ON A RANDOM NUMBER OF DISORDERS

BY

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Abstract. We register a random sequence which has three segments being the homogeneous Markov processes. Each segment has its own one-step transition probability law and the length of the segment is unknown and random. It means that at two random moments θ_1, θ_2, where $0 \leqslant \theta_1 \leqslant \theta_2$, the source of observation is changed. In effect, the number of homogeneous segments is random. The transition probabilities of each process are known and the *a priori* distribution of the disorder moments is given. The former research on such a problem has been devoted to various questions concerning the distribution changes. The random number of distributional segments creates new problems in solutions with relation to analysis of the model with deterministic number of segments. Two cases are presented in detail. In the first one the objective is to stop on or between the disorder moments while in the second one our objective is to find the strategy which immediately detects the distribution changes. Both problems are reformulated to optimal stopping of the observed sequences. The detailed analysis of the problem is presented to show the form of optimal decision function.

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1. INTRODUCTION

Suppose that the process $X = \{X_n, n \in \mathbb{N}\}$, $\mathbb{N} = \{0, 1, 2, \ldots\}$, is observed sequentially. The process is obtained from three Markov processes by switching between them at two random moments of time, θ_1 and θ_2. Our objective is to detect these moments based on observation of X.

Such a model of data appears in many practical problems of the quality control (see Brodsky and Darkhovsky [5], Shewhart [17] and the collection of papers [2]), traffic anomalies in networks (Dube and Mazumdar [6], Tartakovsky

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et al. [22]), epidemiology models (see Baron [1]). In management of manufacture it happens that the plants which produce some details change their parameters, which makes the details change their quality. Production can be divided into three sorts. Assuming that at the beginning of the production process the quality is the highest, from some moment θ_1 the products should be classified to lower sort and beginning with the moment θ_2 the details should be categorized as having the lowest quality. The aim is to recognize the moments of these changes.

Shiryaev [18], [19] solved the disorder problem of independent random variables with one disorder where the mean distance between disorder time and the moment of its detection was minimized. The probability maximizing approach to the problem was used by Bojdecki [3] and the stopping time which is in a given neighborhood of the moment of disorder with maximal probability was found. The disorders in more complicated dependence structures of switched sequences are subjects of investigation by Pelkowitz [14], [15], Yakir [24], Moustakides [11], Lai [9], [10], Fuh [7], Tartakovsky and Veeravalli [23]. The probability maximizing approach to such problems with two disorders was considered by Yoshida [25], Szajowski [20], [21] and Sarnowski and Szajowski [16]. Yoshida [25] investigated the problem of optimal stopping by observation of the process X so as to maximize the probability that the distance between the moments of disorder θ_i and their estimates, the stopping times τ_i, $i = 1, 2$, will not exceed given numbers (for each disorder independently). This question has been reformulated by Szajowski [21] to the simultaneous detection of both disorders under the requirement that the performance of procedure is globally measured for both detections and it has been extended to the case with unknown distribution between disorders by Sarnowski and Szajowski [16] (see also Bojdecki and Hosza [4] for related approach with switching sequences of independent random variables). The method of solution is based on a transformation of the model to the double optimal stopping problem for Markovian function of some statistics (see Haggstrom [8], Nikolaev [12]). The strategy which stops the process between the first and the second disorder with maximal probability has been constructed by Szajowski [20]. The considerations are inspired by the problem regarding how we can protect ourselves against a second fault in a technological system after the occurrence of an initial fault or by the problem of detection at the beginning and the end of an epidemic.

The paper is devoted to a generalization of the double disorder problem considered both in [20] and [21] in which immediate switch from the first preliminary distribution to the third one is possible (i.e. it is possible that the random variables θ_1 and θ_2 are equal with a positive probability). It is also possible that we observe the homogeneous data without disorder when both disorder moments are equal to zero. The extension leads to serious difficulties in the construction of an equivalent double optimal stopping model. The formulation of the problem can be found in Section 2. The main results are subjects of Sections 4 (see Theorem 4.1) and 5.

2. FORMULATION OF DETECTION PROBLEMS

Let $(X_n)_{n\in\mathbb{N}}$ be an observable sequence of random variables defined on the space $(\Omega, \mathcal{F}, \mathbf{P})$ with values in $(\mathbb{E}, \mathcal{B})$, where $\mathbb{E}$ is a Borel subset of $\mathbf{R}$. On $(\mathbb{E}, \mathcal{B})$ there is a σ-additive measure μ. On the same probability space there are defined random variables θ_1, θ_2 with values in $\mathbb{N}$ and the following distributions:

$$(2.1) \qquad \mathbf{P}(\theta_1 = j) = \mathbb{I}_{\{j=0\}}(j)\pi + \mathbb{I}_{\{j>0\}}(j)\bar{\pi}p_1^{j-1}q_1,$$

$$(2.2) \qquad \mathbf{P}(\theta_2 = k \mid \theta_1 = j) = \mathbb{I}_{\{k=j\}}(k)\rho + \mathbb{I}_{\{k>j\}}(k)\bar{\rho}p_2^{k-j-1}q_2,$$

where $j = 0, 1, 2, \ldots$, $k = j, j+1, j+2, \ldots$, $\bar{\pi} = 1 - \pi$, $\bar{\rho} = 1 - \rho$. Additionally, we consider Markov processes $(X_n^i, \mathcal{G}_n^i, \mathbf{P}_x^i)$ on $(\Omega, \mathcal{F}, \mathbf{P})$, $i = 0, 1, 2$, where σ-fields $\mathcal{G}_n^i$ are the smallest σ-fields for which $(X_n^i)_{n=0}^{\infty}$, $i = 0, 1, 2$, are adapted, respectively. Let us define a process $(X_n)_{n\in\mathbb{N}}$ in the following way:

$$(2.3)$$
$$X_n = X_n^0 \mathbb{I}_{\{\theta_1 > n\}} + X_{n-\theta_1+1}^1 \mathbb{I}_{\{X_0^1 = x_{\theta_1-1}^0, \theta_1 \leqslant n < \theta_2\}} + X_{n-\theta_2+1}^2 \mathbb{I}_{\{X_0^2 = x_{\theta_2-\theta_1}^1, \theta_2 \leqslant n\}}.$$

We make inference on θ_1 and θ_2 from the observable sequence $(X_n, n \in \mathbb{N})$ only. It should be emphasized that the sequence $(X_n, n \in \mathbb{N})$ is not Markovian under an admitted assumption as it has been mentioned in [20], [24] and [6]. However, the sequence satisfies the Markov property given θ_1 and θ_2 (see Szajowski [21] and Moustakides [11]). Thus for further consideration we define a filtration $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$, where $\mathcal{F}_n = \sigma(X_0, X_1, \ldots, X_n)$, related to real observation. The variables θ_1, θ_2 are not stopping times with respect to $\mathcal{F}_n$ and σ-fields $\mathcal{G}_n^\bullet$. Moreover, we have knowledge about the distribution of (θ_1, θ_2) independent of any observation of the sequence $(X_n)_{n\in\mathbb{N}}$. This distribution, called the *a priori distribution* of (θ_1, θ_2), is given by (2.1) and (2.2).

It is assumed that the measures $\mathbf{P}_x^i(\cdot)$ on $\mathcal{F}$, $i = 0, 1, 2$, have the following representation. For any $B \in \mathcal{B}$ we have

$$\mathbf{P}_x^i(\omega : X_1^i \in B) = \mathbf{P}(X_1^i \in B | X_0^i = x) = \int_B f_x^i(y)\mu(dy) = \int_B \mu_x^i(dy) = \mu_x^i(B),$$

where the functions $f_x^i(\cdot)$ are different, and $f_x^i(y)/f_x^{(i+1)\bmod 3}(y) < \infty$ for $i = 0, 1, 2$ and all $x, y \in \mathbb{E}$. We assume that the measures μ_x^i, $x \in \mathbb{E}$, are known in advance.

For any $D_n = \{\omega : X_i \in B_i, \ i = 1, \ldots, n\}$, where $B_i \in \mathcal{B}$, and any $x \in \mathbb{E}$ we define

$$\mathbf{P}_x(D_n) = \mathbf{P}(D_n | X_0 = x)$$
$$= \int_{\times_{i=1}^n B_i} S_n(x, \vec{y}_n)\mu(d\vec{y}_n) = \int_{\times_{i=1}^n B_i} \mu_x(d\vec{y}_n) = \mu_x\left(\underset{i=1}{\overset{n}{\times}} B_i\right),$$

where the sequence of functions $S_n : \times_{i=1}^n \mathbb{E} \to \mathbf{R}$ is given by (7.5) in the Appendix.

The presented model has the following heuristic justification: Two disorders take place in the observed sequence (X_n). They affect distributions by changing their parameters. The disorders occur at two random times θ_1 and θ_2, $\theta_1 \leqslant \theta_2$. They split the sequence of observations into segments, at most three ones. The first segment is described by (X_n^0), the second one – for $\theta_1 \leqslant n < \theta_2$ – by $(X_{n-\theta_1+1}^1)$. The third is given by $(X_{n-\theta_2+1}^2)$ and is observed when $n \geqslant \theta_2$. When the first disorder takes place there is a "switch" from the initial distribution to the distribution with the conditional density f_x^i with respect of the measure μ, where $i = 1$ or $i = 2$, when $\theta_1 < \theta_2$ or $\theta_1 = \theta_2$, respectively. Next, if $\theta_1 < \theta_2$, at the random time θ_2 the distribution of observations becomes μ_x^2. We assume that the variables θ_1, θ_2 are unobservable directly.

Let $\mathcal{S}$ denote the set of all stopping times with respect to the filtration $(\mathcal{F}_n)$, $n = 0, 1, \ldots$, and $\mathcal{T} = \{(\tau, \sigma) : \tau \leqslant \sigma,\ \tau, \sigma \in \mathcal{S}\}$. Two problems with three distributional segments are recalled to investigate them under the weaker assumption that there are at most three homogeneous segments.

2.1. Detection of change. Our aim is to stop the observed sequence between the two disorders. This can be interpreted as a strategy for protecting against a second failure when the first has already happened. The mathematical model of this is to control the probability $\mathbf{P}_x(\tau < \infty, \theta_1 \leqslant \tau < \theta_2)$ by choosing the stopping time $\tau^* \in \mathcal{S}$ for which

$$(2.4) \qquad \mathbf{P}_x(\theta_1 \leqslant \tau^* < \theta_2) = \sup_{\tau \in \mathcal{T}} \mathbf{P}_x(\tau < \infty, \theta_1 \leqslant \tau < \theta_2).$$

2.2. Disorders detection. Our aim is to indicate the moments of switching with given precision d_1, d_2 (Problem $D_{d_1 d_2}$). We want to determine a pair of stopping times $(\tau^*, \sigma^*) \in \mathcal{T}$ such that for every $x \in \mathbb{E}$

(2.5)

$$\mathbf{P}_x(|\tau^* - \theta_1| \leqslant d_1, |\sigma^* - \theta_2| \leqslant d_2) = \sup_{\substack{(\tau, \sigma) \in \mathcal{T} \\ 0 \leqslant \tau \leqslant \sigma < \infty}} \mathbf{P}_x(|\tau - \theta_1| \leqslant d_1, |\sigma - \theta_2| \leqslant d_2).$$

The problem has been considered in [21] under natural simplification that there are three segments of data (i.e. there is $0 < \theta_1 < \theta_2$). In Section 5 the problem D_{00} is analyzed.

3. ON SOME *A POSTERIORI* PROCESSES

The formulated problems are translated to the optimal stopping problems for some Markov processes. The important part of the reformulation process is a choice of the *statistics* describing knowledge of the decision maker. The *a posteriori* probabilities of some events play the crucial role. Let us define the following *a posteri-*

ori processes (cf. [25], [20]):

$$(3.1) \qquad \Pi_n^i = \mathbf{P}_x(\theta_i \leqslant n | \mathcal{F}_n),$$

$$(3.2) \qquad \Pi_n^{12} = \mathbf{P}_x(\theta_1 = \theta_2 > n | \mathcal{F}_n) = P_x(\theta_1 = \theta_2 > n | \mathcal{F}_{mn}),$$

$$(3.3) \qquad \Pi_{mn} = \mathbf{P}_x(\theta_1 = m, \theta_2 > n | \mathcal{F}_{mn}),$$

where $\mathcal{F}_{mn} = \mathcal{F}_n$ for $m, n = 1, 2, \ldots$, $m < n$, $i = 1, 2$. For recursive representation of (3.1)–(3.3) we need the following functions:

$$\Pi^1(x, y, \alpha, \beta, \gamma) = 1 - \frac{p_1(1-\alpha)f_x^0(y)}{\mathbf{H}(x, y, \alpha, \beta, \gamma)},$$

$$\Pi^2(x, y, \alpha, \beta, \gamma) = \frac{(q_2\alpha + p_2\beta + q_1\gamma)f_x^2(y)}{\mathbf{H}(x, y, \alpha, \beta, \gamma)},$$

$$\Pi^{12}(x, y, \alpha, \beta, \gamma) = \frac{p_1\gamma f_x^0(y)}{\mathbf{H}(x, y, \alpha, \beta, \gamma)},$$

$$\Pi(x, y, \alpha, \beta, \gamma, \delta) = \frac{p_2\delta f_x^1(y)}{\mathbf{H}(x, y, \alpha, \beta, \gamma)},$$

where $\mathbf{H}(x, y, \alpha, \beta, \gamma) = (1-\alpha)p_1 f_x^0(y) + [p_2(\alpha - \beta) + q_1(1-\alpha-\gamma)]f_x^1(y) + [q_2\alpha + p_2\beta + q_1\gamma]f_x^2(y)$. In the sequel we adopt the following notation:

$$(3.4) \qquad \vec{\alpha} = (\alpha, \beta, \gamma),$$

$$(3.5) \qquad \overrightarrow{\Pi}_n = (\Pi_n^1, \Pi_n^2, \Pi_n^{12}).$$

The basic formulae used in the transformation of the disorder problems to the stopping problems are given in the following

LEMMA 3.1. *For each $x \in \mathbb{E}$ the following formulae for $m, n = 1, 2, \ldots$, $m < n$, hold*:

$$(3.6) \qquad \Pi_{n+1}^1 = \Pi^1(X_n, X_{n+1}, \Pi_n^1, \Pi_n^2, \Pi_n^{12}),$$

$$(3.7) \qquad \Pi_{n+1}^2 = \Pi^2(X_n, X_{n+1}, \Pi_n^1, \Pi_n^2, \Pi_n^{12}),$$

$$(3.8) \qquad \Pi_{n+1}^{12} = \Pi^{12}(X_n, X_{n+1}, \Pi_n^1, \Pi_n^2, \Pi_n^{12}),$$

$$(3.9) \qquad \Pi_{m,n+1} = \Pi(X_n, X_{n+1}, \Pi_n^1, \Pi_n^2, \Pi_n^{12}, \Pi_{mn})$$

with the boundary condition $\Pi_0^1 = \pi$, $\Pi_0^2(x) = \pi\rho$, $\Pi_0^{12}(x) = \bar{\pi}\rho$, and

$$\Pi_{mm} = (1-\rho)\frac{q_1 f_{X_{m-1}}^1(X_m)}{p_1 f_{X_{m-1}}^0(X_m)}(1 - \Pi_m^1).$$

P r o o f. The cases (3.6), (3.7) and (3.9), when $0 < \theta_1 < \theta_2$, have been proved in [25] and [20]. Let $0 \leqslant \theta_1 \leqslant \theta_2$ and suppose $B_i \in \mathcal{B}$, $1 \leqslant i \leqslant n + 1$. Let us assume that $X_0 = x$ and put $D_n = \{\omega : X_i(\omega) \in B_i, 1 \leqslant i \leqslant n\}$.

(3.6) For $A_i = \{\omega : X_i \in B_i\} \in \mathcal{F}_i$, $1 \leqslant i \leqslant n + 1$, and $D_{n+1} \in \mathcal{F}_{n+1}$, by properties of $S_n(\vec{x}_n)$, where $\vec{x}_n = (x_0, \ldots, x_n)$ (see Lemma 7.1), we have

$$
\int\limits_{D_{n+1}} \mathbf{P}_x(\theta_1 > n + 1 | \mathcal{F}_{n+1}) d\mathbf{P}_x = \int\limits_{D_{n+1}} \mathbb{I}_{\{\theta_1 > n+1\}} d\mathbf{P}_x
$$

$$
= \int\limits_{\times_{i=1}^{n+1} B_i} \frac{f_x^{n<\theta_1<\theta_2}(\vec{x}_{1n}) + f_x^{n<\theta_1=\theta_2}(\vec{x}_{1n})}{S_n(\vec{x}_n)} \frac{p_1 f_{x_n}^0(x_{n+1})}{\mathbf{H}(x_n, x_{n+1}, \overrightarrow{\Pi}_n(\vec{x}_n))} \mu_x(d\vec{x}_{1,n+1})
$$

$$
= \int\limits_{D_{n+1}} (1 - \Pi_n^1) \frac{p_1 f_{X_n}^0(X_{n+1})}{\mathbf{H}(X_n, X_{n+1}, \overrightarrow{\Pi}_n)} d\mathbf{P}_x.
$$

Thus, taking into account (3.1) we have $\Pi_{n+1}^1 = 1 - \mathbf{P}_x(\theta_1 > n + 1 \mid \mathcal{F}_{n+1}) = 1 - (1 - \Pi_n^1) p_1 f_{X_n}^0(X_{n+1}) \mathbf{H}^{-1}(X_n, X_{n+1}, \overrightarrow{\Pi}_n)$. This proves the formula (3.6).

(3.7) Under the same notation as in the proof of (3.6) we have, using the notation from Section 7.1 and the results of Lemma 7.3,

$$
\int\limits_{D_{n+1}} \mathbf{P}_x(\theta_2 \leqslant n + 1 \mid \mathcal{F}_{n+1}) d\mathbf{P}_x = \int\limits_{D_{n+1}} \mathbb{I}_{\{\theta_2 \leqslant n+1\}} d\mathbf{P}_x
$$

$$
\overset{(7.1)}{=} \int\limits_{\times_{i=1}^{n+1} B_i} \frac{f_x^{\theta_1 \leqslant \theta_2 \leqslant n+1}(\vec{x}_{1,n+1})}{S_n(\vec{x}_n) \mathbf{H}(x_n, x_{n+1}, \overrightarrow{\Pi}_n(\vec{x}_n))} \mu_x(d\vec{x}_{1,n+1})
$$

$$
= \int\limits_{\times_{i=1}^{n+1} B_i} \frac{[q_2 \Pi_n^1(\vec{x}_{0n}) + p_2 \Pi_n^2(\vec{x}_{0n}) + q_1 \Pi_n^{12}(\vec{x}_{0n})] f_{x_n}^2(x_{n+1})}{\mathbf{H}(x_n, x_{n+1}, \overrightarrow{\Pi}_n(\vec{x}_n))} \mu_x(d\vec{x}_{1,n+1})
$$

$$
= \int\limits_{D_{n+1}} \frac{[q_2 \Pi_n^1 + p_2 \Pi_n^2 + q_1 \Pi_n^{12}] f_{X_n}^2(X_{n+1})}{\mathbf{H}(X_n, X_{n+1}, \overrightarrow{\Pi}_n)} d\mathbf{P}_x.
$$

Thus we get

$$
\Pi_{n+1}^2 = \mathbf{P}_x(\theta_2 \leqslant n + 1 \mid \mathcal{F}_{n+1})
$$
$$
= [(\Pi_n^1 - \Pi_n^2)q_2 + \Pi_n^2 + q_1 \Pi_n^{12}] f_{X_n}^2(X_{n+1}) \mathbf{H}^{-1}(X_n, X_{n+1}, \overrightarrow{\Pi}_n),
$$

which leads to the formula (3.7).

(3.8) By (3.2) and the results of Lemma 7.3 we have

$$\int\limits_{D_{n+1}} \mathbf{P}_x(\theta_2 = \theta_1 > n+1 \mid \mathcal{F}_{n+1})d\mathbf{P}_x = \int\limits_{D_{n+1}} \mathbb{I}_{\{\theta_2=\theta_1\geqslant n+1\}}d\mathbf{P}_x$$

$$= \int\limits_{\times_{i=1}^{n+1} B_i} \frac{f_x^{\theta_1=\theta_2>n}(\vec{x}_{1,n+1})}{S_n(\vec{x}_n)\mathbf{H}(x_n,x_{n+1},\overrightarrow{\overline{\Pi}}_n(\vec{x}_n))}\mu_x(d\vec{x}_{1,n+1})$$

$$= \int\limits_{\times_{i=1}^{n+1} B_i} \frac{\Pi_n^{12}(\vec{x}_n)p_1 f_{x_n}^0(x_{n+1})}{\mathbf{H}(x_n,x_{n+1},\overrightarrow{\overline{\Pi}}_n(\vec{x}_n))}\mu_x(d\vec{x}_{1,n+1})$$

$$= \int\limits_{D_{n+1}} \frac{\Pi_n^{12}p_1 f_{X_n}^0(X_{n+1})}{\mathbf{H}(X_n,X_{n+1},\overrightarrow{\overline{\Pi}}_n)}d\mathbf{P}_x,$$

and, consequently,

$$\Pi_{n+1}^{12} = p_1\Pi_n^{12}f_{X_n}^0(X_{n+1})\mathbf{H}^{-1}(X_n,X_{n+1},\overrightarrow{\overline{\Pi}}_n),$$

which proves the formula (3.8).

(3.9) Similarly, by the definition (3.3) and the results of Lemma 7.3, we get

$$\int\limits_{D_{n+1}} \mathbf{P}_x(\theta_1 = m,\theta_2 > n+1 \mid \mathcal{F}_{n+1})d\mathbf{P}_x = \int\limits_{D_{n+1}} \mathbb{I}_{\{\theta_1=m,\theta_2>n+1\}}d\mathbf{P}_x$$

$$= \int\limits_{\times_{i=1}^{n+1} B_i} \frac{\bar{\pi}\bar{\rho}p_1^{m-1}q_1 p_2^{n+1}\prod_{s=1}^{m-1} f_{x_{s-1}}^0(x_s)\prod_{k=m}^n f_{x_{k-1}}^1(x_k)f_{x_n}^1(x_{n+1})}{S_n(x_{0n})\mathbf{H}(x_n,x_{n+1},\overrightarrow{\overline{\Pi}}_n(\vec{x}_n))}$$

$$\times \mu_x(d\vec{x}_{1,n+1})$$

$$= \int\limits_{\times_{i=1}^{n+1} B_i} \frac{\Pi_{mn}(\vec{x}_n)p_2 f_{x_n}^1(x_{n+1})}{\mathbf{H}(x_n,x_{n+1},\overrightarrow{\overline{\Pi}}_n(\vec{x}_n))}\mu_x(d\vec{x}_{1,n+1}) = \int\limits_{D_{n+1}} \frac{\Pi_{mn}p_2 f_{X_n}^1(X_{n+1})}{\mathbf{H}(X_n,X_{n+1},\overrightarrow{\overline{\Pi}}_n)}d\mathbf{P}_x.$$

It leads to the relation

$$\Pi_{m,n+1} = p_2\Pi_{mn}f_{X_n}^1(X_{n+1})\mathbf{H}^{-1}(X_n,X_{n+1},\overrightarrow{\overline{\Pi}}_n),$$

which proves the formula (3.9).

Further details concerning the recursive formula for conditional probabilities can be found in Remark 7.1 in the Appendix. ∎

REMARK 3.1. *Let us assume that the considered Markov processes have the finite state space and $\vec{x}_n = (x_0, x_1, \ldots, x_n)$, $x_0 = x$, are given. In this case*, (3.9)

follows from the Bayes formula:

$$\mathbf{P}_x(\theta_1 = j, \theta_2 = k | \vec{X}_n = \vec{x}_n)$$

$$= \begin{cases} p_{jk}^{\theta} \prod_{s=1}^{n} f_{x_{s-1}}^0(x_s)\big(S_n(\vec{x}_n)\big)^{-1} & \text{if } j > n, \\ p_{jk}^{\theta} \prod_{s=1}^{j-1} f_{x_{s-1}}^0(x_s) \prod_{t=j}^{n} f_{x_{t-1}}^1(x_t)\big(S_n(\vec{x}_n)\big)^{-1} & \text{if } j \leqslant n < k, \\ p_{jk}^{\theta} \prod_{s=1}^{n} f_{x_{s-1}}^0(x_s) \prod_{t=j}^{k-1} f_{x_{t-1}}^1(x_t) \prod_{u=k}^{n} f_{x_{u-1}}^2(x_u)\big(S_n(\vec{x}_n)\big)^{-1} & \text{if } k \leqslant n, \end{cases}$$

where $p_{jk}^{\theta} = \mathbf{P}(\theta_1 = j, \theta_2 = k)$ *and* $S_n(\cdot)$ *is given by* (7.5).

LEMMA 3.2. *For each* $x \in \mathbb{E}$ *and each Borel function* $u : \mathbb{E} \longrightarrow \Re$ *the following equations are fulfilled*:

$$(3.10) \qquad \mathbf{E}_x\big(u(X_{n+1})(1 - \Pi_{n+1}^1) \mid \mathcal{F}_n\big) = (1 - \Pi_n^1)p_1 \int_{\mathbb{E}} u(y) f_{X_n}^0(y)\mu(dy),$$

$$(3.11) \quad \mathbf{E}_x\big(u(X_{n+1})(\Pi_{n+1}^1 - \Pi_{n+1}^2) \mid \mathcal{F}_n\big)$$
$$= [q_1(1 - \Pi_n^1 - \Pi_n^{12}) + p_2(\Pi_n^1 - \Pi_n^2)] \int_{\mathbb{E}} u(x) f_{X_n}^1(y)\mu(dy),$$

$$(3.12)$$
$$\mathbf{E}_x\big(u(X_{n+1})\Pi_{n+1}^2 \mid \mathcal{F}_n\big) = [q_2\Pi_n^1 + p_2\Pi_n^2 + q_1\Pi_n^{12}] \int_{\mathbb{E}} u(y) f_{X_n}^2(y)\mu(dy),$$

$$(3.13) \qquad \mathbf{E}_x\big(u(X_{n+1})\Pi_{n+1}^{12} \mid \mathcal{F}_n\big) = p_1 \Pi_n^{12} \int_{\mathbb{E}} u(y) f_{X_n}^0(y)\mu(dy),$$

$$(3.14) \qquad \mathbf{E}_x\big(u(X_{n+1}) | \mathcal{F}_n\big) = \int_{\mathbb{E}} u(y)\mathbf{H}(X_n, y, \overrightarrow{\Pi}_n)\mu(dy).$$

P r o o f. The relations (3.10)–(3.13) are consequences of a suitable division of Ω defined by (θ_1, θ_2) and properties established in Lemma 7.3. Let us prove the equation (3.12). To this end define a σ-field $\widetilde{\mathcal{F}}_n = \sigma(\theta_1, \theta_2, X_0, \ldots, X_n)$. Notice that $\mathcal{F}_n \subset \widetilde{\mathcal{F}}_n$. We have

$$\mathbf{E}_x\big(u(X_{n+1})\Pi_{n+1}^2 \mid \mathcal{F}_n\big) = \mathbf{E}_x\big(u(X_{n+1})\mathbf{E}_x(\mathbb{I}_{\{\theta_2 \leqslant n+1\}} \mid \mathcal{F}_{n+1}) \mid \mathcal{F}_n\big)$$

$$= \mathbf{E}_x\big(u(X_{n+1})\mathbb{I}_{\{\theta_2 \leqslant n+1\}} \mid \mathcal{F}_n\big) = \mathbf{E}_x\Big(\mathbf{E}_x\big(u(X_{n+1})\mathbb{I}_{\{\theta_2 \leqslant n+1\}} \mid \widetilde{\mathcal{F}}_n\big) \mid \mathcal{F}_n\Big)$$

$$= \mathbf{E}_x\Big(\mathbb{I}_{\{\theta_2 \leqslant n+1\}}\mathbf{E}_x\big(u(X_{n+1}) \mid \widetilde{\mathcal{F}}_n\big) \mid \mathcal{F}_n\Big)$$

$$= \int_{\mathbb{E}} u(y) f_{X_n}^2(y)\mu(dy)\mathbf{P}_x(\theta_2 \leqslant n+1 \mid \mathcal{F}_n)$$

$$\overset{\text{L.7.3}}{=} (q_2\Pi_n^1 + p_2\Pi_n^2 + q_1\Pi_n^{12}) \int_{\mathbb{E}} u(y) f_{X_n}^2(y)\mu(dy).$$

We used the properties of conditional expectation and Lemma 7.3 (v). Similar transformations give equations (3.10), (3.13) and (3.11) when parts (i) and (ii), part (iv) and part (i) of Lemma 7.3, respectively, are applied. From (3.10)–(3.12) we get (3.14). Thus the proof of the lemma is complete. ∎

4. DETECTION OF A NEW HOMOGENEOUS SEGMENT

4.1. Equivalent optimal stopping problem. For $X_0 = x$ let us define $Z_n = \mathbf{P}_x(\theta_1 \leqslant n < \theta_2 \mid \mathcal{F}_n)$ for $n = 0, 1, 2, \ldots$ We have

(4.1) $$Z_n = \mathbf{P}_x(\theta_1 \leqslant n < \theta_2 \mid \mathcal{F}_n) = \Pi_n^1 - \Pi_n^2,$$

$Y_n = \operatorname{ess\,sup}_{\{\tau \in \mathcal{T}, \ \tau \geqslant n\}} \mathbf{P}_x(\theta_1 \leqslant \tau < \theta_2 \mid \mathcal{F}_n)$ for $n = 0, 1, 2, \ldots$ and

(4.2) $$\tau_0 = \inf\{n \geqslant 0 : Z_n = Y_n\}.$$

Notice that if $Z_\infty = 0$, then $Z_\tau = \mathbf{P}_x(\theta_1 \leqslant \tau < \theta_2 \mid \mathcal{F}_\tau)$ for $\tau \in \mathcal{T}$. Since $\mathcal{F}_n \subseteq \mathcal{F}_\tau$ (when $n \leqslant \tau$), we have

$$Y_n = \operatorname{ess\,sup}_{\tau \geqslant n} \mathbf{E}_x(Z_\tau \mid \mathcal{F}_n).$$

LEMMA 4.1. *The stopping time τ_0 defined by the formula* (4.2) *is the solution of the problem* (2.4).

P r o o f. From the theorems presented in [3] it is enough to show that $\lim_{n \to \infty} Z_n = 0$. For all natural numbers n, k, where $n \geqslant k$, for each $x \in \mathbb{E}$ we have

$$Z_n = \mathbf{E}_x(\mathbb{I}_{\{\theta_1 \leqslant n < \theta_2\}} \mid \mathcal{F}_n) \leqslant \mathbf{E}_x(\sup_{j \geqslant n} \mathbb{I}_{\{\theta_1 \leqslant j < \theta_2\}} \mid \mathcal{F}_n).$$

From Lévy's theorem we obtain $\limsup_{n \to \infty} Z_n \leqslant \mathbf{E}_x(\sup_{j \geqslant k} \mathbb{I}_{\{\theta_1 \leqslant j < \theta_2\}} \mid \mathcal{F}_\infty)$, where $\mathcal{F}_\infty = \sigma\left(\bigcup_{n=1}^\infty \mathcal{F}_n\right)$. It is true that

$$\lim_{k \to \infty} \sup_{j \geqslant k} \mathbb{I}_{\{\theta_1 \leqslant j < \theta_2\}} = 0 \quad \text{a.s.}$$

and by the dominated convergence theorem we get

$$\lim_{k \to \infty} \mathbf{E}_x(\sup_{j \geqslant k} \mathbb{I}_{\{\theta_1 \leqslant j < \theta_2\}} \mid \mathcal{F}_\infty) = 0 \quad \text{a.s.},$$

which completes the proof of the lemma. ∎

The reduction of the disorder problem to optimal stopping of Markov sequence is a consequence of the following lemma.

LEMMA 4.2. *The system $X^x = \{X_n^x\}$, where $X_n^x = (X_{n-1}, X_n, \Pi_n^1, \Pi_n^2, \Pi_n^{12})$, forms a family of random Markov functions.*

P r o o f. Define a function

(4.3) $$\varphi(x_1, x_2, \vec{\alpha}\,; z) = \big(x_2, z, \Pi^1(x_2, z, \vec{\alpha}), \Pi^2(x_2, z, \vec{\alpha}), \Pi^{12}(x_2, z, \vec{\alpha})\big).$$

Observe that

$$X_n^x = \varphi(X_{n-2}, X_{n-1}, \overrightarrow{\Pi}_{n-1}; X_n) = \varphi(X_{n-1}^x; X_n).$$

Hence X_n^x can be interpreted as a function of the previous state X_{n-1}^x and the random variable X_n. Moreover, applying (3.14), we see that the conditional distribution of X_n given σ-field $\mathcal{F}_{n-1}$ depends only on X_{n-1}^x. According to [19] (pp. 102–103) the system X^x is a family of random Markov functions. ■

Lemma 4.2 implies that we can reduce the initial problem (2.4) to the optimal stopping of the five-dimensional process $(X_{n-1}, X_n, \Pi_n^1, \Pi_n^2, \Pi_n^{12})$ with reward

(4.4) $$h(x_1, x_2, \vec{\alpha}) = \alpha - \beta.$$

The reward function results from (4.1). By Lemma 4.2, in the sequel we construct the solution using standard tools of the optimal stopping theory (cf. [19]).

For any Borel function $v \colon \mathbb{E}^2 \times [0,1]^3 \to [0,1]$ and the set $D = \{\omega \colon X_{n-1} = y, X_n = z, \Pi_n^1 = \alpha, \Pi_n^2 = \beta, \Pi_n^{12} = \gamma\}$ we define two operators

$$T_x v(y, z, \vec{\alpha}) = \mathbf{E}_x\big(v(X_n, X_{n+1}, \overrightarrow{\Pi}_{n+1}) \mid D\big),$$

$$\mathbf{Q}_x v(y, z, \vec{\alpha}) = \max\{v(y, z, \vec{\alpha}), \mathbf{T}_x v(y, z, \vec{\alpha})\}.$$

From the well-known theorems of optimal stopping theory (see [19]) we infer that the solution of the problem (2.4) is the Markov time τ_0:

(4.5) $$\tau_0^\star = \inf\{n \geqslant 0 : h(X_n, X_{n+1}, \overrightarrow{\Pi}_{n+1}) \geqslant h^*(X_n, X_{n+1}, \overrightarrow{\Pi}_{n+1})\},$$

where

$$h^*(y, z, \vec{\alpha}) = \lim_{k \to \infty} \mathbf{Q}_x^k h(y, z, \vec{\alpha}).$$

Obviously,

$$\mathbf{Q}_x^k v(y, z, \vec{\alpha}) = \max\{\mathbf{Q}_x^{k-1} v, \mathbf{T}_x \mathbf{Q}_x^{k-1} v\} = \max\{v, \mathbf{T}_x \mathbf{Q}_x^{k-1} v\}.$$

To obtain a clearer formula for $\tau_0^\star$ and the solution of the problem (2.4), we formulate (cf. (3.5) and (3.4)):

THEOREM 4.1. (a) *The solution* (4.5) *of the optimal stopping problem for the stochastic system X^x defined in Lemma* 4.2 *with payoff function* (4.4) *is given by*

(4.6) $$\tau_0^* = \inf\{n \geqslant 0 : (X_n, X_{n+1}, \overrightarrow{\Pi}_{n+1}) \in B^*\},$$

where B^* is the set of the form

$$B^* = \Big\{(y, z, \vec{\alpha}) \colon (\alpha - \beta) \geqslant (1 - \alpha - \gamma)\big[p_1 \int_{\mathbb{E}} R^*\big(y, u, \overrightarrow{\Pi}_1(y, u, \vec{\alpha})\big) f_y^0(u) \mu(du)$$

$$+ q_1 \int_{\mathbb{E}} S^*\big(y, u, \overrightarrow{\Pi}_1(y, u, \vec{\alpha})\big) f_y^1(u) \mu(du)\big]$$

$$+ (\alpha - \beta) p_2 \int_{\mathbb{E}} S^*\big(y, u, \overrightarrow{\Pi}_1(y, u, \vec{\alpha})\big) f_y^1(u) \mu(du)\Big\},$$

and $R^*(y, z, \vec{\alpha}) = \lim_{k \to \infty} R^k(y, z, \vec{\alpha})$, $S^*(y, z, \vec{\alpha}) = \lim_{k \to \infty} S^k(y, z, \vec{\alpha})$. *The functions* R^k *and* S^k *are defined recursively*: $R^1(y, z, \vec{\alpha}) = 0$, $S^1(y, z, \vec{\alpha}) = 1$ *and*

(4.7)
$$R^{k+1}(y, z, \vec{\alpha}) = \left(1 - \mathbb{I}_{\mathcal{R}_k}(y, z, \vec{\alpha})\right) \left[p_1 \int_{\mathbb{E}} R^k\left(y, u, \overrightarrow{\Pi}_1(y, u, \vec{\alpha})\right) f_y^0(u)\mu(du) \right.$$
$$\left. + q_1 \int_{\mathbb{E}} S^k\left(y, u, \overrightarrow{\Pi}_1(y, u, \vec{\alpha})\right) f_y^1(u)\mu(du) \right],$$

(4.8)
$$S^{k+1}(y, z, \vec{\alpha}) = \mathbb{I}_{\mathcal{R}_k}(y, z, \vec{\alpha}) + \left(1 - \mathbb{I}_{\mathcal{R}_k}(y, z, \vec{\alpha})\right)$$
$$\times p_2 \int_{\mathbb{E}} S^k\left(y, u, \overrightarrow{\Pi}_1(y, u, \vec{\alpha})\right) f_y^1(u)\mu(du),$$

where the set $\mathcal{R}_k$ *is*

(4.9)
$$\mathcal{R}_k = \left\{ (y, z, \vec{\alpha}) : h(y, z, \vec{\alpha}) \geqslant \mathbf{T}_x \mathbf{Q}_x^{k-1} h(y, z, \vec{\alpha}) \right\}$$
$$= \left\{ (y, z, \vec{\alpha}) : (\alpha - \beta) \geqslant (1 - \alpha - \gamma) \right.$$
$$\times \left[p_1 \int_{\mathbb{E}} R^k\left(y, u, \overrightarrow{\Pi}_1(y, u, \vec{\alpha})\right) f_y^0(u)\mu(du) \right.$$
$$\left. + q_1 \int_{\mathbb{E}} S^k\left(y, u, \overrightarrow{\Pi}_1(y, u, \vec{\alpha})\right) f_y^1(u)\mu(du) \right]$$
$$\left. + (\alpha - \beta)p_2 \int_{\mathbb{E}} S^k\left(y, u, \overrightarrow{\Pi}_1(y, u, \vec{\alpha})\right) f_y^1(u)\mu(du) \right\}.$$

(b) *The optimal value for* (2.4) *is given by the formula*

$$V(x) = \max\{p_2 \bar{\pi} \rho, V_0(x)\},$$

where

$$V_0(x) = \bar{\pi}\bar{\rho} \left[p_1 \int_{\mathbb{E}} R^*\left(x, u, \overrightarrow{\Pi}_1(x, u, \pi, \rho\pi, \rho\bar{\pi})\right) f_x^0(u)\mu(du) \right.$$
$$\left. + q_1 \int_{\mathbb{E}} S^*\left(x, u, \overrightarrow{\Pi}_1(x, u, \pi, \rho\pi, \rho(1-\pi))\right) f_x^1(u)\mu(du) \right]$$
$$+ \bar{\pi}\rho p_2 \int_{\mathbb{E}} S^*\left(x, u, \overrightarrow{\Pi}_1(x, u, \pi, \rho\pi, \rho(1-\pi))\right) f_x^1(u)\mu(du)$$

and $\tau^\star = 0 \cdot \mathbb{I}_{\{p_2 \bar{\pi}\rho \geqslant V_0(x)\}} + \tau_0^\star \mathbb{I}_{\{p_2 \bar{\pi}\rho < V_0(x)\}}$.

P r o o f. Part (a) results from Lemma 3.2: the problem reduces to the optimal stopping of the Markov process $(X_{n-1}, X_n, \Pi_n^1, \Pi_n^2, \Pi_n^{12})$ with the payoff function

$h(y, z, \vec{\alpha}) = \alpha - \beta$. Given (3.11) with the function u equal to unity we get on the set $D = \{\omega : X_{n-1} = y, X_n = z, \Pi_n^1 = \alpha, \Pi_n^2 = \beta, \Pi_n^{12} = \gamma\}$:

$$
\begin{aligned}
\mathbf{T}_x h(y, z, \vec{\alpha}) &= \mathbf{E}_x(\Pi_{n+1}^1 - \Pi_{n+1}^2 \mid \mathcal{F}_n)\big|_D \\
&= \left[((1 - \Pi_n^1 - \Pi_n^{12})q_1 + (\Pi_n^1 - \Pi_n^2)p_2) \int_{\mathbb{E}} f_{X_n}^1(u)\mu(du) \right]\Big|_D \\
&= (1 - \alpha - \gamma)q_1 + (\alpha - \beta)p_2.
\end{aligned}
$$

From the definition of R^1 and S^1 it is clear that

$$
h(y, z, \vec{\alpha}) = \alpha - \beta = (1 - \alpha - \gamma)R^1(y, z, \vec{\alpha}) + (\alpha - \beta)S^1(y, z, \vec{\alpha}).
$$

Also $\mathcal{R}_1 = \{(y, z, \vec{\alpha}) \colon h(y, z, \vec{\alpha}) \geqslant \mathbf{T}_x h(y, z, \vec{\alpha})\}$. By the definition of $\mathbf{Q}_x$ and the facts above we obtain

$$
\mathbf{Q}_x h(y, z, \vec{\alpha}) = (1 - \alpha - \gamma)R^2(y, z, \vec{\alpha}) + (\alpha - \beta)S^2(y, z, \vec{\alpha}),
$$

where

$$
R^2(y, z, \vec{\alpha}) = q_1\big(1 - \mathbb{I}_{\mathcal{R}_1}(y, z, \vec{\alpha})\big) \text{ and } S^2(y, z, \vec{\alpha}) = p_2 + \big((1 - p_2)\mathbb{I}_{\mathcal{R}_1}(y, z, \vec{\alpha})\big).
$$

Suppose the following induction hypothesis holds:

$$
\mathbf{Q}_x^{k-1} h(y, z, \vec{\alpha}) = (1 - \alpha - \gamma)R^k(y, z, \vec{\alpha}) + (\alpha - \beta)S^k(y, z, \vec{\alpha}),
$$

where R^k and S^k are given by equations (4.7) and (4.8), respectively. We will show that

$$
\mathbf{Q}_x^k h(y, z, \vec{\alpha}) = (1 - \alpha - \gamma)R^{k+1}(y, z, \vec{\alpha}) + (\alpha - \beta)S^{k+1}(y, z, \vec{\alpha}).
$$

From the induction assumption and the equations (3.10), (3.13) and (3.11) we get:

$$
\begin{aligned}
(4.10) \quad \mathbf{T}_x \mathbf{Q}_x^{k-1} & h(y, z, \vec{\alpha}) \\
&= \mathbf{T}_x (1 - \alpha - \gamma)R^k(y, z, \vec{\alpha}) + \mathbf{T}_x(\alpha - \beta)S^k(y, z, \vec{\alpha}) \\
&= (1 - \alpha - \gamma)p_1 \int_{\mathbb{E}} R^k\big(y, u, \overrightarrow{\Pi}_1(y, u, \vec{\alpha})\big) f_y^0(u)\mu(du) \\
&\quad + [(1 - \alpha - \gamma)q_1 + (\alpha - \beta)p_2] \int_{\mathbb{E}} S^k\big(y, u, \overrightarrow{\Pi}_1(y, u, \vec{\alpha})\big) f_y^1(u)\mu(du) \\
&= (1 - \alpha - \gamma)\big[p_1 \int_{\mathbb{E}} R^k\big(y, u, \overrightarrow{\Pi}_1(y, u, \vec{\alpha})\big) f_y^0(u)\mu(du) \\
&\quad + q_1 \int_{\mathbb{E}} S^k\big(y, u, \overrightarrow{\Pi}_1(y, u, \vec{\alpha})\big) f_y^1(u)\mu(du) \big] \\
&\quad + (\alpha - \beta)p_2 \int_{\mathbb{E}} S^k\big(y, u, \overrightarrow{\Pi}_1(y, u, \vec{\alpha})\big) f_y^1(u)\mu(du).
\end{aligned}
$$

Notice that the expression

$$
(1 - \alpha - \gamma)R^{k+1}(y, z, \vec{\alpha}) + (\alpha - \beta)S^{k+1}(y, z, \vec{\alpha})
$$

is equal to $\alpha - \beta = h(y, z, \vec{\alpha}) = \mathbf{Q}_x^k h(y, z, \vec{\alpha})$ for $(y, z, \vec{\alpha}) \in \mathcal{R}_k$ and, by (4.10), it is equal to $\mathbf{T}_x \mathbf{Q}_x^{k-1} h(y, z, \vec{\alpha}) = \mathbf{Q}_x^k h(y, z, \vec{\alpha})$ for $(y, z, \vec{\alpha}) \notin \mathcal{R}_k$, where $\mathcal{R}_k$ is given by (4.9). Finally, we get

$$\mathbf{Q}_x^k h(y, z, \vec{\alpha}) = (1 - \alpha - \gamma) R^{k+1}(y, z, \vec{\alpha}) + (\alpha - \beta) S^{k+1}(y, z, \vec{\alpha}).$$

This proves (4.7) and (4.8). Using the monotone convergence theorem and the theorems of optimal stopping theory (see [19]) we conclude that the optimal stopping time τ_0^* is given by (4.6), which completes the proof of part (a).

(b) First, notice that Π_1^1, Π_1^2 and Π_1^{12} are given by (3.6)–(3.8) and the boundary condition formulated in Lemma 3.1. Under the assumption $\tau^* < \infty$ a.s. we get

$$\mathbf{P}_x(\tau^* < \infty, \theta_1 \leqslant \tau^* < \theta_2) = \sup_\tau \mathbf{E} Z_\tau$$

$$= \mathbf{E} \max\{h(x, X_1, \overrightarrow{\Pi}_1), \mathbf{T}_x h^*(x, X_1, \overrightarrow{\Pi}_1)\} = \mathbf{E} \lim_{k \to \infty} \mathbf{Q}_x^k h(x, X_1, \overrightarrow{\Pi}_1)$$

$$= \mathbf{E}[(1 - \Pi_1^1 - \Pi_1^{12}) R^*(x, X_1, \overrightarrow{\Pi}_1) + (\Pi_1^1 - \Pi_1^2) S^*(x, X_1, \overrightarrow{\Pi}_1)]$$

$$= \bar{\pi} \bar{\rho} p_1 \int_{\mathbb{E}} R^*\big(x, u, \overrightarrow{\Pi}_1(x, u, \pi, \rho\pi, \rho\bar{\pi})\big) f_x^0(u) \mu(du)$$

$$+ (\bar{\pi} \bar{\rho} q_1 + \pi \bar{\rho} p_2) \int_{\mathbb{E}} S^*\big(x, u, \overrightarrow{\Pi}_1(x, u, \pi, \rho\pi, \rho\bar{\pi})\big) f_x^1(u) \mu(du).$$

We used Lemma 3.2 here and simple calculations for Π_1^1, Π_1^2 and Π_1^{12}. This completes the proof of Theorem 4.1. ∎

4.2. Remarks. Observe that the solution of the formulated problem depends only on two-dimensional vector of posterior processes because $\Pi_n^{12} = \rho(1 - \Pi_n^1)$. The obtained formulae are very general and for this reason – quite complicated. We simplify the model by assuming that $P(\theta_1 > 0) = 1$ and $P(\theta_2 > \theta_1) = 1$. However, it seems that some further simplifications can be made in special cases. Further research should be carried out in this direction. From a practical point of view, computer algorithms are necessary to construct B^* – the set in which it is optimally to stop our observable sequence.

5. IMMEDIATE DETECTION OF THE FIRST AND THE SECOND DISORDER

5.1. Equivalent double optimal stopping problem. Let us consider the problem D_{00} formulated in (2.5). A *compound stopping variable* is a pair (τ, σ) of stopping times such that $0 \leqslant \tau \leqslant \sigma$ a.e. The aim is to find a compund stopping variable $(\tau^\star, \sigma^\star)$ such that

$$(5.1) \qquad \mathbf{P}_x\big((\theta_1, \theta_2) = (\tau^\star, \sigma^\star)\big) = \sup_{\substack{(\tau, \sigma) \in \mathcal{T} \\ 0 \leqslant \tau \leqslant \sigma < \infty}} \mathbf{P}_x\big((\theta_1, \theta_2) = (\tau, \sigma)\big).$$

Write $\mathcal{T}_m = \{(\tau, \sigma) \in \mathcal{T} : \tau \geqslant m\}$, $\mathcal{T}_{mn} = \{(\tau, \sigma) \in \mathcal{T} : \tau = m, \sigma \geqslant n\}$ and $\mathcal{S}_m = \{\tau \in \mathcal{S} : \tau \geqslant m\}$. Let us put $\mathcal{F}_{mn} = \mathcal{F}_n$, $m, n \in \mathbb{N}$, $m \leqslant n$. We define a two-parameter stochastic sequence $\xi(x) = \{\xi_{mn}, \ m, n \in \mathbb{N}, \ m < n, \ x \in \mathbb{E}\}$, where

$$\xi_{mn} = \mathbf{P}_x(\theta_1 = m, \theta_2 = n | \mathcal{F}_{mn}).$$

We can consider for every $x \in \mathbb{E}$, $m, n \in \mathbb{N}$, $m < n$, the optimal stopping problem of $\xi(x)$ on $\mathcal{T}_{mn}^+ = \{(\tau, \sigma) \in \mathcal{T}_{mn} : \tau < \sigma\}$. A compound stopping variable (τ^*, σ^*) is said to be *optimal* in $\mathcal{T}_m^+$ (or $\mathcal{T}_{mn}^+$) if

$$(5.2) \qquad \mathbf{E}_x \xi_{\tau^* \sigma^*} = \sup_{(\tau, \sigma) \in \mathcal{T}_m} \mathbf{E}_x \xi_{\tau \sigma}$$

(or $\mathbf{E}_x \xi_{\tau^* \sigma^*} = \sup_{(\tau, \sigma) \in \mathcal{T}_{mn}^+} \mathbf{E}_x \xi_{\tau \sigma}$). Let us define

$$(5.3) \qquad \eta_{mn} = \operatorname*{ess\,sup}_{(\tau, \sigma) \in \mathcal{T}_{mn}^+} \mathbf{E}_x(\xi_{\tau \sigma} | \mathcal{F}_{mn}).$$

If we put $\xi_{m\infty} = 0$, then

$$\eta_{mn} = \operatorname*{ess\,sup}_{(\tau, \sigma) \in \mathcal{T}_{mn}^+} \mathbf{P}_x(\theta_1 = \tau, \theta_2 = \sigma | \mathcal{F}_{mn}).$$

By the theory of optimal stopping for double indexed processes (cf. [8], [13]) the sequence η_{mn} satisfies

$$\eta_{mn} = \max\{\xi_{mn}, \mathbf{E}(\eta_{m,n+1} | \mathcal{F}_{mn})\}.$$

Moreover, if $\sigma_m^* = \inf\{n > m : \eta_{mn} = \xi_{mn}\}$, then (m, σ_n^*) is optimal in $\mathcal{T}_{mn}^+$ and $\eta_{mn} = \mathbf{E}_x(\xi_{m\sigma_n^*} | \mathcal{F}_{mn})$ a.e. The case when there are no segment with the distribution $f_x^1(y)$ appears with probability ρ. It will be taken into account. Define

$$\hat{\eta}_{mn} = \max\{\xi_{mn}, \mathbf{E}(\eta_{m,n+1} | \mathcal{F}_{mn})\} \quad \text{for } n \geqslant m.$$

If $\hat{\sigma}_m^* = \inf\{n \geqslant m : \hat{\eta}_{mn} = \xi_{mn}\}$, then $(m, \hat{\sigma}_m^*)$ is optimal in $\mathcal{T}_{mn}$ and $\hat{\eta}_{mm} = \mathbf{E}_x(\xi_{m\sigma_m^*} | \mathcal{F}_{mm})$ a.e.. For further consideration we put

$$(5.4) \qquad \eta_m = \mathbf{E}_x(\eta_{m,m+1} | \mathcal{F}_m).$$

LEMMA 5.1. *The stopping time σ_m^* is optimal for every stopping problem defined by* (5.3).

P r o o f. It suffices to prove that $\lim_{n \to \infty} \xi_{mn} = 0$ (cf. [3]). For $m, n, k \in \mathbb{N}$, $n \geqslant k > m$, and every $x \in \mathbb{E}$ we have

$$\mathbf{E}_x(\mathbb{I}_{\{\theta_1 = m, \theta_2 = n\}} | \mathcal{F}_{mn}) = \xi_{mn}(x) \leqslant \mathbf{E}_x(\sup_{j \geqslant k} \mathbb{I}_{\{\theta_1 = m, \theta_2 = j\}} | \mathcal{F}_m),$$

where $\mathbb{I}_A$ is the characteristic function of the set A. By Lévy's theorem we get

$$\limsup_{n\to\infty} \xi_{mn}(x) \leqslant \mathbf{E}_x\big(\sup_{j\geqslant k} \mathbb{I}_{\{\theta_1=m,\theta_2=j\}}|\mathcal{F}_{n\infty}\big),$$

where $\mathcal{F}_\infty = \mathcal{F}_{n\infty} = \sigma\big(\bigcup_{n=1}^\infty \mathcal{F}_n\big)$. We have

$$\lim_{k\to\infty} \sup_{j\geqslant k} \mathbb{I}_{\{\theta_1=m,\theta_2=j\}} = 0 \ \text{ a.e.}$$

and, by the dominated convergence theorem,

$$\lim_{k\to\infty} \mathbf{E}_x\big(\sup_{j\geqslant k} \mathbb{I}_{\{\theta_1=m,\theta_2=j\}}|\mathcal{F}_\infty\big) = 0. \quad \blacksquare$$

What is left is to consider the optimal stopping problem for $(\eta_{mn})_{m=0,n=m}^{\infty,\infty}$ on $(\mathcal{T}_{mn})_{m=0,n=m}^{\infty,\infty}$. Let us define

$$(5.5) \qquad\qquad V_m = \operatorname*{ess\,sup}_{\tau\in\mathcal{S}_m} \mathbf{E}_x(\eta_\tau|\mathcal{F}_m).$$

Then $V_m = \max\{\eta_m, \mathbf{E}_x(V_{m+1}|\mathcal{F}_m)\}$ a.e. and we define

$$\tau_n^* = \inf\{k\geqslant n : V_k = \eta_k\}.$$

LEMMA 5.2. *The strategy τ_0^* is the optimal first stop.*

P r o o f. To show that τ_0^* is the optimal first stop strategy we prove that

$$\mathbf{P}_x(\tau_0^* < \infty) = 1.$$

To this end, we argue in the usual manner, i.e. we show that $\lim_{m\to\infty} \eta_m = 0$.
We have

$$\eta_m = \mathbf{E}_x(\xi_{m\sigma_m^*}|\mathcal{F}_m) = \mathbf{E}_x\big(\mathbf{E}_x(\mathbb{I}_{\{\theta_1=m,\theta_2=\sigma_m^*\}}|\mathcal{F}_{m\sigma_m^*})|\mathcal{F}_m\big)$$

$$= \mathbf{E}_x(\mathbb{I}_{\{\theta_1=m,\theta_2=\sigma_m^*\}}|\mathcal{F}_m) \leqslant \mathbf{E}_x\big(\sup_{j\geqslant k} \mathbb{I}_{\{\theta_1=j,\theta_2=\sigma_j^*\}}|\mathcal{F}_m\big).$$

As in the proof of Lemma 5.1 we have

$$\limsup_{m\to\infty} \eta_m(x) \leqslant \mathbf{E}_x\big(\sup_{j\geqslant k} \mathbb{I}_{\{\theta_1=j,\theta_2=\sigma_j^*\}}|\mathcal{F}_\infty\big).$$

Since $\lim_{k\to\infty} \sup_{j\geqslant k} \mathbb{I}_{\{\theta_1=k,\theta_2=\sigma_j^*\}} \leqslant \limsup_{k\to\infty} \mathbb{I}_{\{\theta_1=k\}} = 0$, it follows that

$$\lim_{m\to\infty} \eta_m(x) \leqslant \lim_{k\to\infty} \mathbf{E}_x\big(\sup_{j\geqslant k} \mathbb{I}_{\{\theta_1=j,\theta_2=\sigma_j^*\}}|\mathcal{F}_\infty\big) = 0. \quad \blacksquare$$

Lemmas 5.1 and 5.2 describe the method of solving the "disorder problem" formulated in Section 2 (see (5.1)).

5.2. Solution of the equivalent double stopping problem. For the sake of simplicity we shall confine ourselves to the case $d_1 = d_2 = 0$. It will be easily seen how to generalize the solution of the problem to solve $D_{d_1 d_2}$ for $d_1 > 0$ or $d_2 > 0$. First of all we construct multidimensional Markov chains such that ξ_{mn} and η_m will be the functions of their states. By considerations of Section 3 concerning *a posteriori* processes we get $\xi_{00} = \pi\rho$ and for $m < n$

$$
\begin{aligned}
\xi_{mn}^x &= \mathbf{P}_x(\theta_1 = m, \theta_2 = n | \mathcal{F}_{mn}) \\
&= \bar{\pi}\bar{\rho}\frac{p_1^{m-1}q_1 p_2^{n-m-1}q_2 \prod_{s=1}^{j-1} f_{X_{s-1}}^0(X_s) \prod_{t=j}^{n-1} f_{X_{t-1}}^1(X_t) f_{X_{n-1}}^2(X_n)}{S_n(x_0, X_1, \ldots, X_n)} \\
&= \frac{q_2}{p_2}\Pi_{mn}(x)\frac{f_{X_{n-1}}^2(X_n)}{f_{X_{n-1}}^1(X_n)};
\end{aligned}
$$

moreover, for $n = m$, by Lemma 7.3,

$$
(5.6) \qquad \xi_{mm}^x = \mathbf{P}_x(\theta_1 = m, \theta_2 = m | \mathcal{F}_{mm}) = \rho\frac{q_1}{p_1}\frac{f_{X_{m-1}}^2(X_m)}{f_{X_{m-1}}^0(X_m)}(1 - \Pi_m^1).
$$

We can observe that $(X_n, X_{n+1}, \overrightarrow{\Pi}_{n+1}, \Pi_{m,n+1})$ for $n = m+1, m+2, \ldots$ is a function of $(X_{n-1}, X_n, \overrightarrow{\Pi}_n, \Pi_{mn})$ and X_{n+1}. Besides, the conditional distribution of X_{n+1} given $\mathcal{F}_n$ (cf. (3.14)) depends on X_n, $\Pi_n^1(x)$ and $\Pi_n^2(x)$ only. These facts imply that $\{(X_n, X_{n+1}, \overrightarrow{\Pi}_{n+1}, \Pi_{m,n+1})\}_{n=m+1}^\infty$ forms a homogeneous Markov process (see Chapter 2.15 of [19]). This allows us to reduce the problem (5.3) for each m to the optimal stopping problem of the Markov process $Z_m(x) = \{(X_{n-1}, X_n, \overrightarrow{\Pi}_n, \Pi_{mn}), m, n \in \mathbb{N}, m < n, x \in \mathbb{E}\}$ with the reward function

$$
h(t, u, \vec{\alpha}, \delta) = \frac{q_2}{p_2}\delta\frac{f_t^2(u)}{f_t^1(u)}.
$$

LEMMA 5.3. *A solution of the optimal stopping problem* (5.3) *for* $m = 1, 2, \ldots$ *has the form*

$$
(5.7) \qquad \sigma_m^* = \inf\left\{n > m : \frac{f_{X_{n-1}}^2(X_n)}{f_{X_{n-1}}^1(X_n)} \geqslant R^*(X_n)\right\},
$$

where $R^*(t) = p_2 \int_{\mathbb{E}} r^*(t, s) f_t^1(s)\mu(ds)$. *We have the function* $r^* = \lim_{n \to \infty} r_n$, *where* $r_0(t, u) = f_t^2(u)/f_t^1(u)$ *and*

$$
(5.8) \qquad r_{n+1}(t, u) = \max\left\{\frac{f_t^2(u)}{f_t^1(u)}, p_2\int_{\mathbb{E}} r_n(u, s) f_u^1(s)\mu(ds)\right\}.
$$

Then $r^*(t, u)$ satisfies the equation

$$(5.9) \qquad r^*(t, u) = \max\left\{\frac{f_t^2(u)}{f_t^1(u)}, p_2 \int_{\mathbb{E}} r^*(u, s) f_u^1(s) \mu(ds)\right\}.$$

The value of the problem is

$$(5.10) \qquad \eta_m = \mathbf{E}_x(\eta_{m\,m+1}|\mathcal{F}_m) = \frac{q_1}{p_1} \frac{f_{X_{m-1}}^1(X_m)}{f_{X_{m-1}}^0(X_m)}(1 - \Pi_m^1) R_\rho^\star(X_{m-1}, X_m),$$

where

$$(5.11) \qquad R_\rho^\star(t, u) = \max\left\{\rho\frac{f_t^2(u)}{f_t^1(u)}, \frac{q_2}{p_2}(1 - \rho)R^\star(u)\right\}.$$

P r o o f. For any Borel function $u : \mathbb{E} \times \mathbb{E} \times [0, 1]^4 \to [0, 1]$ and the set $D = \{\omega : X_{n-1} = t, X_n = u, \Pi_n^1(x) = \alpha, \Pi_n^2(x) = \beta, \Pi_n^{12} = \gamma, \Pi_{mn}(x) = \delta\}$ let us define two operators

$$\mathbf{T}_x u(t, u, \vec{\alpha}, \delta) = \mathbf{E}_x\left(u\big(X_n, X_{n+1}, \overrightarrow{\Pi}_{n+1}(x), \Pi_{m,n+1}(x)\big)\big|D\right)$$

and

$$\mathbf{Q}_x u(t, u, \vec{\alpha}, \delta) = \max\{u(t, u, \vec{\alpha}, \delta), \mathbf{T}_x u(t, u, \vec{\alpha}, \delta)\}.$$

By the well-known theorem from the theory of optimal stopping (see [19], [13]) we conclude that the solution of (5.3) is a Markov time:

$$\sigma_m^* = \inf\left\{n > m : h(X_{n-1}, X_n, \overrightarrow{\Pi}_n, \Pi_{mn}) = h^*\big(X_{n-1}, X_n, \overrightarrow{\Pi}_n(x), \Pi_{mn}\big)\right\},$$

where $h^* = \lim_{k \to \infty} \mathbf{Q}_x^k h(t, u, \vec{\alpha}, \delta)$. By (3.9) and (3.14) on $D = \{\omega : X_{n-1} = t, X_n = u, \Pi_n^1 = \alpha, \Pi_n^2 = \beta, \Pi_n^{12} = \gamma, \Pi_{mn} = \delta\}$ we have

$$\begin{aligned}
\mathbf{T}_x h(t, u, \vec{\alpha}, \delta) &= \mathbf{E}_x\left(\frac{q_2}{p_2}\Pi_{m,n+1}\frac{f_{X_n}^2(X_{n+1})}{f_{X_n}^1(X_{n+1})}\bigg|D\right) \\
&= \frac{q_2}{p_2}\delta p_2 \mathbf{E}\left(\frac{f_u^1(X_{n+1})}{H(u, X_{n+1}, \vec{\alpha})}\frac{f_u^2(X_{n+1})}{f_u^1(X_{n+1})}\bigg|\mathcal{F}_n\right)\bigg|_D \\
&\overset{(3.14)}{=} q_2\delta\int_{\mathbb{E}}\frac{f_u^2(s)}{H(u, s, \vec{\alpha})}H(u, s, \vec{\alpha})\mu(ds) = q_2\delta
\end{aligned}$$

and

$$(5.12) \qquad \mathbf{Q}_x h(t, u, \vec{\alpha}, \delta) = \frac{q_2}{p_2}\delta\max\left\{\frac{f_t^2(u)}{f_t^1(u)}, p_2\right\}.$$

Let us define $r_0(t, u) = 1$ and

$$r_{n+1}(t, u) = \max\left\{ \frac{f_t^2(u)}{f_t^1(u)}, p_2 \int\limits_{\mathbb{E}} r_n(u, s) f_u^1(s)\mu(ds) \right\}.$$

We show that

(5.13) $$\mathbf{Q}_x^\ell h(t, u, \vec{\alpha}, \delta) = \frac{q_2}{p_2}\delta r_\ell(t, u)$$

for $\ell = 1, 2, \ldots$ We infer by (5.12) that $\mathbf{Q}_x h = (q_2/p_2)\gamma r_1$. Let us assume (5.13) holds for $\ell \leqslant k$. By (3.14), on the set $D = \{\omega : X_{n-1} = t, X_n = u, \Pi_n^1 = \alpha,$ $\Pi_n^2 = \beta, \Pi_n^{12} = \gamma, \Pi_{mn} = \delta\}$ we have

$$\mathbf{T}_x\mathbf{Q}_x^k h(t, u, \vec{\alpha}, \delta) = \mathbf{E}_x\left(\frac{q_2}{p_2}\Pi_{m,k+1} r_k(X_n, X_{n+1})|D \right)$$
$$= \frac{q_2}{p_2}\delta p_2 \int\limits_{\mathbb{E}} r_k(u, s) f_u^1(s)\mu(ds).$$

It is easy to show (see [19]) that

$$\mathbf{Q}_x^{k+1} h = \max\{h, \mathbf{T}_x\mathbf{Q}_x^k h\} \quad \text{for } k = 1, 2, \ldots$$

Hence $\mathbf{Q}_x^{k+1} h = (q_2/p_2)\delta r_{k+1}$ and (5.13) is proved for $\ell = 1, 2, \ldots$ This gives

(5.14) $$h^*(t, u, \vec{\alpha}, \delta) = \frac{q_2}{p_2}\delta \lim_{k\to\infty} r_k(t, u) = \frac{q_2}{p_2}\delta r^*(t, u)$$

and

$$\eta_{mn} = \operatorname*{ess\,sup}_{(\tau,\sigma)\in\mathcal{T}_{mn}} \mathbf{E}_x(\xi_{\tau,\sigma}|\mathcal{F}_{mn}) = h^*(X_{n-1}, X_n, \overrightarrow{\Pi}_n, \Pi_{mn}).$$

By (5.14) and (3.9) we have

$$\mathbf{T}_x h^*(t, u, \vec{\alpha}, \delta) = \frac{q_2}{p_2}\delta p_2 \int\limits_{\mathbb{E}} r^*(u, s) f_u^1(s)\mu(ds) = \frac{q_2}{p_2}\delta R^*(u)$$

and σ_m^* takes the form (5.7). By (5.4), (5.6) and (3.14) we obtain

(5.15) $$\eta_m = \max\{\xi_{mm}^x, \mathbf{E}(\eta_{m,m+1}|\mathcal{F}_m)\} = f(X_{m-1}, X_m, \overrightarrow{\Pi}_m, \Pi_{mm})$$
$$= \max\left\{ \rho\frac{q_1}{p_1}\frac{f_{X_{m-1}}^2(X_m)}{f_{X_{m-1}}^0(X_m)}(1 - \Pi_m^1), \frac{q_2}{p_2}(1 - \Pi_{mm})R^\star(X_m) \right\}$$
$$\stackrel{\text{L.3.1}}{=} \frac{q_1}{p_1}\frac{f_{X_{m-1}}^1(X_m)}{f_{X_{m-1}}^0(X_m)}(1 - \Pi_m^1)R_\rho^\star(X_{m-1}, X_m). \quad \blacksquare$$

REMARK 5.1. *By the results of Lemma* 5.3 *and properties of the* a posteriori *process* Π_{nm} *we infer that the expected value of success for the second stop when the observer stops immediately at $n = 0$ is $\pi\rho$, and when at least one observation has been made*, *we get*

$$\mathbf{E}(\eta_1|\mathcal{F}_0) = \frac{q_1}{p_1}\mathbf{E}\left((1 - \Pi_1^1)\frac{f_x^1(X_1)}{f_x^0(X_1)}R_\rho^\star(x, X_1)|\mathcal{F}_0\right)$$

$$= \frac{q_1}{p_1}(1 - \pi)p_1 \int_{\mathbb{E}} f_x^1(u)R_\rho^\star(x, u)\mu(du).$$

As a consequence, *we have the optimal second moment*

$$\hat{\sigma}_0^\star = \begin{cases} 0 & \text{if } \pi\rho \geqslant q_1(1 - \pi) \int_{\mathbb{E}} f_x^1(u)R_\rho^\star(x, u)\mu(du), \\ \sigma_0^\star & \text{otherwise.} \end{cases}$$

By Lemmas 5.3 and 3.1 (formula (3.9)) the optimal stopping problem (5.5) has been transformed to the optimal stopping problem for the homogeneous Markov process

$$W = \{(X_{m-1}, X_m, \overrightarrow{\Pi}_m),\ m \in \mathbb{N},\ x \in \mathbb{E}\}$$

with the reward function

(5.16) $$f(t, u, \vec{\alpha}) = \frac{q_1}{p_1}\frac{f_t^1(u)}{f_t^0(u)}(1 - \alpha)R_\rho^\star(t, u).$$

THEOREM 5.1. *A solution of the optimal stopping problem* (5.5) *for $n = 1, 2, \ldots$ has the form*

(5.17) $$\tau_n^* = \inf\{k \geqslant n : (X_{k-1}, X_k, \overrightarrow{\Pi}_k) \in B^*\},$$

where

$$B^* = \left\{(t, u, \vec{\alpha}) : \frac{f_t^2(u)}{f_t^1(u)}R_\rho^\star(t, u) \geqslant p_1 \int_{\mathbb{E}} v^*(u, s)f_u^0(s)\mu(ds)\right\}.$$

We have the function $v^*(t, u) = \lim_{n\to\infty} v_n(t, u)$, where $v_0(t, u) = R_\rho^\star(t, u)$ and

(5.18) $$v_{n+1}(t, u) = \max\left\{\frac{f_t^2(u)}{f_t^1(u)}R_\rho^\star(t, u), p_1 \int_{\mathbb{E}} v_n(u, s)f_u^1(s)\mu(ds)\right\}.$$

Then $v^*(t, u)$ satisfies the equation

(5.19) $$v^*(t, u) = \max\left\{\frac{f_t^2(u)}{f_t^1(u)}R_\rho^\star(t, u), p_1 \int_{\mathbb{E}} v^*(u, s)f_u^1(s)\mu(ds)\right\}.$$

The value of the problem is $V_n = v^*(X_{n-1}, X_n)$.

P r o o f. For any Borel function $u : \mathbb{E} \times \mathbb{E} \times [0,1]^3 \to [0,1]$ and the set $D = \{\omega : X_{n-1} = t, X_n = u, \Pi_n^1(x) = \alpha, \Pi_n^2(x) = \beta, \Pi_n^{12} = \gamma\}$ let us define two operators

$$\mathbf{T}_x u(t, u, \vec{\alpha}) = \mathbf{E}_x\big(u(X_n, X_{n+1}, \overrightarrow{\Pi}_{n+1})|D\big)$$

and

$$\mathbf{Q}_x u(t, u, \vec{\alpha}) = \max\{u(t, u, \vec{\alpha}), \mathbf{T}_x u(t, u, \vec{\alpha})\}.$$

As in the proof of Lemma 5.3 we infer that the solution of (5.5) is a Markov time

$$\tau_m^* = \inf\{n > m : f(X_{n-1}, X_n, \overrightarrow{\Pi}_n) = f^*(X_{n-1}, X_n, \overrightarrow{\Pi}_n)\},$$

where $f^* = \lim_{k \to \infty} \mathbf{Q}_x^k f(t, u, \vec{\alpha})$. By (3.14) and (5.16), on the set $D = \{\omega : X_{n-1} = t, X_n = u, \Pi_n^1 = \alpha, \Pi_n^2 = \beta, \Pi_n^{12} = \gamma\}$ we have

$$
\begin{aligned}
\mathbf{T}_x f(t, u, \vec{\alpha}) &= \mathbf{E}_x\left(\frac{q_1}{p_1}(1 - \Pi_{n+1}^1)\frac{f_{X_n}^1(X_{n+1})}{f_{X_n}^0(X_{n+1})}R_\rho^\star(X_n, X_{n+1})|D\right) \\
&= \frac{q_1}{p_1}(1-\alpha)p_1 \mathbf{E}\left(\frac{f_u^0(X_{n+1})}{H(u, X_{n+1}, \alpha, \beta)}\frac{f_u^1(X_{n+1})}{f_u^0(X_{n+1})}R_\rho^\star(X_n, X_{n+1})|\mathcal{F}_n\right)\bigg|_D \\
&\overset{(3.14)}{=} \frac{q_1}{p_1}(1-\alpha)p_1 \int_{\mathbb{E}} \frac{f_u^1(s)}{H(u, s, \alpha, \beta)}H(u, s, \alpha, \beta)R_\rho^*(u, s)\mu(ds) \\
&= \frac{q_1}{p_1}(1-\alpha)p_1 \int_{\mathbb{E}} R_\rho^*(u, s)f_u^1(s)\mu(ds)
\end{aligned}
$$

and

$$(5.20) \quad \mathbf{Q}_x f(t, u, \vec{\alpha})$$
$$= \frac{q_1}{p_1}(1-\alpha)\max\left\{\frac{f_t^1(u)}{f_t^0(u)}R_\rho^\star(t, u), p_1\int_{\mathbb{E}} R_\rho^\star(u, s)f_u^1(s)\mu(ds)\right\} = \frac{q_1}{p_1}\alpha v_1(t, u).$$

Let us define

$$v_1(t, u) = \max\left\{\frac{f_t^1(u)}{f_t^0(u)}R_\rho^\star(t, u), p_1\int_{\mathbb{E}} R_\rho^*(u, s)f_u^1(s)\mu(ds)\right\}$$

and

$$v_{n+1}(t, u) = \max\left\{\frac{f_t^1(u)}{f_t^0(u)}R_\rho^\star(t, u), p_1\int_{\mathbb{E}} v_n(u, s)f_u^0(s)\mu(ds)\right\}.$$

We show that

$$(5.21) \qquad \mathbf{Q}_x^\ell f(t, u, \vec{\alpha}) = \frac{q_1}{p_1}(1-\alpha)v_\ell(t, u)$$

for $\ell = 1, 2, \ldots$ By (5.20) we have $\mathbf{Q}_x f(t, u, \vec{\alpha}) = (q_1/p_1)(1 - \alpha)v_1(t, u)$ and let us assume (5.21) holds for $\ell \leqslant k$. By (3.14), on $D = \{\omega : X_{n-1} = t, X_n = u, \Pi_n^1 = \alpha, \Pi_n^2 = \beta, \Pi_n^{12} = \gamma\}$ we have

$$\mathbf{T}_x \mathbf{Q}_x^k f(t, u, \vec{\alpha}) = \mathbf{E}_x \left(\frac{q_1}{p_1}(1 - \Pi_{k+1}^1)v_k(X_n, X_{n+1})|D \right)$$
$$= \frac{q_1}{p_1}(1 - \alpha)p_1 \int_{\mathbb{E}} v_k(u, s)f_u^0(s)\mu(ds).$$

Hence we have $\mathbf{Q}_x^{k+1} f = (q_1/p_1)(1 - \alpha)v_{k+1}$ and (5.21) is proved for $\ell = 1, 2, \ldots$ This gives

$$f^*(t, u, \vec{\alpha}) = \frac{q_1}{p_1}(1 - \alpha) \lim_{k \to \infty} v_k(t, u) = \frac{q_1}{p_1}\alpha v^*(t, u)$$

and

$$V_m = \frac{q_1}{p_1}(1 - \Pi_m^1)v^*(X_{m-1}, X_m).$$

We have

$$\mathbf{T}_x f^*(t, u, \vec{\alpha}) = \frac{q_1}{p_1}(1 - \alpha)p_1 \int_{\mathbb{E}} v^*(u, s)f_u^0(s)\mu(ds).$$

Define

$$B^* = \left\{ (t, u, \vec{\alpha}) : \frac{f_t^1(u)}{f_t^0(u)}R_\rho^\star(t, u) \geqslant p_1 \int_{\mathbb{E}} v^*(u, s)f_u^0(s)\mu(ds) \right\}.$$

Then τ_n^* for $n \geqslant 1$ takes the form (5.17). The value of the problem (5.2), (5.5) and (2.5) is equal to

$$v_0(x) = \max\{\pi, \mathbf{E}_x(V_1|\mathcal{F}_0)\} = \max \left\{ \pi, \frac{q_1}{p_1}(1 - \pi)p_1 \int_{\mathbb{E}} v^*(u, s)f_u^0(s)\mu(ds) \right\}$$

and

$$\hat{\tau}_0^* = \begin{cases} 0 & \text{if } \pi \geqslant q_1(1 - \pi) \int_{\mathbb{E}} v^*(u, s)f_u^0(s)\mu(ds), \\ \tau_0^* & \text{otherwise.} \quad \blacksquare \end{cases}$$

By Lemmas 5.3 and 5.1 the solution of the problem D_{00} can be formulated as follows.

THEOREM 5.2. *A compound stopping time* $(\tau^*, \sigma_{\tau^*}^*)$, *where* σ_m^* *is given by* (5.7) *and* $\tau^* = \hat{\tau}_0^*$ *is determined by* (5.17), *is the solution of the problem* D_{00}. *The value of the problem is*

$$\mathbf{P}_x(\tau^* < \sigma^* < \infty, \theta_1 = \tau^*, \theta_2 = \sigma_{\tau^*}^*)$$
$$= \max \left\{ \pi, q_1(1 - \pi) \int_{\mathbb{E}} v^*(u, s)f_u^0(s)\mu(ds) \right\}.$$

REMARK 5.2. *The problem can be extended to optimal detection of more than two successive disorders. The distribution of θ_1, θ_2 may be more general. The general* a priori *distributions of disorder moments lead to more complicated formulae, since the corresponding Markov chains are not homogeneous.*

6. FINAL REMARKS

It is notable that the final optimal solutions turn out to have an unexpectedly simple form. It seems that some further simplifications can be made in special cases. From a practical point of view, computer algorithms are necessary to construct B^* – the set in which we stop our observable sequence. Since we always refer to the transition densities, it is still an open problem of switching between the independent Markov sequences.

7. APPENDIX – USEFUL RELATIONS

7.1. Distributions of disordered samples. Let us introduce the n-dimensional distribution for various configuration of disorders:

$$(7.1) \quad f_x^{\theta_1 \leqslant \theta_2 \leqslant n}(\vec{x}_{1n})$$

$$= \bar{\pi}\rho \sum_{j=1}^{n} \left\{ p_1^{j-1} q_1 \prod_{s=1}^{j-1} f_{x_{s-1}}^0(x_s) \prod_{t=j}^{n} f_{x_{t-1}}^2(x_t) \right\}$$

$$+ \bar{\pi}\bar{\rho} \sum_{j=1}^{n-1} \sum_{k=j+1}^{n} \left\{ p_1^{j-1} q_1 p_2^{k-j-1} q_2 \prod_{s=1}^{j-1} f_{x_{s-1}}^0(x_s) \prod_{t=j}^{k-1} f_{x_{t-1}}^1(x_t) \prod_{u=k}^{n} f_{x_{u-1}}^2(x_u) \right\}$$

$$+ \pi\rho \prod_{s=1}^{n} f_{x_{s-1}}^2(x_s),$$

$$(7.2) \quad f_x^{\theta_1 \leqslant n < \theta_2}(\vec{x}_{1n}) = \bar{\pi}\bar{\rho} \sum_{j=1}^{n} \left\{ p_1^{j-1} q_1 p_2^{n-j} \prod_{s=1}^{j-1} f_{x_{s-1}}^0(x_s) \prod_{t=j}^{n} f_{x_{t-1}}^1(x_t) \right\}$$

$$+ \pi\bar{\rho} \sum_{j=1}^{n} \left\{ p_2^{j-1} q_2 \prod_{s=1}^{j-1} f_{x_{s-1}}^1(x_s) \prod_{t=j}^{n} f_{x_{t-1}}^2(x_t) \right\},$$

$$(7.3) \quad f_x^{\theta_1 = \theta_2 > n}(\vec{x}_{1n}) = \rho\bar{\pi} p_1^n \prod_{s=1}^{n} f_{x_{s-1}}^0(x_s),$$

$$(7.4) \quad f_x^{n < \theta_1 < \theta_2}(\vec{x}_{1n}) = \bar{\rho}\bar{\pi} p_1^n \prod_{s=1}^{n} f_{x_{s-1}}^0(x_s).$$

Let us define a sequence of functions $S_n : \times_{i=1}^n \mathbb{E} \to \mathbf{R}$ as follows: $S_0(x_0) = 1$ and for $n \geqslant 1$

$$(7.5) \quad S_n(\vec{x}_n) = f_x^{\theta_1 \leqslant \theta_2 \leqslant n}(\vec{x}_{1n}) + f_x^{\theta_1 \leqslant n < \theta_2}(\vec{x}_{1n})$$
$$+ f_x^{\theta_1 = \theta_2 > n}(\vec{x}_{1n}) + f_x^{n < \theta_1 < \theta_2}(\vec{x}_{1n}).$$

LEMMA 7.1. *For $n > 0$ the function $S_n(\vec{x}_{1,n})$ follows the recursion*

(7.6) $$S_{n+1}(\vec{x}_{1,n+1}) = \mathbf{H}(x_n, x_{n+1}, \overrightarrow{\Pi}_n)S_n(\vec{x}_{1n}),$$

where

(7.7) $\mathbf{H}(x, y, \alpha, \beta, \gamma) = (1 - \alpha)p_1 f_x^0(y) + [p_2(\alpha - \beta) + q_1(1 - \alpha - \gamma)]f_x^1(y)$
$$+ [q_2\alpha + p_2\beta + q_1\gamma]f_x^2(y).$$

P r o o f. Let $0 \leqslant \theta_1 \leqslant \theta_2$ and suppose $B_i \in \mathcal{B}$, $1 \leqslant i \leqslant n + 1$. Let us assume that $X_0 = x$ and put $D_n = \{\omega : X_i(\omega) \in B_i, 1 \leqslant i \leqslant n\}$. For $A_i = \{\omega : X_i \in B_i\} \in \mathcal{F}_i$, $1 \leqslant i \leqslant n + 1$, by the properties of the density function $S_n(\vec{x})$ with respect to the measure $\mu(\cdot)$ we have

$$\int\limits_{D_{n+1}} d\mathbf{P}_x = \int\limits_{\times_{i=1}^{n+1} B_i} S_{n+1}(\vec{x}_{n+1})\mu(d\vec{x}_{1,n+1})$$
$$= \int\limits_{\times_{i=1}^{n} B_i} \int\limits_{B_{n+1}} f(x_{n+1}|\vec{x}_n)\mu(dx_{n+1})S_n(\vec{x}_{0n})\mu(d\vec{x}_{1n})$$
$$= \int\limits_{\times_{i=1}^{n} B_i} \mathbf{P}(A_{n+1}|\vec{X}_n = x_n)\mu_x(d\vec{x}_{1n})$$
$$= \int\limits_{D_n} \mathbf{P}_x(A_{n+1}|\vec{X}_{1n})d\mathbf{P}_x = \int\limits_{D_n} \mathbf{P}_x(A_{n+1}|\mathcal{F}_n)d\mathbf{P}_x = \int\limits_{D_n} \mathbb{I}_{A_{n+1}}d\mathbf{P}_x.$$

Now we split the conditional probability of A_{n+1} into the following parts:

(7.8) $\mathbf{P}_x(X_{n+1} \in A_{n+1} \mid \mathcal{F}_n) = \mathbf{P}_x(n < \theta_1 < \theta_2, X_{n+1} \in A_{n+1} \mid \mathcal{F}_n)$

(7.9) $+ \mathbf{P}_x(\theta_1 \leqslant n < \theta_2, X_{n+1} \in A_{n+1} \mid \mathcal{F}_n)$

(7.10) $+ \mathbf{P}_x(n < \theta_1 = \theta_2, X_{n+1} \in A_{n+1} \mid \mathcal{F}_n)$

(7.11) $+ \mathbf{P}_x(\theta_1 \leqslant \theta_2 \leqslant n, X_{n+1} \in A_{n+1} \mid \mathcal{F}_n).$

In (7.8) we have

$$\int\limits_{D_n} \mathbf{P}_x(\theta_2 > \theta_1 > n, X_{n+1} \in A_{n+1} \mid \mathcal{F}_n)d\mathbf{P}_x$$
$$= \int\limits_{D_n} (\mathbb{I}_{\{\theta_1 = n+1\}} + \mathbb{I}_{\{\theta_1 > n+1\}})\mathbb{I}_{A_{n+1}}d\mathbf{P}_x$$
$$= \int\limits_{\times_{i=1}^{n+1} B_i} \left[f_x^{n<\theta_1<\theta_2}(\vec{x}_{1n})\left(p_1 f_{x_n}^0(x_{n+1}) + q_1 f_{x_n}^1(x_{n+1})\right)\right]\mu(d\vec{x}_{1,n+1})$$
$$= \int\limits_{\times_{i=1}^{n} B_i} \left[f_x^{n<\theta_1<\theta_2}(\vec{x}_{1n}) \int\limits_{B_{n+1}} \left(p_1 f_{x_n}^0(x_{n+1}) + q_1 f_{x_n}^1(x_{n+1})\right)\mu(dx_{n+1})\right]\mu(d\vec{x}_{1n})$$
$$= \int\limits_{D_n} \mathbf{P}_x(\theta_2 > \theta_1 > n \mid \mathcal{F}_n)[\mathbf{P}_{X_n}^0(A_{n+1})p_1 + q_1\mathbf{P}_{X_n}^1(A_{n+1})]d\mathbf{P}_x.$$

In (7.9), by similar arguments to those for (7.8) we get

$$\mathbf{P}_x(\theta_1 \leqslant n < \theta_2, X_{n+1} \in A_{n+1} \mid \mathcal{F}_n)$$
$$= \mathbf{P}_x(\theta_1 \leqslant n < \theta_2, \theta_2 = n + 1, X_{n+1} \in A_{n+1} \mid \mathcal{F}_n)$$
$$\quad + \mathbf{P}_x(\theta_1 \leqslant n < \theta_2, \theta_2 \neq n + 1, X_{n+1} \in A_{n+1} \mid \mathcal{F}_n)$$
$$= \big(\mathbf{P}_x(\theta_1 \leqslant n \mid \mathcal{F}_n) - \mathbf{P}_x(\theta_2 \leqslant n \mid \mathcal{F}_n)\big)[q_2\mathbf{P}^2_{X_n}(A_{n+1}) + p_2\mathbf{P}^1_{X_n}(A_{n+1})].$$

In (7.11) this part has the form:

$$\mathbf{P}_x(\theta_2 \leqslant n, X_{n+1} \in A_{n+1} \mid \mathcal{F}_n) = \mathbf{P}_x(\theta_2 \leqslant n \mid \mathcal{F}_n)\mathbf{P}^2_{X_n}(A_{n+1}).$$

In (7.10) the conditional probability is equal to

$$\mathbf{P}_x(\theta_1 = \theta_2 > n, X_{n+1} \in A_{n+1} \mid \mathcal{F}_n)$$
$$= \mathbf{P}_x(\theta_1 = \theta_2 > n, \theta_2 = n + 1, X_{n+1} \in A_{n+1} \mid \mathcal{F}_n)$$
$$\quad + \mathbf{P}_x(\theta_1 = \theta_2 > n, \theta_2 \neq n + 1, X_{n+1} \in A_{n+1} \mid \mathcal{F}_n)$$
$$= \mathbf{P}_x(\theta_1 = \theta_2 > n \mid \mathcal{F}_n)[q_1\mathbf{P}^2_{X_n}(A_{n+1}) + p_1\mathbf{P}^0_{X_n}(A_{n+1})].$$

These formulae lead to

$$f(X_{n+1}|\vec{X}_{1,n}) = \mathbf{H}(X_n, X_{n+1}, \Pi^1_n, \Pi^2_n, \Pi^{12}_n).$$

which proves the lemma. ∎

7.2. Conditional probability of various events defined by disorder moments. According to the definition of Π^1_n, Π^2_n, Π^{12}_n we get

LEMMA 7.2. *For the model described in Section* 2 *the following formulae are valid*:

(i) $$\mathbf{P}_x(\theta_2 > \theta_1 > n|\mathcal{F}_n) = 1 - \Pi^1_n - \Pi^{12}_n = \frac{f_x^{n<\theta_1<\theta_2}(\vec{x}_{1n})}{S_n(\vec{x}_n)};$$

(ii) $$\mathbf{P}_x(\theta_2 = \theta_1 > n|\mathcal{F}_n) = \Pi^{12}_n = \frac{f_x^{\theta_1=\theta_2>n}(\vec{x}_{1n})}{S_n(\vec{x}_n)};$$

(iii) $$\mathbf{P}_x(\theta_1 \leqslant n < \theta_2|\mathcal{F}_n) = \Pi^1_n - \Pi^2_n;$$

(iv) $$\mathbf{P}_x(\theta_2 \geqslant \theta_1 > n|\mathcal{F}_n) = 1 - \Pi^1_n = \frac{\bar{\pi}p_1^n \prod_{s=1}^n f^0_{x_{s-1}}(x_s)}{S_n(\vec{x}_n)}.$$

Proof. (i) We have

(7.12) $$\Omega = \{\omega : n < \theta_1 < \theta_2\} \cup \{\omega : \theta_1 \leqslant n < \theta_2\}$$
$$\cup \{\omega : \theta_1 \leqslant \theta_2 \leqslant n\} \cup \{\omega : \theta_1 = \theta_2 > n\}.$$

Hence $1 = \mathbf{P}_x(\omega : n < \theta_1 < \theta_2|\mathcal{F}_n) + (\Pi_n^1 - \Pi_n^2) + \Pi_n^2 + \Pi_n^{12}$ and

$$\mathbf{P}_x(\omega : n < \theta_1 < \theta_2|\mathcal{F}_n) = 1 - \Pi_n^1 - \Pi_n^{12}.$$

Let $B_i \in \mathcal{B}, 1 \leqslant i \leqslant n$, $X_0 = x$ and put $D_n = \{\omega : X_i(\omega) \in B_i, 1 \leqslant i \leqslant n\}$. For $A_i = \{\omega : X_i \in B_i\} \in \mathcal{F}_i, 1 \leqslant i \leqslant n$, and $D_n \in \mathcal{F}_n$ we have

$$\int_{D_n} \mathbb{I}_{\{\theta_2 > \theta_1 > n\}} d\mathbf{P}_x = \int_{D_n} \mathbf{P}_x(\theta_2 > \theta_1 > n|\mathcal{F}_n) d\mathbf{P}_x$$

$$= \int_{D_n} \mathbf{P}_x(\theta_2 > \theta_1 > n|\vec{X}_n) d\mathbf{P}_x = \mathbf{P}_x(\theta_2 > \theta_1 > n, D_n)$$

$$= \int_{\times_{i=1}^n B_i} f_x^{n<\theta_1<\theta_2}(\vec{x}_{1n}) \mu(d\vec{x}_{1n}) = \int_{\times_{i=1}^n B_i} f_x^{n<\theta_1<\theta_2}(\vec{x}_{1n}) \left(S_n(\vec{x}_n)\right)^{-1} \mu_x(d\vec{x}_{1n})$$

$$= \int_{D_n} f_x^{n<\theta_1<\theta_2}(\vec{X}_{1n}) \left(S_n(\vec{X}_n)\right)^{-1} d\mathbf{P}_x.$$

Thus $\mathbf{P}_x(\theta_2 > \theta_1 > n|\mathcal{F}_n) = \bar{\rho}\bar{\pi} p_1^n \prod_{i=1}^n f_{X_{i-1}}^0(X_i) \left(S_n(\vec{X}_n)\right)^{-1}$.

(ii) This formula can be obtained by a similar argument to that of (i).

(iii) Let $\theta_1 \leqslant \theta_2$. Since $\{\omega : \theta_2 \leqslant n\} \subset \{\omega : \theta_1 \leqslant n\}$, we have

$$\mathbf{P}_x(\{\omega : \theta_1 \leqslant n < \theta_n\}|\mathcal{F}_n) = \mathbf{P}_x(\{\omega : \theta_1 \leqslant n\} \setminus \{\omega : \theta_2 \leqslant n\}|\mathcal{F}_n) = \Pi_n^1 - \Pi_n^2.$$

(iv) This is obvious.

Thus the proof of the lemma is complete. ∎

REMARK 7.1. *Let $B_i \in \mathcal{B}, 1 \leqslant i \leqslant n+1$, $X_0 = x$ and $D_n = \{\omega : X_i(\omega) \in B_i, 1 \leqslant i \leqslant n\}$. For $A_i = \{\omega : X_i \in B_i\} \in \mathcal{F}_i, 1 \leqslant i \leqslant n$, and $D_n \in \mathcal{F}_n$ we have*

$$\int_{D_n} \mathbb{I}_{\{\theta_1 > n\}} d\mathbf{P}_x = \int_{D_n} \mathbf{P}_x(\theta_1 > n|\mathcal{F}_n) d\mathbf{P}_x = \int_{D_n} \mathbf{P}_x(\theta_1 > n|\vec{X}_n) d\mathbf{P}_x$$

$$= \mathbf{P}_x(\theta_1 > n, D_n) = \int_{\times_{i=1}^n B_i} p_1^n \prod_{i=1}^n f_{x_{i-1}}^0(x_i) \mu(d\vec{x}_{1n})$$

$$= \int_{\times_{i=1}^n B_i} p_1^n \prod_{i=1}^n f_{x_{i-1}}^0(x_i) \left(S_n(\vec{x}_n)\right)^{-1} \mu_x(d\vec{x}_{1n}).$$

Thus $\mathbf{P}_x(\theta_1 > n|\mathcal{F}_n) = p_1^n \prod_{i=1}^n f_{X_{i-1}}^0(X_i) \left(S_n(\vec{X}_n)\right)^{-1}$. Moreover,

$$1 - \Pi_{n+1}^1 = p_1 f_{X_n}^0(X_{n+1})(1 - \Pi_n^1) S_n(\vec{X}_n) \left(S_{n+1}(\vec{X}_{n+1})\right)^{-1}$$

and $S_{n+1}(\vec{X}_{n+1}) = \mathbf{H}(X_n, X_{n+1}, \overrightarrow{\Pi}_n^1) S_n(\vec{X}_n)$. Hence

$$\Pi_{n+1}^1 = 1 - \frac{p_1 f_{X_n}^0(X_{n+1})(1 - \Pi_n^1)}{\mathbf{H}(X_n, X_{n+1}, \overrightarrow{\Pi}_n)}.$$

7.3. Some recursive formulae. In derivation of the formulae in Theorem 3.1 the form of the distribution of some random vectors is taken into account.

LEMMA 7.3. *For the model described in Section 2 the following formulae are valid*:

(i) $$\mathbf{P}_x(\theta_2 = \theta_1 > n+1|\mathcal{F}_n) = p_1\Pi_n^{12} = p_1\rho(1 - \Pi_n^1);$$

(ii) $$\mathbf{P}_x(\theta_2 > \theta_1 > n+1|\mathcal{F}_n) = p_1(1 - \Pi_n^1 - \Pi_n^{12});$$

(iii) $$\mathbf{P}_x(\theta_1 \leqslant n+1|\mathcal{F}_n) = \mathbf{P}_x(\theta_1 \leqslant n+1 < \theta_2|\mathcal{F}_n) + \mathbf{P}_x(\theta_2 \leqslant n+1|\mathcal{F}_n);$$

(iv) $$\mathbf{P}_x(\theta_1 \leqslant n+1 < \theta_2|\mathcal{F}_n) = q_1(1 - \Pi_n^1 - \Pi_n^{12}) + p_2(\Pi_n^1 - \Pi_n^2);$$

(v) $$\mathbf{P}_x(\theta_2 \leqslant n+1|\mathcal{F}_n) = q_2\Pi_n^1 + p_2\Pi_n^2 + q_1\Pi_n^{12};$$

(vi) $$\mathbf{P}_x(\theta_1 = m, \theta_2 > n+1|\mathcal{F}_n) = p_2\Pi_{mn}.$$

P r o o f. (i) On the set $D = \{\omega: X_0 = x, X_1 \in A_1, X_2 \in A_2, \ldots, X_n \in A_n\} \in \mathcal{F}_n$ we have

$$\int_D \mathbb{I}_{\{\theta_2 = \theta_1 > n+1\}} d\mathbf{P}_x = \mathbf{P}_x(D)\mathbf{P}_x(\theta_2 = \theta_1 > n+1|D)$$

$$= \rho\bar{\pi} \sum_{j=n+2}^{\infty} p_1^{j-1} q_1 \int_{\times_{i=1}^n A_i} \prod_{i=1}^n f_{x_{i-1}}^0(x_i)\mu(d\vec{x}_{1n})$$

$$= p_1\rho\bar{\pi}p_1^n \int_{\times_{i=1}^n A_i} \prod_{i=1}^n f_{x_{i-1}}^0(x_i)\mu(d\vec{x}_{1n})$$

$$= p_1\mathbf{P}_x(D)\mathbf{P}_x(\theta_2 = \theta_1 > n|D) = p_1\int_D \mathbb{I}_{\{\theta_2 = \theta_1 > n\}} d\mathbf{P}_x.$$

Consequently, by (3.2) and the definition of the conditional probability we have

$$\mathbf{P}_x(\theta_2 = \theta_1 > n+1|\mathcal{F}_n) = p_1\Pi_n^{12}.$$

Next, we get

$$\int_D \mathbb{I}_{\{\theta_1 > n\}} d\mathbf{P}_x = \mathbf{P}_x(D)\mathbf{P}_x(\theta_1 > n|D)$$

$$= \bar{\pi} \sum_{j=n+1}^{\infty} p_1^{j-1} q_1 \int_{\times_{i=1}^n A_i} \prod_{i=1}^n f_{x_{i-1}}^0(x_i)\mu(d\vec{x}_{1n})$$

$$= \frac{1}{\rho}\mathbf{P}_x(D)\mathbf{P}_x(\theta_2 = \theta_1 > n|D) = \frac{1}{\rho}\int_D \mathbb{I}_{\{\theta_2 = \theta_1 > n\}} d\mathbf{P}_x,$$

which proves (i) of the lemma.

(ii) Similarly to the above we get

$$\int\limits_{D} \mathbb{I}_{\{\theta_2>\theta_1>n+1\}}d\mathbf{P}_x = \mathbf{P}(D)\mathbf{P}_x(\theta_2>\theta_1>n+1|D)$$

$$= p_1\rho\bar{\pi}p_1^n \int\limits_{\times_{i=1}^n A_i} \prod_{i=1}^n f_{x_{i-1}}^0(x_i)\mu(d\vec{x}_{1n})$$

$$= p_1\mathbf{P}(D)\mathbf{P}_x(\theta_2>\theta_1>n|D) = p_1\int\limits_{D}\mathbb{I}_{\{\theta_2>\theta_1>n\}}d\mathbf{P}_x.$$

Therefore, by Lemma 7.2 (ii) we obtain the formula (ii) of Lemma 7.3.

(iii) It is obvious by the assumption $\theta_1 \leqslant \theta_2$.

(iv) On the set D we have

$$\int\limits_{D} \mathbb{I}_{\{\theta_1\leqslant n+1<\theta_2\}}d\mathbf{P}_x = \mathbf{P}(D)\mathbf{P}_x(\theta_1\leqslant n+1<\theta_2|D)$$

$$\overset{(2.1),(2.2)}{=} \sum_{j=0}^{n+1}\mathbf{P}(\omega:\theta_1=j)\sum_{k=n+2}^{\infty}\bar{\rho}p_2^{k-j-1}q_2$$

$$\times \int\limits_{\times_{i=1}^n A_i}\prod_{s=1}^{j-1}f_{x_{s-1}}^0(x_s)\prod_{r=j}^n f_{x_{r-1}}^1(x_r)\mu(d\vec{x}_{1n})$$

$$= \bar{\pi}p_1^n q_1(1-\rho)\int\limits_{\times_{i=1}^n A_i}\prod_{s=1}^n f_{x_{s-1}}^0(x_s)\mu(d\vec{x}_{1n})$$

$$+ p_2\sum_{0}^n\mathbf{P}(\omega:\theta_1=j)p_2^{n+1-j}\int\limits_{\times_{i=1}^n A_i}\prod_{s=1}^{j-1}f_{x_{s-1}}^0(x_s)\prod_{r=j}^n f_{x_{r-1}}^1(x_r)\mu(d\vec{x}_{1n})$$

$$\overset{(\mathrm{L.7.2})}{=} q_1\mathbf{P}(D)\mathbf{P}_x(\theta_2>\theta_1>n|D) + p_2\mathbf{P}(D)\mathbf{P}_x(\theta_1\leqslant n<\theta_2|D)$$

$$= q_1\int\limits_{D}\mathbb{I}_{\{\theta_2>\theta_1>n\}}d\mathbf{P}_x + p_2\int\limits_{D}\mathbb{I}_{\{\theta_1\leqslant n<\theta_2\}}d\mathbf{P}_x.$$

(v) If we substitute $n+1$ for n in (7.12), then we obtain

$$\mathbf{P}_x(\theta_2\leqslant n+1|\mathcal{F}_n) = 1 - \mathbf{P}_x(n+1<\theta_1=\theta_2|\mathcal{F}_n)$$

$$- \mathbf{P}_x(n+1<\theta_1<\theta_2|\mathcal{F}_n) - \mathbf{P}_x(\theta_1\leqslant n+1<\theta_2|\mathcal{F}_n)$$

$$= 1 - p_1\Pi_n^{12} - p_1(1-\Pi_n^1-\Pi_n^{12}) - q_1(1-\Pi_n^1-\Pi_n^{12})$$

$$+ p_2(\Pi_n^2-\Pi_n^1) = q_2\Pi_n^1 + p_2\Pi_n^2 + q_1\Pi_n^{12}.$$

(vi) We have

$$\int_D \mathbb{I}_{\{\theta_1=m,\theta_2>n+1\}} d\mathbf{P}_x = \mathbf{P}_x(D)\mathbf{P}_x(\theta_1=m,\theta_2>n+1|D)$$

$$= \bar{\pi}\bar{\rho}p_1^{m-1}q_1 \sum_{j=n+2}^{\infty} p_2^{j-m-1}q_2 \int_{\times_{i=1}^n B_i} \prod_{i=1}^m f_{x_{i-1}}^0(x_i) \prod_{j=m+1}^n f_{x_{j-1}}^1(x_j)\mu(d\vec{x}_{1n})$$

$$= p_2\bar{\pi}\bar{\rho}p_1^{m-1}q_1 p_2^{n-m} \int_{\times_{i=1}^n B_i} \prod_{i=1}^m f_{x_{i-1}}^0(x_i) \prod_{j=m+1}^n f_{x_{j-1}}^1(x_j)\mu(d\vec{x}_{1n})$$

$$= p_2\mathbf{P}_x(D)\mathbf{P}_x(\theta_1=m,\theta_2>n|D) = p_2 \int_D \mathbb{I}_{\{\theta_1=m,\theta_2>n\}} d\mathbf{P}_x$$

and, consequently, by (3.3) and the definition of conditional probability we obtain

$$\mathbf{P}_x(\theta_2=m,\theta_1>n+1|\mathcal{F}_n) = p_2\Pi_{nm}.$$

This proves the part (vi) of the lemma. ∎

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