L_2^n} < \varepsilon$. For any bounded open set $U_i \subseteq (0,\infty)\times\mathbb{R}$ one can find a sequence of 'half-open rectangles' $Q_{i,k} = (s_k^i, t_k^i] \times (a_k^i, b_k^i] = T_k^i \times A_k^i$ such that $U_i = \bigcup_{k=1}^\infty Q_{i,k}$ (taking half-open rectangles $Q_x \subseteq U_i$ with rational 'end points' containing the point $x \in U_i$ gives $U_i = \bigcup_{Q_x\subseteq U_i} Q_x$).

Hence for sufficiently large K_i's we have

$$\|\mathbb{1}_{U_1\times\dots\times U_n} - \mathbb{1}_P\|_{L_2^n} < \varepsilon, \quad \text{where } P := \bigcup_{k=1}^{K_1} Q_{1,k} \times \dots \times \bigcup_{k=1}^{K_n} Q_{n,k}$$

and where the $Q_{i,1}, \dots, Q_{i,K_i}$ can now be chosen such that they are disjoint. This implies that the linear span of $\mathbb{1}_{Q_1\times\dots\times Q_n}$, where the Q_i's are of the form $T_i \times A_i$, is dense in L_2^n.

2° For the convenience of the reader we recall the idea of the proof of Lemma 2 in [4] to show that the intervals T_i can be chosen disjoint. Consider the situation (all other cases can be treated similarly) where for the set

$$(T_1 \times A_1) \times \dots \times (T_n \times A_n)$$

we have $T_1 = T_2$. To shorten the notation we write

$$Q := (T_3 \times A_3) \times \dots \times (T_n \times A_n).$$

Choosing an equidistant partition $(E_j)_{j=1}^k$ of T_1 we have

$$\mathbb{1}_{(T_1\times A_1)\times(T_1\times A_2)\times Q} = \sum_{j\neq l} \mathbb{1}_{(E_j\times A_1)\times(E_l\times A_2)\times Q} + \sum_{j=1}^k \mathbb{1}_{(E_j\times A_1)\times(E_j\times A_2)\times Q}.$$

It can be easily checked that $\big\| \sum_{j=1}^k \mathbb{1}_{(E_j\times A_1)\times(E_j\times A_2)\times Q}\big\|_{L_2^n} \to 0$ as $k \to \infty$.

3^{o} The denseness of $\mathcal{H}_n$ in L_2^n implies that $\mathcal{L}$ is dense in L_2 and $\mathbb{D}_{1,2}$. The remaining cases follow from the fact that

$$\|f_n \mathbb{1}_{(\mathbb{R}_+\times\{0\})\times(\mathbb{R}_+\times\mathbb{R})^{n-1}}\|_{L_2^n} \leqslant \|f_n\|_{L_2^n}$$

and

$$\|f_n \mathbb{1}_{(\mathbb{R}_+\times\mathbb{R}_0)\times(\mathbb{R}_+\times\mathbb{R})^{n-1})}\|_{L_2^n} \leqslant \|f_n\|_{L_2^n}. \quad \blacksquare$$

3. THE MALLIAVIN DERIVATIVE AS OPERATOR ON $\mathcal{S}$

Let $C_c^\infty(\mathbb{R}^n)$ denote the space of smooth functions $f:\mathbb{R}^n \to \mathbb{R}$ with compact support.

DEFINITION 3.1. A random variable of the form $F = f(X_{t_1},\ldots,X_{t_n})$, where $f \in C_c^\infty(\mathbb{R}^n)$, $n \in \mathbb{N}$, and $t_1,\ldots,t_n \geqslant 0$, is said to be a *smooth random variable*. The set of all smooth random variables is denoted by $\mathcal{S}$.

DEFINITION 3.2. For $F = f(X_{t_1},\ldots,X_{t_n}) \in \mathcal{S}$ we define the *Malliavin derivative operator* $\boldsymbol{D}$ as a map from $\mathcal{S}$ into $L_2(\mathbb{m}\otimes\mathbb{P})$ by

$$\begin{aligned} &\boldsymbol{D}_{t,x} f(X_{t_1},\ldots,X_{t_n}) \\ &:= \sum_{i=1}^n \frac{\partial f}{\partial x_i}(X_{t_1},\ldots,X_{t_n})\mathbb{1}_{[0,t_i]\times\{0\}}(t,x) \\ &\quad + \frac{f\big(X_{t_1}+x\mathbb{1}_{[0,t_1]}(t),\ldots,X_{t_n}+x\mathbb{1}_{[0,t_n]}(t)\big) - f(X_{t_1},\ldots,X_{t_n})}{x}\mathbb{1}_{\mathbb{R}_0}(x) \end{aligned}$$

for $(t,x) \in \mathbb{R}_+\times\mathbb{R}$.

The following lemma holds true:

LEMMA 3.1. *We have* $\boldsymbol{D}F = DF$ *in* $L_2(\mathbb{m}\otimes\mathbb{P})$ *for all* $F \in \mathcal{S}$.

Since for $f(X_{t_1},\ldots,X_{t_n}) \in \mathcal{S}$ we get

$$\mathbb{E}\int_{\mathbb{R}_+} |\boldsymbol{D}_{t,0} f(X_{t_1},\ldots,X_{t_n})|^2 dt < \infty$$

and

$$\mathbb{E}\int_{\mathbb{R}_+\times\mathbb{R}_0} |\boldsymbol{D}_{t,x} f(X_{t_1},\ldots,X_{t_n})|^2 d\mathbb{m}(t,x) < \infty,$$

Lemma 3.1 follows for the canonical Lévy space from Propositions 3.5 and 5.5 in [12].

A proof of Lemma 3.1 for the situation where the Lévy process (X_t) is a square integrable pure jump process which has an absolutely continuous distribution can be found in [7].

An outline of the proof in the general case is given in the Appendix. Like in [7], Proposition 8, one can derive from the proof an explicit form for the functions (f_n) of the chaos expansion $f(X_{t_1},\ldots,X_{t_k})=\sum_{n=0}^{\infty} I_n(f_n)$,

$$f_n\big((s_1,x_1),\ldots,(s_n,x_n)\big)$$
$$=\mathbb{E}\sum_{I\subset\{1,\ldots,n\}\cup\emptyset}\frac{(-1)^{n-|I|}}{n!}\frac{f\big(X_{t_1}+\sum_{i\in I}x_i\mathbb{1}_{[0,t_1]}(s_i),\ldots,X_{t_k}+\sum_{i\in I}x_i\mathbb{1}_{[0,t_k]}(s_i)\big)}{x_1\ldots x_n},$$

with the convention that to get $f_n\big((s_1,x_1),\ldots,(s_i,0),\ldots,(s_n,x_n)\big)$ one has to take the limit $\lim_{|x_i|\downarrow 0} f_n\big((s_1,x_1),\ldots,(s_n,x_n)\big)$.

Especially, since any $F\in L_2\supseteq\mathcal{S}$ has a unique chaos expansion, we conclude that also $\boldsymbol{D}F$ does not depend on the representation $F=f(X_{t_1},\ldots,X_{t_n})$. Using the equality of $\boldsymbol{D}$ and D on $\mathcal{S}$ and the fact that $\mathcal{S}$ is closed with respect to multiplication we are now able to reformulate Proposition 5.1 of [12] for our situation:

COROLLARY 3.1. *For F and G in $\mathcal{S}$ we have*

$$D_{t,x}(FG)=GD_{t,x}F+FD_{t,x}G+xD_{t,x}FD_{t,x}G$$

for $\mathbb{m}\otimes\mathbb{P}$*-a.e.* $(t,x,\omega)\in\mathbb{R}_+\times\mathbb{R}\times\Omega$.

4. THE CLOSURE OF THE MALLIAVIN DERIVATIVE OPERATOR

The operator $\boldsymbol{D}:\mathcal{S}\to L_2(\mathbb{m}\otimes\mathbb{P})$ is *closable* if for any sequence $(F_n)\subseteq\mathcal{S}$ which converges to 0 in L_2 such that $\boldsymbol{D}(F_n)$ converges in $L_2(\mathbb{m}\otimes\mathbb{P})$ it follows that $(\boldsymbol{D}F_n)$ converges to 0 in $L_2(\mathbb{m}\otimes\mathbb{P})$. As we know from the previous section that D and $\boldsymbol{D}$ coincide on $\mathcal{S}\subseteq\mathbb{D}_{1,2}$, it is clear that $\boldsymbol{D}$ is closable and the closure of the domain of definition of $\boldsymbol{D}$ with respect to the norm

$$\|F\|_{\boldsymbol{D}}:=[\mathbb{E}|F|^2+\mathbb{E}\|\boldsymbol{D}F\|^2_{L_2(\mathbb{m})}]^{1/2}$$

is contained in $\mathbb{D}_{1,2}$. What remains to show is that the closure is equal to $\mathbb{D}_{1,2}$.

THEOREM 4.1. *The closure of $\mathcal{S}$ with respect to the norm $\|\cdot\|_{\boldsymbol{D}}=\|\cdot\|_{\mathbb{D}_{1,2}}$ is the space $\mathbb{D}_{1,2}$.*

Theorem 4.1 implies that the Malliavin derivative D defined *via* Itô's chaos expansion and the closure of the operator $L_2\supseteq\mathcal{S}\overset{\boldsymbol{D}}{\to}L_2(\mathbb{m}\otimes\mathbb{P})$ coincide. Before we start with the proof we formulate a lemma for later use.

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LEMMA 4.1. *For $\varphi \in C_c^\infty(\mathbb{R})$ and partitions $\pi_n := \{s = t_0^n < t_1^n < \ldots < t_n^n = u\}$ of the interval $[s,u]$ it follows for $\psi(x) := x\varphi(x)$ that*

$$\mathbb{D}_{1,2} - \lim_{|\pi_n|\to 0} \Big(\sum_{j=1}^n \psi(X_{t_j^n} - X_{t_{j-1}^n}) - \mathbb{E} \sum_{j=1}^n \psi(X_{t_j^n} - X_{t_{j-1}^n}) \Big) = \int\limits_{(s,u]\times\mathbb{R}} \varphi(x)\, dM(t,x),$$

where $|\pi_n| := \max_{1\leqslant i\leqslant n} |t_i^n - t_{i-1}^n|$.

P r o o f. To keep the notation simple, we drop the n of the partition points t_j^n. Notice that

$$\int\limits_{(s,u]\times\mathbb{R}} \varphi(x)\, dM(t,x) = I_1(\mathbb{1}_{(s,u]} \otimes \varphi).$$

We set

$$G^n := \sum_{j=1}^n \psi(X_{t_j} - X_{t_{j-1}}) - \mathbb{E} \sum_{j=1}^n \psi(X_{t_j} - X_{t_{j-1}})$$

and

$$G := \int\limits_{(s,u]\times\mathbb{R}} \varphi(x)\, dM(t,x).$$

In general, $\psi(X_{t_j} - X_{t_{j-1}}) \notin \mathcal{S}$ but we can conclude from Lemma 3.1 that

$$\boldsymbol{D}_{t,x}\psi(X_{t_j} - X_{t_{j-1}}) = D_{t,x}\psi(X_{t_j} - X_{t_{j-1}})$$

$\mathbb{m} \otimes \mathbb{P}$-a.e. using a suitable approximation of $\psi(X_{t_j} - X_{t_{j-1}})$ by a sequence of smooth random variables from $\mathcal{S}$. So we can write $D_{t,x}G^n$ explicitly as

$$D_{t,x}G^n = \sum_{j=1}^n \psi'(X_{t_j} - X_{t_{j-1}}) \mathbb{1}_{(t_{j-1},t_j]\times\{0\}}(t,x) + \sum_{j=1}^n \frac{\psi(X_{t_j} - X_{t_{j-1}} + x) - \psi(X_{t_j} - X_{t_{j-1}})}{x} \mathbb{1}_{(t_{j-1},t_j]\times\mathbb{R}_0}(t,x).$$

Moreover, we have $D_{t,x}I_1(\mathbb{1}_{(s,u]} \otimes \varphi) = \mathbb{1}_{(s,u]}(t)\varphi(x)$ $\mathbb{m}$-a.e. Using the general fact that for any $F \in \mathbb{D}_{1,2}$ with expectation zero the inequality

$$\|F\|^2_{\mathbb{D}_{1,2}} \leqslant 2\|DF\|^2_{L_2(m\otimes\mathbb{P})}$$

holds true, we obtain

$$\begin{aligned}\|G-G^n\|_{\mathbb{D}_{1,2}}^2 &\leqslant 2\|DG-DG^n\|_{L_2(\mathbb{m}\otimes\mathbb{P})}^2\\ &=2\sigma^2\mathbb{E}\int_{\mathbb{R}_+}\sum_{j=1}^n \mathbb{1}_{(t_{j-1},t_j]}(t)[\varphi(0)-\psi'(X_{t_j}-X_{t_{j-1}})]^2dt\\ &\quad+2\mathbb{E}\int_{\mathbb{R}_+\times\mathbb{R}_0}\sum_{j=1}^n \mathbb{1}_{(t_{j-1},t_j]}(t)[\psi(x)-\psi(X_{t_j}-X_{t_{j-1}}+x)\\ &\quad+\psi(X_{t_j}-X_{t_{j-1}})]^2dtd\nu(x)\\ &\to 0\end{aligned}$$

as $n\to\infty$ because of dominated convergence and the a.s. càdlàg property of the paths of (X_t). ■

P r o o f o f T h e o r e m 4.1. According to Lemma 2.1 it is sufficient to show that an expression like $M(T_1\times A_1)\ldots M(T_n\times A_n)$, where the A_i's are bounded Borel sets and the T_i's finite disjoint intervals, can be approximated in $\mathbb{D}_{1,2}$ by a sequence $(F_k)_k\subseteq\mathcal{S}$.

1° In this step we want to show that it is enough to approximate

$$I_1(\mathbb{1}_{T_1}\otimes\varphi_1)\ldots I_1(\mathbb{1}_{T_n}\otimes\varphi_n) \tag{4.1}$$

by $(F_k)_k\subseteq\mathcal{S}$, where $\varphi_i\in C_c^\infty(\mathbb{R})$. Since the intervals T_i are disjoint, the definition of the multiple integral implies that

$$M(T_1\times A_1)\ldots M(T_n\times A_n)=I_n(\mathbb{1}_{T_1\times A_1}\otimes\ldots\otimes\mathbb{1}_{T_n\times A_n})\quad\text{a.s.}$$

By the same reason,

$$I_1(\mathbb{1}_{T_1}\otimes\varphi_1)\ldots I_1(\mathbb{1}_{T_n}\otimes\varphi_n)=I_n\big((\mathbb{1}_{T_1}\otimes\varphi_1)\otimes\ldots\otimes(\mathbb{1}_{T_n}\otimes\varphi_n)\big)\quad\text{a.s.}$$

We have

$$\begin{aligned}&\big\|I_n(\mathbb{1}_{(T_1\times A_1)\times\ldots\times(T_n\times A_n)})-I_n\big((\mathbb{1}_{T_1}\otimes\varphi_1)\otimes\ldots\otimes(\mathbb{1}_{T_n}\otimes\varphi_n)\big)\big\|_{\mathbb{D}_{1,2}}^2\\ &\leqslant(n+1)!\|\mathbb{1}_{(T_1\times A_1)\times\ldots\times(T_n\times A_n)}-(\mathbb{1}_{T_1}\otimes\varphi_1)\otimes\ldots\otimes(\mathbb{1}_{T_n}\otimes\varphi_n)\|_{L_2^n}^2\\ &\leqslant(n+1)!|T_1|\ldots|T_n|\|\mathbb{1}_{A_1\times\ldots\times A_n}-\varphi_1\otimes\ldots\otimes\varphi_n\|_{L_2^n(\mu^{\otimes n})}^2.\end{aligned}$$

The last expression can be made arbitrarily small by choosing φ_i such that the expression $\|\mathbb{1}_{A_i}-\varphi_i\|_{L_2^1(\mu)}$ is small. Indeed, for each i there are compact sets $C_1^i\subseteq C_2^i\subseteq\ldots\subseteq A_i$ and open sets $U_1^i\supseteq U_2^i\supseteq\ldots\supseteq A_i$ such that

$$\mu(U_k^i\setminus C_k^i)\to 0$$

as $k \to \infty$. By the C^∞ Urysohn lemma ([3], p. 237) there is for each k a function $\varphi_k^i \in C_c^\infty(\mathbb{R})$ such that $0 \leqslant \varphi_k^i \leqslant 1$, $\varphi_k^i = 1$ on C_k^i and $\operatorname{supp}(\varphi_k^i) \subset U_k^i$. Then

$$\| \mathbb{1}_{A_i} - \varphi_i^k \|^2_{L_2^1(\mu)} \leqslant \mu(U_k^i \setminus C_k^i) \to 0$$

as $k \to \infty$.

2° Now we use Lemma 4.1 to approximate the expression (4.1) by a sequence $(F_k)_k \subseteq \mathcal{S}$. For $i = 1, \ldots, n$ set $\psi_i(x) := x\varphi_i(x)$ and

$$G_i^k := \sum_{j=1}^{k} \mathbb{1}_{\{t_j, t_{j-1} \in \bar{T}_i\}} \psi_i(X_{t_j} - X_{t_{j-1}}) - \mathbb{E} \sum_{j=1}^{k} \mathbb{1}_{\{t_j, t_{j-1} \in \bar{T}_i\}} \psi_i(X_{t_j} - X_{t_{j-1}}).$$

The partition $\pi_k = \{0 \leqslant t_0^k \leqslant \ldots \leqslant t_k^k\}$ can be chosen such that all end points of the closed intervals $\bar{T}_i$ belong to π_k. Put

$$f_k(X_{t_0}, \ldots, X_{t_k}) := \prod_{i=1}^{n} G_i^k$$

and notice that $f_k \in C^\infty(\mathbb{R}^{k+1})$. Let us choose functions $\beta_m \in C_c^\infty(\mathbb{R})$ such that $0 \leqslant \beta_m \leqslant 1$ and $\beta_m(x) = 1$ for $|x| \leqslant m$, the support of β_m is contained in $\{x; |x| \leqslant m+2\}$ and $\|\beta_m'\|_\infty \leqslant 1$. Setting $x_{-1} := 0$ and

$$\alpha_m(x_0, \ldots, x_k) := \prod_{i=0}^{k} \beta_m(x_i - x_{i-1}),$$

we have $f_k(x)\alpha_m(x) \in C_c^\infty(\mathbb{R}^{k+1})$. By dominated convergence one can show that

$$\mathbb{D}_{1,2} - \lim_{m \to \infty} f_k(X_{t_0}, \ldots, X_{t_k}) \alpha_m(X_{t_0}, \ldots, X_{t_k}) = f_k(X_{t_0}, \ldots, X_{t_k}).$$

Because the intervals (T_i) are disjoint, it follows that the product rule holds in our case:

$$D \prod_{i=1}^{n} G_i^k = \sum_{i=1}^{n} G_1^k \ldots G_{i-1}^k (D G_i^k) G_{i+1}^k \ldots G_n^k \quad \mathbbm{m} \otimes \mathbb{P}\text{-a.e.} \tag{4.2}$$

Indeed, because of $D_{t,x} G_i^k = (D_{t,x} G_i^k) \mathbb{1}_{T_i}(t)$ we have

$$x(D_{t,x} G_i^k) \mathbb{1}_{T_i}(t) (D_{t,x} G_j^k) \mathbb{1}_{T_j}(t) = 0 \quad \mathbbm{m} \otimes \mathbb{P}\text{-a.e.}$$

for any $i \neq j$. Equation (4.2) follows then by induction. Let

$$G_i := I_1(\mathbb{1}_{T_i} \otimes \varphi_i).$$

We observe that $G_1^k, \ldots, G_n^k$ as well as $G_1^k, \ldots, G_{i-1}^k, DG_i^k, G_{i+1}^k, \ldots, G_n^k$ are mutually independent by construction. Hence to show L_2-convergence of these products it is enough to prove L_2-convergence for each factor. From Lemma 4.1 we obtain $G_i^k \to G_i$ in $\mathbb{D}_{1,2}$ for all $i = 1, \ldots, n$, so that

$$L_2(\mathbb{m} \otimes \mathbb{P}) - \lim_{|\pi_k| \to 0} G_1^k \ldots G_{i-1}^k (DG_i^k) G_{i+1}^k \ldots G_n^k$$
$$= G_1 \ldots G_{i-1} (DG_i) G_{i+1} \ldots G_n.$$

Consequently, we have found a sequence $(F_k) \subseteq \mathcal{S}$ given by

$$F_k = f_k(X_{t_0}, \ldots, X_{t_k}) \alpha_{m_k}(X_{t_0}, \ldots, X_{t_k}),$$

where the m_k's are chosen in a suitable way, that converges to expression (4.1) in $\mathbb{D}_{1,2}$. ∎

COROLLARY 4.1. *The set $\mathcal{S}$ of smooth random variables is dense in L_2, $\mathbb{D}_{1,2}^0$ and $\mathbb{D}_{1,2}^J$.*

P r o o f. The denseness in L_2 is clear. To show that $\mathcal{S}$ is dense in $\mathbb{D}_{1,2}^0$ assume $F \in \mathbb{D}_{1,2}^0$ has the representation $F = \sum_{n=0}^{\infty} I_n(f_n)$. For a given $\varepsilon > 0$ fix N_ε such that $\| \sum_{n=N_\varepsilon}^{\infty} I_n(f_n) \|_{\mathbb{D}_{1,2}^0} < \varepsilon$. From $F \in L_2$ we conclude

$$F^{N_\varepsilon} := \sum_{n=0}^{N_\varepsilon} I_n(f_n) \in \mathbb{D}_{1,2}.$$

By Theorem 4.1 we can find a sequence $(F_k) \subseteq \mathcal{S}$ converging to F^{N_ε} in $\mathbb{D}_{1,2}$, and therefore also in $\mathbb{D}_{1,2}^0$. In the same way one can see that $\mathcal{S}$ is dense in $\mathbb{D}_{1,2}^J$. ∎

5. LIPSCHITZ FUNCTIONS OPERATE ON $\mathbb{D}_{1,2}$

LEMMA 5.1. *Assume that $g : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous with Lipschitz constant L_g.*

(a) *If $\sigma > 0$, then $g(F) \in \mathbb{D}_{1,2}^0$ for all $F \in \mathbb{D}_{1,2}^0$ and*

$$D_{t,0} g(F) = G D_{t,0} F \quad dt \otimes \mathbb{P}\text{-a.e.}, \tag{5.1}$$

where G is a random variable which is a.s. bounded by L_g.

(b) *If $\nu \neq 0$, then $g(F) \in \mathbb{D}_{1,2}^J$ for all $F \in \mathbb{D}_{1,2}^J$, where*

$$D_{t,x} g(F) = \frac{g(F + x D_{t,x} F) - g(F)}{x} \tag{5.2}$$

for $\mathbb{m} \otimes \mathbb{P}$-a.e. $(t, x, \omega) \in \mathbb{R}_+ \times \mathbb{R}_0 \times \Omega$.

Proof. (a) We will adapt the proof of Proposition 1.2.4 in [8] to our situation. Corollary 4.1 implies that there exists a sequence $(F_n) \subseteq \mathcal{S}$ of the form $F_n = f_n(X_{t_1}, \dots, X_{t_n})$ which converges to F in $\mathbb{D}^0_{1,2}$. Like in [8], we choose a non-negative $\psi \in C_c^\infty(\mathbb{R})$ such that $\operatorname{supp}(\psi) \subseteq [-1,1]$ and $\int_{\mathbb{R}} \psi(x)dx = 1$ and define the approximation of unity $\psi_m(x) := m\psi(mx)$. Then $g_m := g * \psi_m$ is smooth and converges uniformly to g as $m \to \infty$. Moreover, $\|g'_m\|_\infty \leqslant L_g$. Hence $g_m(F_n) - g_m(0) \in \mathcal{S}$ and $\big(g_n(F_n)\big)$ converges to $g(F)$ in L_2. Moreover,

$$\mathbb{E} \int_{\mathbb{R}_+} |D_{t,0} g_n(F_n)|^2 dt \leqslant L_g^2 \, \|F_n\|^2_{\mathbb{D}^0_{1,2}}.$$

Since $\big(g_n(F_n)\big)$ converges to $g(F)$ in L_2 and

$$\sup_n \|g_n(F_n)\|^2_{\mathbb{D}^0_{1,2}} < \infty,$$

Lemma 1.2.3 in [8] states that $g(F) \in \mathbb{D}^0_{1,2}$ and that $\big(D_{\cdot,0}\, g_n(F_n)\big)$ converges to $D_{\cdot,0}\, g(F)$ in the weak topology of $L_2\big(\Omega; L_2(\mathbb{R}_+ \times \{0\})\big)$. The obvious inequality $\mathbb{E}|g'_n(F_n)|^2 \leqslant L_g^2$ implies the existence of a subsequence $\big(g'_{n_k}(F_{n_k})\big)_k$ which converges to some $G \in L_2$ in the weak topology of L_2. One can show that $|G| \leqslant L_g$ a.s. Hence for any element $\alpha \in L_\infty\big(\Omega; L_2(\mathbb{R}_+ \times \{0\})\big)$ we have

$$\lim_{k\to\infty} \mathbb{E} \int_{\mathbb{R}_+} g'_{n_k}(F_{n_k})(D_{t,0}\, F_{n_k})\alpha(t)dt = \mathbb{E}\Big(G \int_{\mathbb{R}_+} (D_{t,0}\, F)\alpha(t)dt\Big).$$

Consequently, $D_{t,0}\, g(F) = GD_{t,0}F \; dt \otimes \mathbb{P}$-a.e.

(b) Let $(F_n)_n \subseteq \mathcal{S}$ be a sequence such that $\mathbb{D}^J_{1,2} - \lim F_n = F$. Since the expression

$$Z(t,x) := \frac{g(F + xD_{t,x}F) - g(F)}{x} \mathbb{1}_{\mathbb{R}_+ \times \mathbb{R}_0}(t,x)$$

is in $L_2(\mathbb{m} \otimes \mathbb{P})$, it is enough to show that the sequence $\big(Dg_n(F_n)\mathbb{1}_{\mathbb{R}_+ \times \mathbb{R}_0}\big)$ converges in $L_2(\mathbb{m} \otimes \mathbb{P})$ to Z, where (g_n) is the sequence constructed in (a). Choose $T > 0$ and $L > 0$ large enough and $\delta > 0$ sufficiently small such that

$$\limsup_n \mathbb{E} \int_{([0,T]\times\{\delta \leqslant |x| \leqslant L\})^c} |Z(t,x)|^2 + |D_{t,x} g_n(F_n)|^2 d\mathbb{m}(t,x) < \varepsilon.$$

Then, for $n \geqslant n_0$,

$$\begin{aligned}
&\|Z - Dg_n(F_n)\mathbb{1}_{\mathbb{R}_+\times\mathbb{R}_0}\|^2_{L_2(\mathbb{m}\otimes\mathbb{P})} \\
&\leqslant \varepsilon + 2\mathbb{E} \int_{[0,T]\times\{\delta \leqslant |x| \leqslant L\}} \left| Z(t,x) - \frac{g(F_n + xD_{t,x}F_n) - g(F_n)}{x} \right|^2 d\mathbb{m}(t,x) \\
&\quad + 8\delta^{-2} T \mu(\{\delta \leqslant |x| \leqslant L\}) \|g - g_n\|^2_\infty.
\end{aligned}$$

Hence we obtain (5.2) from the Lipschitz continuity of g and the uniform convergence of g_n to g. ■

PROPOSITION 5.1. *Let $g: \mathbb{R} \to \mathbb{R}$ be Lipschitz continuous. Then $F \in \mathbb{D}_{1,2}$ implies $g(F) \in \mathbb{D}_{1,2}$, where $Dg(F)$ is given by* (5.1) *and* (5.2).

Proof. The assertion is an immediate consequence of Lemma 5.1 and the equality (2.2). ■

6. APPENDIX

Proof of Lemma 3.1. We denote by $J_n(f_n)$ the multiple integral

$$\int\limits_{\mathbb{R}_+\times\mathbb{R}} \int\limits_{[0,t_n)\times\mathbb{R}} \cdots \int\limits_{[0,t_2)\times\mathbb{R}} f_n\big((t_1,x_1),\ldots,(t_n,x_n)\big)\, dM(t_1,x_1)\ldots dM(t_n,x_n),$$

where for the definition of a stochastic integral with respect to M we refer to [1]. We have

$$I_n(\tilde{f}_n) = n! J_n(\tilde{f}_n). \tag{6.1}$$

Let us first prove on $\mathcal{S}$ a Clark–Ocone–Haussman type formula for the operator $\boldsymbol{D}$. By the Fourier inversion formula (see, for example, [1]) we infer for $f \in C_c^\infty(\mathbb{R}^k)$ that

$$f(X_{t_1},\ldots,X_{t_k}) = \int\limits_{\mathbb{R}^k} \hat{f}(u)\exp\big(2\pi i \sum_{j=1}^k u_j X_{t_j}\big)du = \int\limits_{\mathbb{R}^k} \hat{f}(u)e^{\eta(u,T)}Y_T(u)du,$$

where $e^{\eta(u,t)} = \mathbb{E}\exp\big(2\pi i \sum_{j=1}^k (u_j X_{t_j \wedge t})\big)$ and

$$Y_t(u) = \exp\big(2\pi i \sum_{j=1}^k u_j X_{t_j\wedge t} - \eta(u,t)\big) \quad \text{for } 0 \leqslant t \leqslant T := \max\{t_1,\ldots,t_k\}.$$

We rewrite $Y_T(u)$ by Itô's formula using $\xi(u,s) := 2\pi i \sum_{j=1}^k u_j \mathbb{1}_{[0,t_j]}(s)$ and get

$$\begin{aligned} f(X_{t_1},&\ldots,X_{t_k}) \\ &= \int\limits_{\mathbb{R}^k} \hat{f}(u)e^{\eta(u,T)}\,du \\ &\quad + \int\limits_{\mathbb{R}^k} \hat{f}(u)e^{\eta(u,T)}\Big(\int\limits_0^T Y_{s-}(u)\xi(u,s)\,\sigma dW_s\Big)du \\ &\quad + \int\limits_{\mathbb{R}^k} \hat{f}(u)e^{\eta(u,T)}\Big(\int\limits_{(0,T]\times\mathbb{R}_0} Y_{s-}(u)(e^{x\xi(u,s)}-1)d\tilde{N}(s,x)\Big)du. \end{aligned} \tag{6.2}$$

It follows by Fubini's theorem that

$$\int\limits_{\mathbb{R}^k} \hat{f}(u)e^{\eta(u,T)}\,du = \mathbb{E}\int\limits_{\mathbb{R}^k} \hat{f}(u)\exp\big(2\pi i \sum_{j=1}^k u_j X_{t_j}\big)\,du = \mathbb{E}f(X_{t_1},\ldots,X_{t_k}).$$

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Now we deal with the second term on the right-hand side of (6.2). Using the fact that the process $(Y_t)_{t\in[0,T]}$ is a square integrable martingale, we infer by the conditional theorem of Fubini (see, e.g., [1]) and Fubini's theorem for stochastic integrals (see, e.g., [10]) that it can be written as

$$\int_0^T \mathbb{E}\Big[\int_{\mathbb{R}^k} Y_T(u)\hat{f}(u)e^{\eta(u,T)}\xi(u,s)\,du\Big|\mathcal{F}_{s-}\Big]\sigma dW_s.$$

Applying Theorem 8.22 (e) of [3] and the Fourier inversion formula we rewrite the inner integral as follows:

$$\begin{aligned}\int_{\mathbb{R}^k} & Y_T(u)\hat{f}(u)e^{\eta(u,T)}\xi(u,s)\,du \\ &= \sum_{j=1}^k \mathbb{1}_{[0,t_j]}(s)\int_{\mathbb{R}^k} 2\pi i u_j \hat{f}(u)\exp\big(2\pi i\sum_{j=1}^k u_j X_{t_j}\big)du \\ &= \sum_{j=1}^k \mathbb{1}_{[0,t_j]}(s)\frac{\partial f}{\partial x_j}(X_{t_1},\ldots,X_{t_k}).\end{aligned}$$

Similarly, one can write the last term on the right-hand side of (6.2) as

$$\int_{(0,T]\times\mathbb{R}_0} \mathbb{E}\Big[\int_{\mathbb{R}^k}\hat{f}(u)e^{\eta(u,T)}Y_T(u)(e^{x\xi(u,s)}-1)du\Big|\mathcal{F}_{s-}\Big]d\tilde{N}(s,x),$$

where

$$\begin{aligned}&\int_{\mathbb{R}^k}\hat{f}(u)e^{\eta(u,T)}Y_T(u)(e^{x\xi(u,s)}-1)du \\ &= \int_{\mathbb{R}^k}\hat{f}(u)\Big(\exp\big[2\pi i\sum_{j=1}^k u_j\big(X_{t_j}+x\mathbb{1}_{[0,t_j]}(s)\big)\big]-\exp\big(2\pi i\sum_{j=1}^k u_j X_{t_j}\big)\Big)du \\ &= f\big(X_{t_1}+x\mathbb{1}_{[0,t_1]}(s),\ldots,X_{t_k}+x\mathbb{1}_{[0,t_k]}(s)\big)-f(X_{t_1},\ldots,X_{t_k}).\end{aligned}$$

Consequently, for $F=f(X_{t_1},\ldots,X_{t_k})\in\mathcal{S}$ the Clark–Ocone–Haussman type formula holds true:

$$F=\mathbb{E}F+\int_{\mathbb{R}_+\times\mathbb{R}}\mathbb{E}\left[\boldsymbol{D}_{t,x}F|\mathcal{F}_{t-}\right]dM(t,x). \tag{6.3}$$

Since $\boldsymbol{D}_{t,x}f(X_{t_1},\ldots,X_{t_k})\in\mathcal{S}$ for any $(t,x)\in\mathbb{R}_+\times\mathbb{R}$, iterating equation (6.3) we obtain

$$f(X_{t_1},\ldots,X_{t_k})=\mathbb{E}f(X_{t_1},\ldots,X_{t_k})+\sum_{n=1}^\infty J_n\big(\mathbb{E}\boldsymbol{D}^n f(X_{t_1},\ldots,X_{t_k})\big),$$

where $\boldsymbol{D}^n:=\boldsymbol{D}\ldots\boldsymbol{D}$.

Notice that $\mathbb{E}\boldsymbol{D}^n f(X_{t_1}, \ldots, X_{t_k})$ is a symmetric function on $(\mathbb{R}_+ \times \mathbb{R})^n$. The relation (6.1) between the multiple and the iterated integral and equation (2.1) together with $\boldsymbol{D}_{t,x} f(X_{t_1}, \ldots, X_{t_k}) \in L_2(\mathbb{m} \otimes \mathbb{P})$ imply that

$$\begin{aligned} D_{t,x} f(X_{t_1}, \ldots, X_{t_k}) &= \sum_{n=1}^{\infty} J_{n-1}\left(\mathbb{E}\boldsymbol{D}^{n-1}\boldsymbol{D}_{t,x} f(X_{t_1}, \ldots, X_{t_k})\right) \\ &= \boldsymbol{D}_{t,x} f(X_{t_1}, \ldots, X_{t_k}) \ \ \mathbb{m} \otimes \mathbb{P}\text{-a.e.} \quad \blacksquare \end{aligned}$$

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