

UNBIASED ESTIMATES FOR LINEAR REGRESSION WITH ROUND-OFF ERROR

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Abstract. We consider the linear regression model, where the residuals have zero mean and an otherwise unspecified distribution F . Suppose that least squares estimates are formed by using *rounded* values of the dependent variables. We show that these are still unbiased, and that unbiased estimates for the moments and cumulants of F are given by applying Sheppard's corrections to their estimates.

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1. INTRODUCTION

The linear regression model is one of the most popular models in statistics. It is also one of the simplest models in statistics. It has received applications in almost every area of science, engineering and medicine.

The data on the dependent variables are often *rounded off*. There may be many reasons that the data are rounded off, for example: recording standard, faulty equipments, human error or the lack of technology to measure variables accurately.

Many authors have considered the important problem: the estimates of linear regression with the rounded values used instead of the actual ones. For most excellent reviews of the literature, see Heitjan [5] and Schneeweiss et al. [8]. However, much of the work has focused on correcting for bias of the estimates under suitable conditions.

In this short note, we show that the estimates of linear regression can, in fact, be unbiased if the rounded values are used instead of the actual ones; see Theorem 2.1 in Section 2. We also give unbiased estimates for the r th moment and the r th cumulant of the residuals of linear regression for any r ; see Theorem 2.2 in Section 2. The related proofs are given in Section 4. We assume throughout that Sheppard's corrections hold exactly.

We believe it is the first time that a general result like Theorem 2.2 has been proven. This result could have potentially wide spread applications. The first four moments and the first two cumulants have obvious physical interpretations and use in statistical inference. There are often situations, where one also needs higher moments and higher cumulants. For example, there are many situations in insurance and economics that require moments of orders higher than four. We mention:

- Taleb [9] suggests using moments of order higher than four to measure the risk of an option. For example, the fifth moment is presumed as the asymmetry sensitivity of the fourth one. The seventh moment is suggested as the sign of the convexity change as the underlying asset moves up or down.

- Avramidis and Matzinger [1] show that an estimator for pricing American options can be improved using moments of order higher than four. An example in the area of Cornish–Fisher expansions (Cornish and Fisher [2]; Fisher and Cornish [3]) is described in Section 3.

2. MAIN RESULTS

Suppose we observe $Y_N = \mathbf{x}'_N \boldsymbol{\beta} + e_N$, $1 \leq N \leq n$, where \mathbf{x}_N and $\boldsymbol{\beta}$ are known and unknown q -vectors, respectively, and $\{e_N\}$ are independently and identically distributed according to F , an unknown distribution with zero mean and unknown central moments and cumulants $\{\mu_r, \kappa_r, r \geq 2\}$.

Suppose we record not $\mathbf{Y}' = (Y_1, \dots, Y_n)$ but $\mathbf{Z}' = (Z_1, \dots, Z_n)$, where $Z_N = T_h(Y_N)$, say, is Y_N rounded to the nearest integral multiple of a given $h < 0$, with mid-values rounded down (or F continuous). Let $\hat{\boldsymbol{\beta}}_h$ denote the least squares estimate about $\{Z_N\}$:

$$\hat{\boldsymbol{\beta}}_h = (\mathbf{X}' \mathbf{X})^{-1} \mathbf{X}' \mathbf{Z}, \quad \text{where } \mathbf{X}' = (x_1, \dots, x_n).$$

THEOREM 2.1. *We have $\mathbb{E}[\hat{\boldsymbol{\beta}}_h] = \boldsymbol{\beta}$.*

In Withers and Nadarajah [12], unbiased estimates (UEs) were given for $\{\mu_r, \kappa_r, r \geq 2\}$, say $\hat{\mu}_r = \hat{\mu}_r(Y)$, $\hat{\kappa}_r = \hat{\kappa}_r(Y)$. In particular,

$$\hat{\mu}_2 = |\mathbf{Y} - \mathbf{X} \hat{\boldsymbol{\beta}}|^2 / (n - q),$$

as is well known.

THEOREM 2.2. *UEs of $\{\mu_r, \kappa_r, r \geq 2\}$ are given by applying Sheppard's corrections to $\{\hat{\mu}_{r,h}, \hat{\kappa}_{r,h}, r \geq 2\}$, where $\hat{\mu}_{r,h} = \hat{\mu}_r(\mathbf{Z})$, $\hat{\kappa}_{r,h} = \hat{\kappa}_r(\mathbf{Z})$. That is, UEs are given by*

$$\begin{aligned} \tilde{\mu}_2 &= \hat{\mu}_{2,h} - H/3, \\ \tilde{\mu}_3 &= \hat{\mu}_{3,h}, \\ \tilde{\mu}_4 &= \hat{\mu}_{4,h} - 2H\hat{\mu}_{2,h} + 7H^2/5, \\ \tilde{\mu}_5 &= \hat{\mu}_{5,h} - 10H\hat{\mu}_{3,h}/3, \end{aligned}$$

where $H = h^2/4$. In general, for $r \geq 2$

$$\begin{aligned} \tilde{\mu}_r &= \sum_{0 \leq m \leq r/2} b_m H^m \binom{r}{2m} \hat{\mu}_{r-2m,h}, \\ \tilde{\kappa}_r &= \hat{\kappa}_{r,h} - h^r B_r/r, \end{aligned}$$

where

$$\begin{aligned} \hat{\mu}_{r,h} &= \sum_{0 \leq m \leq r/2} (2m+1)^{-1} H^m \binom{r}{2m} \tilde{\mu}_r, \\ b_m &= (2 - 2^{2-2m}) B_{2m}, \end{aligned}$$

and B_r is the r th Bernoulli number.

The conditions on F in these theorems are as for Sheppard's corrections with $r = 1$ for Theorem 2.1. Basically, this requires sufficient regularity of F in its tails.

3. EXAMPLE

Here, we describe an example, where the results of Theorem 2.2 can be applied.

Suppose we wish to make inferences about a parameter θ . Let $\hat{\theta}$ denote an estimator of θ . Suppose the cumulants of $\hat{\theta}$ satisfy the standard expansion

$$(3.1) \quad \kappa_r(\hat{\theta}) = \sum_{j=r-1}^{\infty} a_{r,j} n^{-j}$$

for $r \geq 1$ with the coefficients $\{a_{r,j}\}$ bounded as $n \rightarrow \infty$, $a_{2,1}$ bounded away from zero. This assumption is key to obtaining Cornish–Fisher expansions for $\hat{\theta}$ and is known to be true for a wide variety of estimates, including smooth functions of sample means.

By (3.1), the mean and variance of $\hat{\theta}$ are asymptotic to $a_{1,0}$ and $a_{2,1}/n$, respectively. So, a central limit approximation is to assume that

$$(3.2) \quad Y_n = (n/a_{2,1})^{1/2} (\hat{\theta} - a_{1,0})$$

has the standard normal distribution for n sufficiently large. But this approximation can be crude.

It would be better to use corrections for the central limit version. Withers [10] (see also Kolassa and McCullagh [7]) showed that under the assumption (3.1) the asymptotic expansions of Cornish and Fisher [2] and Fisher and Cornish [3] for

the distribution and quantiles of $\hat{\theta}$ reduce to the form

$$(3.3) \quad P_n(x) = P(Y_n \leq x) = \Phi(x) - \phi(x) \sum_{r=1}^{\infty} n^{-r/2} h_{0,r}(x),$$

$$(3.4) \quad \Phi^{-1}(P_n(x)) = x - \sum_{r=1}^{\infty} n^{-r/2} f_r(x),$$

$$(3.5) \quad P_n^{-1}(\Phi(x)) = x + \sum_{r=1}^{\infty} n^{-r/2} g_r(x),$$

where $\Phi(x)$ and $\phi(x)$ are the distribution and density, respectively, of a standard normal random variable. The functions $h_{0,r}(x)$, $f_r(x)$, $g_r(x)$ are certain polynomials of degree $3r - 1$, $r + 1$, $r + 1$ in x , respectively, and $A_{r,i} = a_{r,i}/a_{2,1}^{r/2}$.

The infinite sums on the right-hand sides of (3.3)–(3.5) can be truncated to provide better inferences for θ than the central limit version given by (3.2). The polynomials $h_{0,r}(x)$, $f_r(x)$, $g_r(x)$ involve the r th order cumulants of $\hat{\theta}$. Consequently, higher order inferences for θ would require knowing estimates for higher order cumulants of $\hat{\theta}$.

Withers and Nadarajah [11] have demonstrated how (3.1) and (3.3)–(3.5) can be applied to construct improved confidence intervals for parameters of interest when the data are rounded off.

4. PROOFS

Proof of Theorem 2.1. Sheppard showed (see Kendall and Stuart [6], p. 76) that if $X \sim F$ is rounded to produce $X_h = T_h \sim F_h$, say, then their noncentral moments and their cumulants are connected by

$$(4.1) \quad \mu'_{r,h} = \mu'_r(F_h) = \sum_{0 \leq m} (2m+1)^{-1} H^m \binom{r}{2m} \mu'_r,$$

$$(4.2) \quad \mu'_r = \mu'_r(F) = \sum_{0 \leq m} b_m H^m \binom{r}{2m} \mu'_{r,h}$$

and, for $r \neq 1$,

$$(4.3) \quad \kappa_r = \kappa_{r,h} - h^r B_r / r.$$

In particular,

$$(4.4) \quad \mathbb{E}(X_h) = \mathbb{E}(X).$$

So, $\mathbb{E}(Z) = \mathbb{E}(Y)$ and Theorem 2.1 holds. ■

To prove Theorem 2.2 we need the following lemma.

LEMMA 4.1. For any constant c , $T_h(X - c)$ and $T_h(X) - c$ have the same moments.

Proof. By (4.4), we have $\mathbb{E}[T_h(X - c)] = \mathbb{E}(X) - c = \mathbb{E}[T_h(X)] - c$. Let $G(y) = P(X - c \leq y) = F(y + c)$ and G_h be the distribution of $T_h(X - c)$. Note that (4.1) and (4.2) hold for central moments, i.e. with the dashes dropped. So, for $r \geq 2$,

$$\begin{aligned} \mu_r(G_h) &= \sum (2m + 1)^{-1} H^m \binom{r}{2m} \mu_r(G) \\ &= \sum (2m + 1)^{-1} H^m \binom{r}{2m} \mu_r(F) \\ &= \mu_r(F_h) = \mu_r[T_h(X) - c], \end{aligned}$$

where F, F_h are the distributions of $X, T_h(X)$, respectively, and

$$\mu_r[X] = \mu_r(F). \quad \blacksquare$$

Proof of Theorem 2.2. Set $e_{h,N} = Z_N - \mathbf{x}'_N \boldsymbol{\beta}$. Then, by equations (1.2), (2.2), (3.1) of Withers and Nadarajah [12], $\hat{\mu}_{r,h}$ and $\hat{\kappa}_{r,h}$ each have the form

$$\sum R_{i_1, \dots, i_r}(X) e_{h,i_1} \dots e_{h,i_r}.$$

Note that $\mathbb{E}[e_{h,i}] = \mathbb{E}[e_i] = 0$ by (4.4), so

$$\mathbb{E}[e_{h,i_1} \dots e_{h,i_r}] = \prod_j \mathbb{E}[e_{h,L_j}^{a_j}],$$

where $\{L_j\}$ are the distinct values of $\{i_j\}$ and a_j is the number of i 's equal to L_j . By Theorem 2.1, $\mathbb{E}[e_{h,L}^a] = \mu_a(F_h)$, where F_h is the distribution of $T_h(e_N)$, $N \geq 1$. So,

$$\mathbb{E}[e_{h,i_1} \dots e_{h,i_r}] = \prod_j \mu_{a_j}(F_h).$$

But in the case of no round-off,

$$\mathbb{E}[e_{i_1} \dots e_{i_r}] = \prod_j \mu_{a_j}(F)$$

and $\mu_r(F) = \mathbb{E}[\hat{\mu}_r] = \mu_r(F)$, $\mathbb{E}[\hat{\kappa}_r] = \mu_r(F)$. So, $\mathbb{E}[\hat{\mu}_{r,h}] = \mu_r(F_h)$ and $\mathbb{E}[\hat{\kappa}_{r,h}] = \mu_r(F_h)$. Thus, the result for moments follows by the fact that Sheppard's corrections apply to central moments, and the result for cumulants is implied by (4.3). \blacksquare

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