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RANDOM WALKS ON THE NONNEGATIVE INTEGERS WITH A LEFT-BOUNDED GENERATOR

BY

CHARLES DELORME AND JEAN-MARC RINKEL (ORSAY)

Abstract. This paper studies the random walks $S_0 + \sum X_i$ on the nonnegative integers, where the X_i's are independent identically distributed random variables with generating function of type $\Phi(z) = \sum_{i \geqslant -s} c_i z^i$, s a positive integer, with a convergence radius greater than 1. We infer from a link between the number of zeros of $z \mapsto 1 - \Phi(z)$ inside the unit disc and $\inf X_i$ a factorisation of the symbol $f(\theta) = 1 - \Phi(e^{i\theta})$ which allows a geometrical computation of the potentials associated with these random walks. Examples illustrate this theory.

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1. INTRODUCTION

1.1. Context. Among works about discrete random walks two threads can be seen, namely the search of recurrence or transience criteria, and the computation of potentials. In this case, the domain of the walk may be bounded or unbounded. If it is bounded, its size can be used as a parameter in asymptotic evaluation of potentials. Most papers deal with one-dimensional walks. Let us cite Spitzer and Stone [17] for their proving that potentials on a segment tend to the values of a Green kernel when the size of the segment tends to infinity. In their case the symbol $f(\theta) = 1 - \Phi(e^{i\theta})$ associated with the generating function Φ of the random walk has just a zero of order 2 on the unit circle. Kesten in [7] extends this to random walks with a generating function admitting a zero of order α with $0 < \alpha < 2$. These two works ([17], [7]) deal with symmetric random walks, that is, jumps of equal lengths in opposite directions occur with the same probability.

In the same vein, any symbol of the kind of $f(\theta) = (1 - \cos\theta)|f_1(e^{i\theta})|^2$, where $f_1(z)$ belongs to a class of holomorphic functions on an open disc containing the torus $\mathbb{T}$, provides a much finer asymptotic estimation of potentials; see [9] and [10]. Higher dimension begins to appear within the same theme (potentials and their limits as a Green kernel and, more generally, their asymptotic behaviour) for a rectangular domain (see [4]). On the other hand, a recent result in [5] allows an investigation in the domain of nonsymmetric random walks: in that work the number of zeros inside the unit disc of the symbol extended to some part of the complex plane is evaluated. A simpler version of this result (Theorem 1.1) is given with its corollary (Theorem 1.2), i.e. the factorisation theorem in Section 1.2. Fundamental ideas about random walks are recalled there. Potentials appear as entries of inverses of Toeplitz matrices; conditions of inversibility of these matrices are presented in Section 1.3. The main results, Theorems 1.4 and 1.5, deal with approximation of potentials according to the length of the interval where the random walk is performed. They are explained in Section 1.4. Examples illustrate these results. The proofs of Theorems 1.4 and 1.5 are detailed in Sections 3 and 4.

For nonsymmetric random walks on a segment we intend to study here the asymptotic behaviour of their potentials when the rightmost end of the segment tends to infinity. Examples are provided.

1.2. Random walks on the integers generated by a left-bounded variable. In this paper, we study a family of random walks on the integers. First, we give some notation. A random walk over the positive integers is defined by $S_n = S_0 + X_1 + \ldots + X_n$, where X_i's are independent identically distributed random variables. The X_i's have the same distribution as a variable X, the *generator of the random walk*, and S_0 is an integer-valued random variable distributed on the interval $[0, N]$, independent of X_i's. We denote by Φ the generating function, that is

$$\Phi(z) = \sum_{k\in\mathbb{Z}} \mathbf{P}(X = k)z^k. \tag{1.1}$$

When the random walk occurs on the interval $[0, N]$, where $N \in [0, +\infty)$, we put (as in [16], p. 107): for all $k, l \in [0, N]$

$$Q(k, l) = \mathbf{P}(X = l - k).$$

The probability that a "particle" starting from k at time 0 will reach l at time n, without leaving the interval $[0, N]$ in the meantime, is $Q_N^n(k, l)$ given by:

$$Q_N^0(k, l) = \begin{cases} 1 & \text{if } k = l, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad Q_N^{n+1}(k, l) = \sum_{t=0}^{N} Q_N^n(k, t)Q(t, l).$$

The probability that the particle arrives at l at time n is $\sum_{t=0}^{N} Q_N^n(t, l)\mathbf{P}(S_0 = t)$.

This admits an algebraic interpretation: Q_N^n is the n-th power of the matrix Q_N^1, and if the entries of the $1\times(N+1)$-matrix L_0 are the probabilities $\mathbf{P}(S_0) = t$, $t \in [0, N]$, the probabilities that the particle comes to l, $0 \leqslant l \leqslant N$, at instant n are given by the matrix product $L_n = L_0 Q_N^n$. This interpretation will be useful in Proposition 1.1.

For $k, l \in [0, N]$, let $\mathcal{N}_N(k, l)$ be the number of visits to l of the process S_n before the particle leaves the interval $[0, N]$ when $S_0 = k$. We denote by $g_N(k, l)$

the expectation $E\big(\mathcal{N}_N(k,l)\big)$ of $\mathcal{N}_N(k,l)$. These real numbers $g_N(k,l)$ are called *potentials*. We have

$$(1.2)\qquad g_N(k,l) = \begin{cases} \sum_{n\geqslant 0} Q_N^n(k,l) & \text{if } k,l \in [0,N],\\ 0 & \text{otherwise.}\end{cases}$$

A proof of (1.2) can be found in [9]. The computation of the numbers $g_N(k,l)$ is fundamental. These potentials indeed allow to compute many interesting probabilistic quantities like those presented in [16] or [9]. Similarly, when the segment $[0,N]$ is replaced by the half-line $[0,+\infty)$, we denote by $\mathcal{N}_\infty(k,l)$ the number of visits at l starting from k, before the mobile leaves the half-line, and by $g_\infty(k,l)$ the expected value of $\mathcal{N}_\infty(k,l)$. Setting $Q_\infty^0(k,l) = 1$ if $k = l$, $Q_\infty^0(k,l) = 0$ if $k \neq l$, and $Q_\infty^n(k,l) = \sum_{t=0}^{\infty} Q_\infty^n(k,t)Q(t,l)$, we get in the same way

$$(1.3)\qquad g_\infty(k,l) = \begin{cases} \sum_{n\geqslant 0} Q_\infty^n(k,l) & \text{if } k,l \text{ are nonnegative,}\\ 0 & \text{otherwise.}\end{cases}$$

The potential $g_\infty(k,l)$ is the limit of $g_N(k,l)$ when $N \to \infty$. We give a short proof, provided by the referee.

LEMMA 1.1. *Let X be an integer-valued random variable generating random walks with potentials $g_N(k,l)$ and $g_\infty(k,l)$. Then, for all $k,l \in \mathbb{N}$,*

$$\lim_{N\to\infty} g_N(k,l) = g_\infty(k,l).$$

Proof. If τ_N denotes the time of a first exit from $[0,N]$ $(0 \leqslant N \leqslant \infty)$, then τ_N increases to τ_∞.

Hence $E\big(\sum_{j=0}^{\tau_N} \mathbf{1}_{\{l\}}(S_j)|S_0 = k\big)$ converges to $E\big(\sum_{j=0}^{\tau_\infty} \mathbf{1}_{\{l\}}(S_j)|S_0 = k\big)$. In other words, $g_N(k,l) \to g_\infty(k,l)$ as $N \to \infty$. ■

The aim of this article is to give an asymptotical estimation of the potentials of random walks on $[0,N]$, satisfying the following assumptions:

(1.4)
- There exists $s \in \mathbb{N}$ such that the variable X is integer-valued and is distributed on $[-s,+\infty)$ and $\mathbf{P}(X = -s) > 0$, so that $\inf X = -s$.
- The convergence radius of $\sum_{k=-s}^{\infty} \mathbf{P}(X = k)z^k$ is greater than 1, and therefore X has finite expectation and variance.
- $\gcd\{k \in [-s,+\infty)$ such that $\mathbf{P}(X = k) \neq 0\} = 1$.

A recent result stated by the authors in [5] gives a relation between the expectation of the random variable X that generates the random walk and the number ζ of zeros of the function $z \mapsto 1 - \Phi(z)$ inside the unit disc, where Φ is the generating function of X. This relation provides a way towards the computation of potentials as explained later. It is described by the following theorem that comes from [5] with some simplifications.

THEOREM 1.1. *Let X be an integer-valued random variable. Let ζ denote the number of zeros of* $1-\Phi(z)$ *inside the unit disc. Under the assumptions* (1.4), *we have*

$$\zeta = \begin{cases} -\inf X & \text{if } E(X) > 0, \\ -1-\inf X & \text{if } E(X) \leqslant 0. \end{cases}$$

A direct calculation is of course not appropriate to obtain such a result. It can be obtained by classical complex analysis arguments. From this result we derive a factorisation of the restriction of $1-\Phi$ to the unit circle (or torus $\mathbb{T}$), that is to say, we factorise the symbol $f(\theta) = 1-\Phi(e^{i\theta})$. This factorisation reveals itself useful in [9].

We have to introduce some notation from functional analysis to describe that factorisation.

• D denotes the unit open disc and $\overline{D}$ its closure.

• Let $\mathbb{T}$ denote the one-dimensional torus and σ be the Haar measure on $\mathbb{T}$. Let $L^2(\mathbb{T})$ be the space of square-integrable complex-valued functions on $\mathbb{T}$, and denote by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|_2$ the corresponding inner product and norm: $\langle f, g\rangle = \int_{\mathbb{T}} f\bar{g}d\sigma$, where for all $g \in L^2(\mathbb{T})$, $\bar{g}$ denotes the function on $\mathbb{T}$ defined by $\bar{g}(\chi) = \overline{g(\chi)}$ (see [14]).

• $H^+ = \{h \in L^2(\mathbb{T})$ such that $\hat{h}(s) = 0$ for $s < 0\}$, where $\hat{h}(s)$ is the s-th Fourier coefficient of h and H^- denotes the orthogonal complement of H^+ in $L^2(\mathbb{T})$. The orthogonal projectors of $L^2(\mathbb{T})$ on H^+ and H^- will be denoted by π_+ and π_-, respectively.

• $H^\infty = H^+ \cap L^\infty(\mathbb{T})$, where $L^\infty(\mathbb{T})$ is the space of the essentially bounded measurable functions endowed with distance d_∞ associated with the norm $\|\cdot\|_\infty$.

• χ denotes $e^{i\theta}$ ($\theta \in \mathbb{R}$), $\mathcal{P}_{[-N,N]}$ denotes the vector space $\mathrm{Vect}\{\chi^h,\ -N \leqslant h \leqslant N\} \subset L^2(\mathbb{T})$, $\mathcal{P}_{[0,N]}$ means the vector space $\mathrm{Vect}\{\chi^h,\ 0 \leqslant h \leqslant N\} \subset L^2(\mathbb{T})$ and π_N is the orthogonal projector of $L(\mathbb{T})$ on $\mathcal{P}_{[0,N]}$.

Now we can give a factorisation of the symbol.

THEOREM 1.2 (Factorisation theorem). *Let* $\Phi(z) = \sum_{k\in\mathbb{Z}} \mathbf{P}(X = k)z^k$ *be the generating function of a random variable X satisfying the assumptions* (1.4). *Denote the zeros of* $1-\Phi$ *in D by* $x_1, \ldots, x_s$ *when* $E(X) > 0$ *and by* $x_1, \ldots, x_{s-1}$ *when* $E(X) \leqslant 0$. *Then there exists a function h holomorphic on an open disc with radius* $\rho > 1$, *without zero on* $\overline{D}$, *such that the symbol* $f(\theta) = 1-\Phi(\chi)$ *satisfies*

$$f(\theta) = g_1(\chi)g_2(\chi),$$

where $g_1(\chi)$ *and* $g_2(\chi)$ *are given by Table* 1.

Theorem 1.2 is a direct consequence of Theorem 1.1. Theorem 1.2 allows the computation of potentials; the tool is the inversion of truncated Toeplitz operators. Let us now explain how these things are related.

TABLE 1. Decomposition of symbols

$g_1(\chi)$	$g_2(\chi)$	condition
$(1-\chi)h(\chi)$	$\prod_{i=1}^{s}(1-x_i\bar{\chi})$	$E(X)>0$
$(1-\chi)h(\chi)$	$\prod_{i=1}^{s-1}(1-x_i\bar{\chi})(1-\bar{\chi})$	$E(X)=0$
$h(\chi)$	$\prod_{i=1}^{s-1}(1-x_i\bar{\chi})(1-\bar{\chi})$	$E(X)<0$

1.3. Truncated Toeplitz operators and potentials. Let f be a function integrable on the one-dimensional torus. The *truncated Toeplitz operator associated with* f is the operator $T_N(f)$ of $\mathcal{P}_{[0,N]}$ defined for all $q \in \mathcal{P}_{[0,N]}$ by $T_N(f)(q) = \pi_N(fq)$. Its matrix in the basis $(1, \chi, \ldots, \chi^N)$ is $\big(\hat{f}(l-k)\big)_{0\leqslant k,l\leqslant N}$ denoted in this paper by $\big(T_N(f)(k,l)\big)_{0\leqslant k,l\leqslant N}$.

It is also said that $T_N(f)$ is a *Toeplitz operator with symbol* f. Let us now describe the link with random walks. Denote by I_N, Q_N, G_N the $(N+1)\times(N+1)$-matrices whose entries in k-th row and l-th column are, respectively, $\delta(k,l)$ (identity matrix), $Q(k,l), g_N(k,l)$ and let $f(\theta) = 1 - \Phi(\chi)$, where Φ is the generating function. Then

$$T_N(f) = I_N - Q_N. \tag{1.5}$$

PROPOSITION 1.1 (Spitzer [16], Chapter V). *If the walk is not the trivial one* $(\mathbf{P}(X=0)=1)$, *in other words* $Q_N \neq I_N$, *then* $T_N(f) = I_N - Q_N$ *is invertible.*

From now on, only the nontrivial random walks are considered. The potentials appear in the formula:

$$G_N = \sum_{n=0}^{\infty} Q_N^n = \big(T_N(f)\big)^{-1}. \tag{1.6}$$

A proof of equation (1.6) can be found in [9].

Consequently, instead of the computation of the sum in (1.2) we may perform the computation of the inverse of a Toeplitz matrix. In this paper, we expose a geometrical inversion structure which appears, for example, in [1] and [15] for positive symbols and has been developed in [12] in a way that allows the estimation of the extremal eigenvalues of Toeplitz operators. Surprisingly, the structure described in [12] is the most suitable to compute potentials for random walks satisfying the assumptions (1.4). Theorem 1.3 as stated here appeared first in [11] in order to evaluate the potentials of a random walk in a part of the plane. We emphasize here the necessity of some regularity of the symbol (namely, equation (1.9)). This condition was not explicitly stated in [12] because the symbols there satisfy it obviously.

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With the previous assumptions and the notation of Theorem 1.2, let us put the following functions of χ:

$$(1.7)\quad \tilde{X}_{N,k} = \pi_+\left(\tilde{\Phi}_N \pi_+\left(\frac{\chi^k}{g_2}\right)\right) \quad \text{and} \quad \tilde{Y}_{N,l} = \pi_+\left(\bar{\Phi}_N \pi_+\left(\frac{\chi^l}{\overline{g_1}}\right)\right),$$

where

$$\Phi_N = \frac{g_1}{g_2}\chi^{N+1}, \quad \tilde{\Phi}_N = \frac{g_2}{g_1}\bar{\chi}^{N+1},$$

and g_1, g_2 are the functions of χ given in Table 1.

The Hankel operators H_{Φ_N} and $H_{\tilde{\Phi}_N}$ are defined by

$$(1.8)\quad \begin{aligned} H_{\Phi_N}(\psi) &= \pi_-(\Phi_N\, \psi) && \text{for all } \psi \in H^+, \\ H_{\tilde{\Phi}_N}(\varphi) &= \pi_+(\tilde{\Phi}_N\, \varphi) && \text{for all } \varphi \in H^-. \end{aligned}$$

THEOREM 1.3. *Suppose that* $f(\chi) = g_1(\chi)g_2(\chi)$, *where* g_1 *and* g_2 *do not vanish on* $\mathbb{T}$, *and* $g_1, g_1^{-1}, \overline{g_2}, \overline{g_2}^{-1}$ *belong to* H^∞ *and satisfy*

(1.9)

$$\lim_{N\to+\infty} d_\infty\left(\frac{1}{|g_2|^2}, \mathcal{P}_{[-N,N]}\right) = 0 \quad \textit{and} \quad d_\infty\left(\frac{1}{|g_1|^2}, \mathcal{P}_{[-N,N]}\right) \textit{ bounded}$$

or

$$\lim_{N\to+\infty} d_\infty\left(\frac{1}{|g_1|^2}, \mathcal{P}_{[-N,N]}\right) = 0 \quad \textit{and} \quad d_\infty\left(\frac{1}{|g_2|^2}, \mathcal{P}_{[-N,N]}\right) \textit{ bounded.}$$

Then the operator $H_{\tilde{\Phi}_N} H_{\Phi_N}$ *is bounded on* $L^2(\mathbb{T})$ *with norm* (*strictly*) *less than* 1. *Furthermore*, *the matrix* $T_N(f)^{-1}$ *is given by*

$$(1.10)\quad \big(T_N(f)^{-1}\big)(k,l) = \mathfrak{T}_1(k,l) - \mathfrak{T}_{2,N}(k,l),$$

where

$$(1.11)\quad \mathfrak{T}_1(k,l) = \left\langle \pi_+\left(\frac{\chi^k}{g_2}\right),\ \pi_+\left(\frac{\chi^l}{\overline{g_1}}\right)\right\rangle,$$

$$(1.12)\quad \mathfrak{T}_{2,N}(k,l) = \langle (I - H_{\tilde{\Phi}_N} H_{\Phi_N})^{-1}\tilde{X}_{N,k},\ \tilde{Y}_{N,l}\rangle.$$

This theorem is the one-dimensional counterpart of the inversion theorem given in [11], p. 76. $\mathfrak{T}_1(k,l)$ and $\mathfrak{T}_{2,N}(k,l)$ are called there the *first term* and the *second term of the inversion formula*. Let us comment the formula (1.10). Under sensible assumptions on X including the conditions (1.4), we show that $\mathfrak{T}_{2,N}(k,l) \to 0$ as $N \to \infty$ and estimate the speed of convergence. Since $g_N(k,l) = \mathfrak{T}_1(k,l) - \mathfrak{T}_{2,N}(k,l)$ (cf. (1.10)), Lemma 1.1 gives then $\mathfrak{T}_1(k,l) = g_\infty(k,l)$ and $\mathfrak{T}_{2,N}(k,l) = g_\infty(k,l) - g_N(k,l)$. Thus under these assumptions, we obtain with $\mathfrak{T}_{2,N}(k,l)$ an evaluation of $g_\infty(k,l) - g_N(k,l)$.

REMARK 1.1. *Let us suppose, with the notation of Theorem* 1.3, *that* f *has a factorisation* $f = g_1 g_2$ *satisfying the assumptions of this theorem. Then* f *also admits the factorisation* $f = G_1 G_2$, *where* $G_1 = g_1/\lambda$ *and* $G_2 = \lambda g_2$, λ *being a nonnull complex. The functions* G_1 *and* G_2 *satisfy also the same assumptions as* g_1 *and* g_2 *and the inversion formula remains the same no matter what factorisation* $f = g_1 g_2$ *or* $f = G_1 G_2$ *is used.*

The proof is immediate.

If the symbol f of Theorem 1.3 vanishes on the torus, it is not possible to just apply the inversion Theorem 1.3 in order to compute potentials. Therefore, we "regularise" the symbol (see Proposition 2.1). We are now in a position to give the main theorems of this paper.

1.4. Main results with examples. The estimates of $g_\infty(k,l) - g_N(k,l)$ come from computations of $\mathfrak{T}_{2,N}(k,l)$ defined in (1.12). It is given in Theorems 1.4 and 1.5 where the generator X satisfies the conditions (1.4) and extra conditions on the generating function Φ of X and $\inf X$, respectively. In both cases, it turns out that if $E(X) > 0$, then $g_\infty(k,l) - g_N(k,l)$ has order x^N, where x is the root of $1 - \Phi$ inside D with maximum modulus. In order to justify the terms of the first assumption of Theorem 1.4, let us state the following lemma.

LEMMA 1.2. *Let* X *be an integer-valued random variable with mean value* $E(X) > 0$ *and* $\inf X < 0$. *If* Φ *denotes its generating function, then* $1 - \Phi$ *admits a unique real zero* x *on the interval* $(0,1)$. *If* z *is another zero of* $1 - \Phi$ *in the unit open disc, then* $|z| \leqslant x$.

Proof. Since $E(X) > 0$, we have $(1 - \Phi)'(1) < 0$. From $\lim_{0^+}(1 - \Phi) = -\infty$ we infer the existence of a zero, denoted by x, in the interval $(0,1)$. The fact that $(1 - \Phi)'' < 0$ on $(0,1)$ proves the uniqueness. Now let z be a complex number such that $x < |z| < 1$. Then

$$|1 - \Phi(z)| \geqslant \big|1 - |\Phi(z)|\big| > 0.$$

Indeed,

$$|\Phi(z)| \leqslant \Phi(|z|) < 1. \quad \blacksquare$$

THEOREM 1.4 (Asymptotic 1). *Let* X *be an integer-valued random variable satisfying the assumptions* (1.4) *with generating function* Φ *and such that the roots of* $1 - \Phi$ *inside* D *are simple. With the notation of Theorem* 1.2 *and Table* 1, *we assume also the following*:

(i) *In the case* $E(X) \geqslant 0$, *the function* $1/h$ *is a polynomial without zero inside* D.

(ii) *In the case* $E(X) < 0$, *the function* h *admits a unique zero* $1/\alpha$ *outside* D *so that* $g_1(\chi) = h(\chi) = (1 - \alpha\chi)\tilde{h}(\chi)$ *and* $\tilde{h}(z)$ *is the inverse of a polynomial without zero inside* D.

Then there exist constants $c_{k,l}$ such that for N large enough the following three relations hold:

1. *If* $E(X) > 0$, $|\mathfrak{T}_{2,N}(k,l)| \leqslant c_{k,l}x^N$, *where x is the largest modulus of the zeros of* $1-\Phi(z)$ *inside* D, *in other words*, *x is the positive zero of* $1-\Phi(z)$ *between* 0 *and* 1.
2. *If* $E(X) = 0$, $|\mathfrak{T}_{2,N}(k,l)| \leqslant c_{k,l}/N$.
3. *If* $E(X) < 0$, $|\mathfrak{T}_{2,N}(k,l)| \leqslant c_{k,l}|\alpha|^N$.

This theorem will be proved in Section 3. As an illustration of the previous result, let us consider the random walk generated by X with *geometrical distribution*:

$$\mathbf{P}(X=k) = qp^{k+s}, \quad p+q=1, p>0, q>0,\ k \geqslant -s. \tag{1.13}$$

The generating function $\Phi(z)$ of X satisfies the assumptions of Theorem 1.4. Indeed,

$$\Phi(z) = \frac{q}{z^s(1-pz)} \quad \text{and} \quad E(X) = \frac{p}{q} - s.$$

A straightforward computation gives

$$1-\Phi(z) = -\frac{z-1}{z^s(1-pz)}\big(z^s - q\sum_{i=0}^{s} z^i\big).$$

Theorem 1.1 ensures the existence of s zeros x_i inside D for $1-\Phi(z)$ when $E(X)>0$, that is, $p>qs$, and in this case we obtain $f(\theta)=1-\Phi(\chi)=g_1(\chi)g_2(\chi)$ with

$$g_1(\chi) = p\frac{1-\chi}{1-p\chi} \quad \text{and} \quad g_2(\chi) = \prod_{i=1}^{s}(1-x_i\bar{\chi}).$$

When $E(X) \leqslant 0$, the function $1-\Phi(z)$ has $s-1$ zeros inside D, denoted by x_i. If $E(X)<0$, that is, $p<sq$, then $1-\Phi$ has furthermore a unique zero $x>1$. A direct computation gives indeed

$$1-\Phi(z) = \frac{p(1-z)}{z^s(1-pz)}\big(z^s - q\sum_{i=0}^{s} z^i\big).$$

Let us put $L(z) = z^s - q\sum_{i=0}^{s} z^i$. The other zeros of L are the x_i's. Hence we obtain the decomposition $f(\theta) = g_1(\chi)g_2(\chi)$, where

$$g_1(\chi) = p\frac{x-\chi}{1-p\chi} \quad \text{and} \quad g_2(\chi) = (1-\bar{\chi})\prod_{i=1}^{s-1}(1-x_i\bar{\chi}).$$

If $E(X)=0$, that is, $p=qs$, a straightforward computation gives the decomposition $f(\theta) = g_1(\chi)g_2(\chi)$ with

$$g_1(\chi) = p\frac{1-\chi}{1-p\chi} \quad \text{and} \quad g_2(\chi) = (1-\bar{\chi})\prod_{i=1}^{s-1}(1-x_i\bar{\chi}).$$

In order to advertise the method we give here some asymptotic estimation of the potentials. The proof is straightforward from the inversion formula (1.11) and Theorem 1.4.

EXAMPLE 1.1 (Potentials for some geometrical distributions). The distribution of X is given by equation (1.13). Then there exist constants $c_{k,l}$ satisfying:

1. In the case $\inf X = -1$, that is, $s = 1$, we have $E(X) = p/q - 1$ and if $E(X) = 0$,

$$g_N(k,l) = \begin{cases} 2(1+l/2) + O(1/N) & \text{if } k \geqslant l, \\ k+1+O(1/N) & \text{if } k < l; \end{cases}$$

if $E(X) > 0$,

$$g_N(k,l) = \begin{cases} \alpha^{k-l}(1-\alpha)^{-1}(1-\alpha^{l+2}) + O(\alpha^N) & \text{if } k \geqslant l, \\ \alpha(1-\alpha^{k+1})(1-\alpha)^{-1} + O(\alpha^N) & \text{if } k < l, \end{cases}$$

where $\alpha = q/p$;

if $E(X) < 0$,

$$g_N(k,l) = \begin{cases} q^{-1}\big(1+p^2(q-p)^{-1}(1-\beta^l)\big) + O(\beta^N) & \text{if } k \geqslant l, \\ p^2[q(q-p)]^{-1}\beta^{l-k-1}(1-\beta^{k+1}) + O(\beta^N) & \text{if } k < l, \end{cases}$$

where $\beta = p/q$.

2. In the case $\inf X = -2$, that is, $s = 2$, we have $E(X) = p/q - 2$. Let us give details for $E(X) > 0$, i.e. $0 < q < \frac{1}{3}$:

$$g_N(k,l) = \begin{cases} [p(x-y)]^{-1}\big(A(y) - A(x)\big) + O(y^N) & \text{if } k \geqslant l > 1, \\ q[p(x-y)]^{-1}\big(B(x) - B(y)\big) + O(y^N) & \text{if } 1 < k < l, \end{cases}$$

where

$$x = \frac{q - \sqrt{q(q+4p)}}{2p}, \quad y = \frac{q + \sqrt{q(q+4p)}}{2p},$$
$$A(t) = \frac{qt^{k+2} - (1-pt)t^{k-l+1}}{1-t}, \quad B(t) = \frac{1-t^{k+2}}{1-t}.$$

We have to make compromises between the assumptions involving the symbol and those about $\inf X$. In Theorem 1.4 no constraint exists on $\inf X$, even if it means restricting the class of the symbols. In Theorem 1.5, however, restrictions involve $\inf X$.

THEOREM 1.5 (Asymptotic 2). *Let X be an integer-valued random variable satisfying the assumptions* (1.4) *with* $\inf X = -1$. *Then there exist constants $c_{k,l}$ such that the following two relations hold*:

1. *If* $E(X) > 0$, $|\mathfrak{T}_{2,N}(k,l)| \leqslant c_{k,l}x^N$, *where x is the real positive zero of* $1 - \Phi(z)$ *inside D.*
2. *If* $E(X) = 0$, $|\mathfrak{T}_{2,N}(k,l)| \leqslant c_{k,l}/N$.

REMARK 1.2. *The case* $E(X) < 0$ *is more intricated and we will keep assuming the conditions of Theorem* 1.4 *in this case.*

Theorem 1.5 will be proved in Section 4. As an illustration for it, let us consider a shifted Poissonian random variable X with parameter λ:

$$\mathbf{P}(X = k) = e^{-\lambda}\frac{\lambda^{(k+1)}}{(k+1)!} \quad \text{with } \lambda > 0 \text{ and } k \geqslant -1. \tag{1.14}$$

Here we obtain $E(X) = \lambda - 1, \Phi(z) = z^{-1}\exp\big(\lambda(z-1)\big)$ and we get then the symbol $f(\theta) = 1 - e^{\lambda(\chi-1)}\bar{\chi}$. A direct calculation gives the following factorisation.

(i) When $\lambda > 1$, we denote by x the unique zero with modulus less than 1 of the function $z \mapsto 1 - \Phi(z)$ and we put

$$h(z) = \frac{z - \exp\big(\lambda(z-1)\big)}{(1-z)(z-x)}.$$

Then:

$$f(\theta) = \underbrace{(1-\chi)h(\chi)}_{g_1}\underbrace{(1-x\bar{\chi})}_{g_2}.$$

(ii) When $\lambda = 1$, we put $h(z) = (z-1)^{-2}\big(z - \exp(z-1)\big)$. Then:

$$f(\theta) = \underbrace{(1-\chi)h(\chi)}_{g_1}\underbrace{(1-\bar{\chi})}_{g_2}.$$

By Theorem 1.3 (inversion formula) and Theorem 1.5, the next proposition gives an example of potentials.

EXAMPLE 1.2. Let the distribution of X be given by equation (1.14). There exist constants $c_{k,l}$ such that, for $\lambda > 1$ and N large enough, the following relations hold: if $k \geqslant l$, then

$$g_N(k,l) = x^{k-l}(1-x)^{-1}\big(\sum_{i=0}^{l}\mu_i x^i - x^{l+1}\sum_{i=0}^{l}\mu_i\big) + O(x^N),$$

and if $k < l$, then

$$g_N(k,l) =$$
$$(1-x^{k+1})(1-x)^{-1}\sum_{i=0}^{l-k}\mu_i + x^{k+1}(1-x)^{-1}\sum_{i=l-k+1}^{l}\mu_i(x^{i-l-1}-1) + O(x^N),$$

where μ_i is the coefficient of order i in the Taylor expansion at 0 of the function $h(z)^{-1} = (1-z)(z-x)/\big(z - \exp(z-1)\big)$.

For simplicity we omit the case $\lambda = 1$.

2. PRELIMINARIES

For a family of complex numbers $[a_1, \ldots, a_s]$, let $\sigma_{i,j}$ denote the j-th elementary symmetric function of the family $[a_1, \ldots, a_s]$ with a_i removed, so that $\prod_{j\neq i}(1-a_j\chi) = \sum_{j=0}^{s-1}\sigma_{i,j}\chi^j$. Furthermore, let $\mu_i = \prod_{j\neq i}(a_i-a_j)^{-1}$; then $\prod_{i=1}^{s}(1-a_i\chi)^{-1} = \sum_{i=1}^{s}\mu_i a_i^{s-1}(1-a_i\chi)^{-1}$.

The goal of the following two lemmas is to provide a list of elementary relations that will be useful for proving Theorems 1.4 and 1.5.

LEMMA 2.1. *Suppose that $a, b \in D, k \in \mathbb{N}, j \in \mathbb{Z}$.*

(i) *We have the equalities*

$$\pi_+\left(\frac{\chi^k}{1-a\bar{\chi}}\right) = \frac{\chi^k - a^{k+1}\bar{\chi}}{1-a\bar{\chi}}, \quad \pi_-\left(\frac{\chi^k}{1-a\bar{\chi}}\right) = a^{k+1}\frac{\bar{\chi}}{1-a\bar{\chi}}, \tag{2.1}$$

$$\pi_+\left(\frac{\bar{\chi}^k}{1-a\chi}\right) = a^k\frac{1}{1-a\chi}. \tag{2.2}$$

(ii) *For a function $\sum_{i=0}^{\infty}\alpha_i z^i$ holomorphic in $D(0,\rho)$, with $\rho > 1$,*

$$\pi_+\left(\frac{\bar{\chi}^k}{1-a\chi}\sum_{i=0}^{\infty}\alpha_i\chi^i\right) = \Big(\sum_{i=0}^{k-1}\alpha_i a^{k-i} + \sum_{j=0}^{\infty}\alpha_{j+k}\,\chi^j\Big)\frac{1}{1-a\chi}. \tag{2.3}$$

Furthermore,

$$\left\langle\frac{\chi^j}{1-a\chi}, \frac{1}{1-b\chi}\right\rangle = \begin{cases}\bar{b}^j/(1-a\bar{b}) & \text{if } j \geqslant 0,\\ a^{-j}/(1-a\bar{b}) & \text{if } j \leqslant 0.\end{cases} \tag{2.4}$$

Finally, for $h \in H^+$

$$\pi_-\left(\frac{h(\chi)}{1-a\bar{\chi}}\right) = ah(a)\frac{\bar{\chi}}{1-a\bar{\chi}}, \quad \pi_+\left(\frac{h(\bar{\chi})}{1-a\chi}\right) = \frac{h(a)}{1-a\chi}. \tag{2.5}$$

P r o o f. These equalities are straightforward. ■

LEMMA 2.2. *Assume that $h \in H^+, b \in D$ and that $a_1, a_2, \ldots, a_s \in D$ are distinct. Define $P(\chi) = \prod_{i=1}^{s}(1-a_i\chi)$. Then*

$$\pi_-\left(\frac{h(\chi)}{P(\bar{\chi})}\right) = \frac{1}{P(\bar{\chi})}\sum_{j=0}^{s-1}\Big(\sum_{i=1}^{s}\mu_i a_i^s h(a_i)\sigma_{i,j}\Big)\bar{\chi}^{j+1}, \tag{2.6}$$

$$\pi_+\left(\frac{h(\bar{\chi})}{P(\chi)}\right) = \frac{1}{P(\chi)}\sum_{j=0}^{s-1}\Big(\sum_{i=1}^{s}\mu_i a_i^{s-1} h(a_i)\sigma_{i,j}\Big)\chi^{j}, \tag{2.7}$$

$$\pi_+\left(\frac{h(\chi)}{P(\bar{\chi})}\right) = \frac{1}{P(\bar{\chi})}\Big(h(\chi) - \sum_{j=0}^{s-1}\Big(\sum_{i=1}^{s}\mu_i a_i^s h(a_i)\sigma_{i,j}\Big)\bar{\chi}^{j+1}\Big), \tag{2.8}$$

$$b^j\sum_{i=1}^{s}\frac{\mu_i a_i^{s-1}}{1-a_i b} = \left\langle\frac{1}{1-b\chi}, \frac{\chi^j}{\prod_{i=1}^{s}(1-\bar{a}_i\chi)}\right\rangle. \tag{2.9}$$

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Proof. We give the proof of the first item only. The other proofs are quite similar. From the decomposition $\prod_{i=1}^{s}(1-a_i\bar{\chi})^{-1} = \sum_{i=1}^{s} \mu_i a_i^{s-1}/(1-a_i\bar{\chi})$ we deduce, for all $k \in \mathbb{N}$,

$$\pi_-\left(\frac{\chi^k}{P(\bar{\chi})}\right) = \sum_{i=1}^{s} \mu_i a_i^{s-1} \pi_-\left(\frac{\chi^k}{1-a_i\bar{\chi}}\right) = \sum_{i=1}^{s} \mu_i a_i^{k+s} \frac{\bar{\chi}}{1-a_i\bar{\chi}}$$

(see Lemma 2.1, equation (2.1)). Hence

$$\pi_-\left(\frac{h(\chi)}{P(\bar{\chi})}\right) = \sum_{i=1}^{s} \mu_i a_i^s h(a_i) \frac{\bar{\chi}}{1-a_i\bar{\chi}} = \frac{\sum_{i=1}^{s} \mu_i a_i^s h(a_i)\bar{\chi}\prod_{j\neq i}(1-a_j\bar{\chi})}{P(\bar{\chi})}$$
$$= \frac{1}{P(\bar{\chi})}\sum_{j=0}^{s-1}\big(\sum_{i=1}^{s} \mu_i a_i^s h(a_i)\sigma_{i,j}\big)\bar{\chi}^{j+1}. \quad \blacksquare$$

The symbols present in Table 1 have zeros on $\mathbb{T}$. In Table 2, where $0 < r < 1$, we replace these symbols by regularised ones. Let us notice that the regularised symbol $f_r = g_{1,r}g_{2,r}$ satisfies the assumptions of Theorem 1.3.

TABLE 2. Regularised factorisation

$g_{1,r}(\chi)$	$g_{2,r}(\chi)$	condition
$(1-r\chi)h(\chi)$	$\prod_{i=1}^{s}(1-x_i\bar{\chi})$	$E(X) > 0$
$(1-r\chi)h(\chi)$	$\prod_{i=1}^{s-1}(1-x_i\bar{\chi})(1-r\bar{\chi})$	$E(X) = 0$
$h(\chi)$	$\prod_{i=1}^{s-1}(1-x_i\bar{\chi})(1-r\bar{\chi})$	$E(X) < 0$

PROPOSITION 2.1. *Let* $f = g_1g_2$ *be the symbol from Table* 1, *and* $f_r = g_{1,r}g_{2,r}$ *the regularised symbol from Table* 2. *Assume that for all k and l fixed in* $\{0, \ldots, N\}$, *and N fixed*, *the limit* $\lim_{r\to 1} T_N(f_r)^{-1}(k,l)$ *exists. Then*

$$\lim_{r\to 1} T_N(f_r)^{-1}(k,l) = T_N(f)^{-1}(k,l). \tag{2.10}$$

Proof. Let us consider the case $E(X) > 0$. Here $g_{2,r}(\chi) = g_2(\chi)$. First, let us prove that $\lim_{r\to 1} T_N(f-f_r)(k,l) = 0$ uniformly with respect to (k,l) and N. Indeed,

$$T_N(f-f_r)(k,l) = \langle \pi_N\big((f-f_r)\chi^k\big), \chi^l\rangle = (r-1)\langle \pi_N(\chi^{k+1}hg_2), \chi^l\rangle$$
$$= (r-1)\langle \chi^{k+1}hg_2, \pi_N(\chi^l)\rangle = (r-1)\langle \chi^{k+1}hg_2, \chi^l\rangle$$

(π_N is self-adjoint). Hence $|T_N(f-f_r)(k,l)| \leqslant (1-r)\|hg_2\|_\infty$. The equality

$$T_N(f_r)^{-1}T_N(f) - I = T_N(f_r)^{-1}T_N(f-f_r)$$

leads to the following:

$$\begin{aligned}\big|\big(T_N(f_r)^{-1}T_N(f)-I\big)(k,l)\big| &\leqslant \sum_{t=0}^{N}|T_N(f_r)^{-1}(k,t)|\,|T_N(f-f_r)(t,l)|\\ &\leqslant (1-r)\|hg_2\|_\infty\sum_{t=0}^{N}|T_N(f_r)^{-1}(t,l)|\end{aligned}$$

and, consequently,

$$\lim_{r\to 1}\big|\big(T_N(f_r)^{-1}T_N(f)-I\big)(k,l)\big| \leqslant \lim_{r\to 1}(1-r)\sum_{t=0}^{N}\lim_{r\to 1}T_N(f_r)(t,l)=0.$$

The two other cases are dealt similarly. ∎

3. PROOF OF THEOREM 1.4 (ASYMPTOTIC 1)

From now on, N,k,l are nonnegative integers and $k\leqslant N, l\leqslant N$.

3.1. Case $E(X)>0$**.** Theorem 1.2 gives the decomposition $f(\theta)=g_1(\chi)g_2(\chi)$ with $g_1(\chi)=(1-\chi)h(\chi)$ and $g_2(\chi)=\prod_{i=1}^{s}(1-x_i\bar{\chi})$. In order to use Theorem 1.3 we introduce, according to Proposition 2.1, the regularised symbol $f_r(\theta)=g_{1,r}(\chi)g_{2,r}(\chi)$ with $g_{1,r}(\chi)=(1-r\chi)h(\chi)$ and $g_{2,r}(\chi)=g_2(\chi)$. We need only to check the existence of $\lim_{r\to 1}\mathfrak{T}_{2,N,r}$, and at the same time the asymptotic behaviour of $\mathfrak{T}_{2,N}$. The formula for $\lim_{r\to 1}\mathfrak{T}_{1,r}$ is quite easily found with similar techniques. Since we concentrate on the meaning of the theorem, we do not present them. $\mathfrak{T}_1(k,l)=\lim_{r\to 1}\mathfrak{T}_{1,r}(k,l)$ is present only in Examples 1.1 and 1.2.

By assumption, $g_{1,r}=(1-r\chi)h(\chi)$, h holomorphic on an open neighbourhood of $\bar{D}$, with $\big(h(z)\big)^{-1}=\sum_{i=0}^{d}b_iz^i$. Note that if $d<l$, we can write $\big(h(\chi)\big)^{-1}=\sum_{i=0}^{l}b_i\chi^i$ with $b_i=0$ if $i>d$. We write also $h^*(z)^{-1}=\sum_{i=0}^{d}\bar{b}_iz^i$ in order to have $\overline{h(\chi)}=h^*(\bar{\chi})$. Moreover, $g_2(\chi)=\prod_{i=1}^{s}(1-x_i\bar{\chi})$ has s simple roots that can be sorted by their modulus, say $0<|x_1|\leqslant\ldots\leqslant|x_{s-1}|\leqslant x_s<1$, where x_s is the real zero of $1-\Phi$ on $(0,1)$ (it was called x in the first point of the assertions in Theorem 1.4).

Evaluation of $\tilde{Y}_{N,l,r}(\chi)$ and $\tilde{X}_{N,k,r}(\chi)$. From equation (2.1) in Lemma 2.1 we obtain

$$\pi_+\left(\frac{\chi^l}{\bar{g_1}}\right)=\frac{\sum_{i=0}^{l}\bar{b}_i\chi^{l-i}-\bar{\chi}\sum_{i=0}^{l}\bar{b}_ir^{l+1-i}}{1-r\bar{\chi}}.$$

Then, by definition (see equation (1.7)),

$$\begin{aligned}\tilde{Y}_{N,l,r}(\chi)&=\pi_+\left(\frac{\bar{\chi}^{N+1}\overline{h(\chi)}\big(\sum_{i=0}^{l}\bar{b}_i\chi^{l-i}-\bar{\chi}\sum_{i=0}^{l}\bar{b}_ir^{l+1-i}\big)}{\prod_{i=1}^{s}(1-\bar{x}_i\chi)}\right)\\ &=\pi_+\left(\frac{G_{N,l,r}(\bar{\chi})}{\prod_{i=1}^{s}(1-\bar{x}_i\chi)}\right),\end{aligned}$$

where $G_{N,l,r}(z) = z^{N+1-l}h^*(z)P_l(z,r)$ and $P_l(z,r)$ is a polynomial in z and r: more precisely, $P_l(z,r) = \sum_{i=0}^{l} \bar{b}_i z^i - z^{l+1} \sum_{i=0}^{l} \bar{b}_i r^{l+r-i}$. Hence, for each i in $\{1,2,\ldots,s\}$, $G_{N,l,r}(\bar{x}_i) = O(x_s^N)$ uniformly in $r \in (0,1)$ as $N \to \infty$. By equation (2.7) of Lemma 2.2,

$$\tilde{Y}_{N,l,r}(\chi) = \frac{1}{\prod_{i=1}^{s}(1-\bar{x}_i\chi)} \sum_{j=0}^{s-1} S_{N,l,j,r}\chi^j,$$

where (with μ_i and $\sigma_{i,j}$ defined for $a_i = x_i$)

$$S_{N,l,j,r} = \sum_{i=1}^{s} \bar{\mu}_i \bar{x}_i^{s-1} G_{N,l,r}(\bar{x}_i)\bar{\sigma}_{i,j} = O(x_s^N)$$

uniformly in $r \in (0,1)$ as $N \to \infty$. Furthermore, the limit of $S_{N,l,j,r}$ as $r \to 1^-$ exists. Using equation (2.8) in Lemma 2.2, we get

$$\pi_+\left(\frac{\chi^k}{g_2}\right) = \left(\chi^k - Q_k(\bar{\chi})\right)\frac{1}{\prod_{i=1}^{s}(1-x_i\bar{\chi})},$$

where $Q_k(z) = \sum_{j=0}^{s-1}\left(\sum_{i=1}^{s}\mu_i\sigma_{i,j}x_i^{k+s}\right)z^{j+1}$ is a polynomial of degree s. Then

$$\tilde{X}_{N,k,r}(\chi) = \pi_+\left(\tilde{\Phi}_N \pi_+\left(\frac{\chi^k}{g_2}\right)\right) = \pi_+\left(\frac{\bar{\chi}^{N+1-k} - \bar{\chi}^{N+1}Q_k(\bar{\chi})}{h(\chi)(1-r\chi)}\right).$$

By assumption, $h(z)^{-1}$ is a polynomial $\sum_{i=0}^{d} b_i z^i$ with no root in the open unit disc. Thus $[\bar{\chi}^{N+1-k} - \bar{\chi}^{N+1}Q_k(\bar{\chi})]/h(\chi)$ is a polynomial $\tilde{G}_N$ with indeterminate $\bar{\chi}$, namely

$$\tilde{G}_N(\bar{\chi}) = \sum_{i=0}^{d} b_i \bar{\chi}^{N+1-k-i} - Q_k(\bar{\chi})\sum_{i=0}^{d} b_i\bar{\chi}^{N+1-i},$$

of degree $N+1+s$.

Then by Lemma 2.1, the second equation of (2.5), we have

$$\tilde{X}_{N,k,r}(\chi) = \frac{\tilde{G}_N(r)}{1-r\chi}. \tag{3.1}$$

Note that $\tilde{G}_N(r) = r^{N+1-k}R_k(r)$, where $R_k(r) = h(1/r)^{-1}\left(1 - r^k Q_k(r)\right)$ and $\tilde{G}_N(r)$ is bounded on the interval $(0,1)$ uniformly with respect to N and has a left limit at 1.

LEMMA 3.1. *It follows that* $\langle \tilde{X}_{N,k,r}(\chi), \tilde{Y}_{N,l,r}(\chi)\rangle = O(x_s^N)$ *uniformly in* $r \in (0,1)$ *as* $N \to \infty$.

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P r o o f. Using (2.9) of Lemma 2.2 and, finally, (2.4) of Lemma 2.1, we have

$$\begin{aligned}
\langle \tilde{X}_{N,k,r}(\chi), \tilde{Y}_{N,l,r}(\chi)\rangle &= \tilde{G}_N(r)\sum_{j=0}^{s-1}\bar{S}_{N,l,j,r}\left\langle \frac{1}{1-r\chi}, \frac{\chi^j}{\prod_{i=1}^{s}(1-\bar{x}_i\chi)}\right\rangle \\
&= \tilde{G}_N(r)\sum_{j=0}^{s-1}\bar{S}_{N,l,j,r}\sum_{i=1}^{s}\mu_i x_i^{s-1}\left\langle \frac{1}{1-r\chi}, \frac{\chi^j}{1-\bar{x}_i\chi}\right\rangle \\
&= r^{N+1-k}R_k(r)\sum_{j=0}^{s-1}\bar{S}_{N,l,j,r}r^j\sum_{i=1}^{s}\mu_i\frac{x_i^{s-1}}{1-x_i r}.
\end{aligned}$$

The last equality leads to the conclusion. ∎

To achieve the proof of the case $E(X) > 0$, we introduce the following lemma.

LEMMA 3.2. *For $N \geqslant d$, the function $\tilde{X}_{k,N,r}(\chi)$ is an eigenvector of the operator $H_{\tilde{\Phi}_N}H_{\Phi_N}$, and the corresponding eigenvalue $\lambda_{N,r}$ satisfies $\lambda_{N,r} = O(x_s^N)$ uniformly in $r \in (0,1)$ as $N \to \infty$. Furthermore, $\lambda_{N,r}$ converges as $r \to 1^-$.*

P r o o f. We have

$$H_{\Phi_N}\left(\frac{1}{1-r\chi}\right) = \pi_-\left(\frac{g_1}{g_2}\chi^{N+1}\frac{1}{1-r\chi}\right) = U_N(\bar{\chi})\frac{1}{\prod_{i=1}^{s}(1-x_i\bar{\chi})},$$

where, by equation (2.6) in Lemma 2.2,

$$U_N(z) = \sum_{j=1}^{s-1}\big(\sum_{i=1}^{s}\mu_i x_i^{s+N+1}h(x_i)\sigma_{i,j}\big)z^{j+1}.$$

Hence, using again the second equation of (2.5) in Lemma 2.1, we obtain at last

$$\begin{aligned}
H_{\tilde{\Phi}_N}H_{\Phi_N}\left(\frac{1}{1-r\chi}\right) &= \pi_+\left(\frac{\bar{\chi}^{N+1}U_N(\bar{\chi})}{(1-r\chi)h(\chi)}\right) \\
&= \pi_+\left(\frac{U_N(\bar{\chi})\sum_{i=0}^{d}b_i\bar{\chi}^{N+1-i}}{1-r\chi}\right) = \frac{r^{N+1}U_N(r)\big(h(1/r)\big)^{-1}}{1-r\chi}.
\end{aligned}$$

Clearly, $\lambda_{N,r} = r^{N+1}U_N(r)\big(h(1/r)\big)^{-1}$. It remains to observe that $U_N(r) = O(x_s^N)$ uniformly in $r \in (0,1)$ as $N \to \infty$. ∎

By Lemmas 3.1 and 3.2, we get $\mathfrak{T}_{2,N,r} = (1-\lambda_{N,r})^{-1}\langle \tilde{X}_{N,k,r}(\chi), \tilde{Y}_{N,l,r}(\chi)\rangle$ $= O(x_s^N)$ uniformly in $r \in (0,1)$ as $N \to \infty$.

This completes the proof of the case $E(X) > 0$.

3.2. Case $E(X) = 0$. By Proposition 2.1, the regularised symbol $f_r(\theta)$ can be written as $g_{1,r}(\chi)g_{2,r}(\chi)$. According to Table 2, $g_{1,r}(\chi) = (1 - r\chi)h(\chi)$ and $g_{2,r}(\chi) = (1 - r\bar{\chi})\prod_{i=1}^{s-1}(1 - x_i\bar{\chi})$. By assumption, $1/h(z) = \sum_{i=0}^{d} a_i z^i$, a polynomial of degree d. Let ε be a real with

$$0 < \varepsilon < 1 - \max_{1 \leqslant i \leqslant s-1} |x_i| \tag{3.2}$$

and consider a regularisation with $r \in (1 - \varepsilon, 1)$ only and put $x_s = r$. So $g_{2,r}(\chi)$ may be written as $\prod_{i=1}^{s}(1 - x_i\bar{\chi})$ as in Subsection 3.1. Evaluating $\tilde{X}_{N,k,r}$, $\tilde{Y}_{N,l,r}$ and their inner product as in equation (1.7), we obtain

$$\langle \tilde{X}_{N,k,r}, \tilde{Y}_{N,l,r} \rangle = r^{N+1-k}\big(h(1/r)\big)^{-1}\big(1 - r^k Q_k(r)\big)\sum_{j=0}^{s-1} S_{N,l,j,r} r^j \sum_{i=1}^{s} \frac{\mu_i x_i^{s-1}}{1 - x_i r},$$

where

$$Q_k(r) = \sum_{j=0}^{s-1}\Big(\sum_{i=1}^{s}\mu_i\sigma_{i,j}x_i^{k+s}\Big)r^{j+1} = \sum_{i=1}^{s}\mu_i x_i^{s-1} r \prod_{j \neq i}(1 - x_i r),$$

$$S_{N,l,j,r} = \sum_{i=1}^{s}\bar{\mu}_i\bar{x}_i^{s-1}G_{N,l,r}(\bar{x}_i)\bar{\sigma}_{i,j},$$

$$G_{N,l,r}(z) = z^{N+1-l}\bar{h}(z)\Big(\sum_{i=0}^{l}\bar{b}_i z^i - z^{l+1}\sum_{i=0}^{l}\bar{b}_i r^{l+1-i}\Big).$$

We have

$$\begin{aligned}
&\frac{1}{1-r}\langle \tilde{X}_{N,k,r}, \tilde{Y}_{N,l,r} \rangle \\
&= \frac{\big(r^{N+1-k} - r^{N+1}Q_k(r)\big)\big(h(1/r)\big)^{-1}}{1-r}\frac{1}{\prod_{i=1}^{s}(1 - x_i r)}\sum_{j=0}^{s-1} S_{N,l,j,r} r^j \\
&= \frac{1}{\prod_{i=1}^{s}(1 - x_i r)}\frac{r^{N+1-k}\big(1 - r^k Q_k(r)\big)\big(h(1/r)\big)^{-1}}{1-r^2}\frac{1}{1-r}\sum_{j=0}^{s-1} S_{N,l,j,r} r^j.
\end{aligned}$$

We claim that the above quantity is bounded uniformly in $r \in (1 - \varepsilon, 1)$ and N and converges as $r \to 1^-$. In fact, we have

$$\begin{aligned}
1 - r^k Q_k(r) &= \Big(\prod_{i=1}^{s}(1 - x_i r)\sum_{i=1}^{s}\mu_i x_i^{s-1}(1 - x_i r)^{-1}\Big) - r^k Q_k(r) \\
&= \sum_{i=1}^{s}\Big(\mu_i x_i^{s-1}(1 - x_i^{k+1}r^{k+1})\prod_{j \neq i}(1 - x_j r)\Big).
\end{aligned}$$

Observe that each summand in the previous sum either contains a factor $1 - x_s r = 1 - r^2$ (when $i < s$) or $1 - x_s^{k+1} r^{k+1} = 1 - r^{2k+2}$ (when $i = s$). Hence this sum is divisible by $1 - r$ with a bounded quotient. It follows that also the quantity

$$(1-r)^{-1} r^{N+1-k} \big(1 - r^k Q_k(r)\big) \big(h(1/r)\big)^{-1}$$

is bounded uniformly in $r \in (1-\varepsilon, 1)$ and N and converges as $r \to 1^-$.

Let us now study $\sum_{j=0}^{s-1} S_{N,l,j,r} r^j$:

$$\begin{aligned}\sum_{j=0}^{s-1} S_{N,l,j,r} r^j &= \sum_{i=1}^{s} \bar{\mu}_i \bar{x}_i^{s-1} G_{N,l,r}(\bar{x}_i) \sum_{j=0}^{s-1} \bar{\sigma}_{i,j} r^j \\ &= \sum_{i=1}^{s} \bar{\mu}_i \bar{x}_i^{s-1} G_{N,l,r}(\bar{x}_i) \prod_{j \neq i} (1 - x_j r).\end{aligned}$$

When $i < s$, the corresponding term on the right-hand side contains again a factor $1 - x_s r = 1 - r^2$. When $i = s$, we have

$$\begin{aligned}G_{N,l,r}(\bar{x}_s) &= r^{N+1-l} \bar{h}(r) \big(\sum_{i=0}^{l} \bar{b}_i r^i - r^{l+1} \sum_{i=0}^{l} \bar{b}_i r^{l+1-i} \big) \\ &= r^{N+1-l} \bar{h}(r) \sum_{i=0}^{l} \bar{b}_i r^i (1 - r^{2l-2i+2}).\end{aligned}$$

The last sum is a polynomial divisible by $1 - r$. It follows that also the quantity $(1-r)^{-1} \sum_{j=0}^{s-1} S_{N,l,j,r} r^j$ is uniformly bounded in $r \in (1-\varepsilon, 1)$ and N and has a limit as $r \to 1^-$. Our claim is proved.

LEMMA 3.3. *For $N \geqslant d$, the function $\tilde{X}_{N,k,r}(\chi)$ is an eigenvector of the operator $H_{\tilde{\Phi}_N} H_{\Phi_N}$ corresponding to an eigenvalue $0 < \lambda_{N,r} < 1$ and $\lambda_{N,r} \to 1$ as $r \to 1^-$. Furthermore, with ε defined as in* (3.2), *we have*

$$\frac{1-r}{1-\lambda_{N,r}} = O\left(\frac{1}{N}\right) \qquad \textit{uniformly in } r \in (1-\varepsilon, 1) \textit{ as } N \to \infty.$$

Proof. Using the proof of Lemma 3.2, we obtain

$$H_{\tilde{\Phi}_N} H_{\Phi_N} (\tilde{X}_{N,k,r}) = \lambda_{n,r} \tilde{X}_{N,k,r},$$

where $\lambda_{N,r} = r^{N+1} U_N(r) h(1/r)^{-1}$ and

$$\begin{aligned}&U_N(r) = \\ &= \sum_{j=0}^{s-1} \big(\sum_{i=1}^{s} \mu_i x_i^{s+N+1} h(x_i) \sigma_{i,j} \big) r^{j+1} = r \sum_{i=1}^{s} \mu_i x_i^{s+N+1} h(x_i) \prod_{j \neq i} (1 - x_j r) \\ &= r \sum_{i=1}^{s-1} \mu_i x_i^{s+N+1} h(x_i) \prod_{j \neq i} (1 - x_i r) + r^{s+N+2} h(r) \prod_{j=1}^{s-1} \frac{1 - x_j r}{r - x_j}.\end{aligned}$$

Therefore, we have $1-\lambda_{N,r} = 1 - r^{2N+3+s}A_N(r) - (1-r^2)B_N(r)$, where

$$A_N(r) = h(r)h(1/r)^{-1}\prod_{j=1}^{s-1}\frac{1-x_jr}{r-x_j}$$

and

$$B_N(r) = r^{N+2}h(1/r)^{-1}\sum_{i=1}^{s-1}\mu_i x_i^{s+N+1}h(x_i)\prod_{j\neq i, j\neq s}(1-x_jr).$$

Both $A_N(r)$ and its derivative $A'_N(r)$ with respect to r are bounded uniformly in $r\in(1-\varepsilon,1)$ and $A(r)$ tends to $A(1)=1$ as $r\to 1^-$. Also $B_N(r)$ is bounded uniformly in $r\in(1-\varepsilon,1)$ and $N\geqslant d$, and $B_N(r)$ has a limit as $r\to 1^-$. Therefore, for some $\xi_r\in(1-\varepsilon,1)$,

$$\begin{aligned}\frac{1-\lambda_{N,r}}{1-r} &= \frac{1-r^{2N+3+s}A_N(r)}{1-r} - (1+r)B_N(r)\\ &= (2N+3+s)\xi_r^{2N+3+s}A_N(\xi_r) + \xi_r^{2N+3+s}A'_N(\xi_r) - (1+r)B_N(r),\end{aligned}$$

and the lemma follows. ■

The proof of the case $E(X)=0$ is a straightforward application of the earlier results and of the formula

$$\mathfrak{T}_{2,N,r} = \frac{1-r}{1-\lambda_{N,r}}\frac{1}{1-r}\langle \tilde{X}_{N,k,r}, \tilde{Y}_{N,l,r}\rangle.$$

3.3. Case $E(X)<0$. According to Proposition 2.1, Table 2 provides the expression of the regularised symbol f_r as the product $f_r(\theta) = g_{1,r}(\chi)g_{2,r}(\chi)$, where $g_{1,r} = h$ and $g_{2,r}(\chi) = (1-r\chi)\prod_{i=1}^{s-1}(1-x_i\bar{\chi})$. By assumption, with the notation of Theorem 1.4, $g_1(\chi) = h(\chi) = (1-\alpha\chi)\tilde{h}(\chi)$, where $1/\tilde{h}(z)$ is a polynomial. Let ε be a real defined as in (3.2) and consider a regularisation with $r\in(1-\varepsilon,1)$ only and put $x_s = r$. So, as in the previous subsection, $g_{2,r}(\chi)$ will be $\prod_{i=1}^{s}(1-x_i\bar{\chi})$. Repeating the argument of Subsection 3.1, we obtain the same expressions for $\tilde{X}_{N,k,r}$ and $\tilde{Y}_{N,l,r}$ as in the other two cases, but with r and h replaced by α and $\tilde{h}$, respectively. Now $\tilde{X}_{N,k,r}$ and $\tilde{Y}_{N,l,r}$ depend on r through x_s, μ_i and $\sigma_{i,j}$ only. Therefore, we have (see the proof of Lemma 3.1):

$$\langle \tilde{X}_{N,k,r}(\chi), \tilde{Y}_{N,l,r}(\chi)\rangle = \alpha^{N+1-k}R_k(\alpha)\big(\sum_{j=0}^{s-1}S_{N,l,j,r}\alpha^j\big)\left(\sum_{i=1}^{s}\mu_i\frac{x_i^{s-1}}{1-x_i\alpha}\right),$$

where $R_k(\alpha) = h(1/\alpha)^{-1}\big(1-\alpha^kQ_k(\alpha)\big)$. Then the quantity $\langle \tilde{X}_{N,k,r}(\chi), \tilde{Y}_{N,l,r}(\chi)\rangle$ is $O(|\alpha|^N)$ uniformly in $r\in(1-\varepsilon,1)$. As in Lemma 3.2, $\tilde{X}_{N,k,r}$ is an eigenfunction of $H_{\tilde{\Phi}_N} H_{\Phi_N}$ with eigenvalue $\lambda_{N,\alpha} < 1$. Note that $\lambda_{N,\alpha}$ depends on r through x_s; however, we have

$$\lambda_{N,\alpha} = \frac{\alpha^{N+1}}{\tilde{h}(1/\alpha)} \sum_{j=0}^{s-1} \Big(\sum_{i=1}^{s} \mu_i x_i^{s+N+1} \tilde{h}(x_i) \sigma_{i,j} \Big) \alpha^{j+1}$$

which is $O(|\alpha|^N)$ uniformly in $r \in (1-\varepsilon, 1)$ as $N \to \infty$ and has a limit as $r \to 1^-$. Hence, $(1 - \lambda_{\alpha,N})^{-1}$ is bounded in $r \in (1 - \varepsilon, 1)$ for N large enough, and so $\mathfrak{T}_{2,N,r}(k,l)$ is $O(|\alpha|^N)$ uniformly in $r \in (1 - \varepsilon, 1)$, and $\lim_{r \to 1^-} \mathfrak{T}_{2,N,r}(k,l)$ exists.

4. PROOF OF THEOREM 1.5 (ASYMPTOTIC 2)

This proof, as well as that of Theorem 1.4, is based on Lemmas 2.1 and 2.2. The calculations used are quite similar to the previous ones. Therefore, we detail only the succession of the steps involved in the proof. When $E(X) > 0$, Theorem 1.2 allows us to write $f(\theta) = g_1(\chi) g_2(\chi)$ with $g_1(\chi) = (1 - \chi) h(\chi)$ and $g_2(\chi) = 1 - x\bar{\chi}$, and when $E(X) = 0$, it provides $f(\theta) = g_1(\chi) g_2(\chi)$ with $g_1(\chi) = (1 - \chi) h(\chi)$ and $g_2(\chi) = 1 - \bar{\chi}$. In accordance with Table 2 we consider the regularisation

$$g_{1,r}(\chi) = (1 - r\chi) h(\chi), \quad g_{2,r}(\chi) = 1 - x\bar{\chi} \quad \text{if } E(X) > 0 \tag{4.1}$$

and

$$g_{1,r}(\chi) = (1 - r\chi) h(\chi), \quad g_{2,r}(\chi) = 1 - r\bar{\chi} \quad \text{if } E(X) = 0. \tag{4.2}$$

Note that $g_{2,r}$ depends on r only in the case $E(X) = 0$. The similarity of the two cases $E(X) > 0$ and $E(X) = 0$ allows us to write $g_{2,r}$ and $g_{1,r}$ as in equation (4.1) and to consider that $x = r$ when $E(X) = 0$. According to Theorem 1.2, h is holomorphic in an open disc $D(0, \rho)$ with $\rho > 1$ and does not vanish in $\overline{D}$. Hence, decreasing ρ if necessary, we may assume that h has no zero in $D(0, \rho)$. Let us put for $|z| < \rho$

$$h(z) = \sum_{j=0}^{\infty} a_j z^j \quad \text{and} \quad \frac{1}{h(z)} = \sum_{j=0}^{\infty} b_j z^j. \tag{4.3}$$

This notation is kept in all this section. By Cauchy's inequality ([3], p. 81) there exists a constant M such that for all $j \in \mathbb{N}$ the inequalities $\rho^j |a_j| < M$ and $\rho^j |b_j| < M$ hold true.

Computation of the functions $\tilde{X}_{N,k,r}(\chi)$ and $\tilde{Y}_{N,l,r}(\chi)$.

LEMMA 4.1. *With the notation as above, the following assertions hold*:

(i) *There exists a holomorphic function* $V_{N,k,r}$ *on the open disc* $D(0, \rho)$ *whose restriction to the torus* $\mathbb{T}$ *has a norm in* $L^\infty(\mathbb{T})$ *satisfying* $\|V_{N,k,r}\|_\infty = O(1/\rho^N)$

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uniformly in $r \in (0,1)$ *such that*

$$\tilde{X}_{N,k,r}(\chi) = \frac{\sum_{j=0}^{N-k} b_j (r^{N+1-(j+k)} - x^{1+k} r^{N+2-j}) + V_{N,k,r}(\chi)}{1 - r\chi}.$$

(ii) *There exists a polynomial* $P_l(x,r)$ *independent of* N *such that*

$$\tilde{Y}_{l,N,r}(\chi) = x^{N+1} \bar{h}(x) P_l(x,r) \frac{1}{1 - x\chi}.$$

From now on, we put

$$\beta_{N,k,r}(z) = \sum_{j=0}^{N-k} b_j (r^{N+1-(j+k)} - x^{1+k} r^{N+2-j}) + V_{N,k,r}(z).$$

For a function $\delta_{N,r}(z) = \sum_{j=0}^{N+1} b_j r^{N+2-j} + \sum_{j=N+2}^{\infty} b_j z^{N+2-j}$, holomorphic on $D(0,\rho)$, we have

$$H_{\tilde{\Phi}_N} H_{\Phi_N} \left(\frac{\beta(\chi)}{1 - r\chi} \right) = x^{N+2} h(x) \beta(x) \frac{\delta_{N,r}(\chi)}{1 - r\chi}$$

for all functions β holomorphic in $D(0,\rho)$. Hence, for all positive integers p,

$$\begin{aligned}(H_{\tilde{\Phi}_N} H_{\Phi_N})^p \big(\tilde{X}_{N,k}(\chi) \big) \\ = x^{N+2} h(x) \beta_{N,k,r}(x) \big(x^{N+2} h(x) \delta_{N,r}(x) \big)^{p-1} \frac{\delta_{N,r}(\chi)}{1 - r\chi}.\end{aligned}$$

We conclude that

$$\begin{aligned}\mathfrak{T}_{2,N,r}(k,l) \\ = \langle \tilde{X}_{N,k}(\chi), \tilde{Y}_{N,l}(\chi) \rangle + \frac{x^{N+2} h(x) \beta_{N,k,r}(x)}{1 - x^{N+2} h(x) \delta_{N,r}(x)} \left\langle \frac{\delta_{N,r}(\chi)}{1 - r\chi}, \tilde{Y}_{N,l}(\chi) \right\rangle.\end{aligned} \tag{4.4}$$

Consider first the case $E(X) > 0$. Note that, by Lemma 2.1 and summability of b_j, $\langle (1 - r\chi)^{-1} \delta_{N,r}, \tilde{Y}_{N,l,r} \rangle$ is bounded uniformly in $r \in (0,1)$ and N and converges as $N \to \infty$. It follows that $\mathfrak{T}_{2,N,r}(k,l) = O(x^N)$ as $N \to \infty$ uniformly in $r \in (0,1)$, and taking the limit as $r \to 1^-$ proves the theorem.

When $E(X) = 0$, we have the equation (4.4) with $x = r$ and $P_l(r,r)$ is divisible by $1 - r$. So the quantities

$$(1 - r)^{-1} \langle \tilde{X}_{N,k}(\chi), \tilde{Y}_{N,l}(\chi) \rangle \quad \text{and} \quad (1 - r)^{-1} \left\langle \frac{\delta_{N,r}(\chi)}{1 - r\chi}, \tilde{Y}_{N,l}(\chi) \right\rangle$$

are bounded in r and N. Furthermore, $\delta_{N,r}(r)$ tends to $1/h(1)$ as $r \to 1^-$, so that $(1 - r)^{-1} \big(1 - r^{N+2} h(r) \delta_{N,r}(r) \big)$ is bounded by $O(1/N)$ uniformly in r as $N \to \infty$ as in the previous section.

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LRI, Université Paris-Sud
91405 Orsay cedex France
E-mail: Charles.Delorme@lri.fr

Département de Mathématiques
Université Paris-Sud
91405 Orsay cedex France
E-mail: jean-marc.rinkel@math.u-psud.fr

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