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CHAIN DEPENDENT CONTINUOUS TIME RANDOM WALK

BY

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Abstract. An asymptotic behavior of a continuous time random walk is investigated in the case when the sequence of pairs of jump vectors and times between jumps is chain dependent.

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1. INTRODUCTION

The aim of the paper is to investigate an asymptotic behavior of a continuous time random walk (CTRW), i.e. a random walk in which both spatial vectors representing jumps and moments of jumps are random. Such a process is defined by a sequence of jump vectors and a sequence of random waiting times separating successive jumps. We consider a situation when pairs of jump vectors and waiting times between jumps form a chain dependent sequence.

Let $\{(Y_k, J_k), k \geqslant 1\}$ be a sequence of random vectors in $\mathbb{R}^d \times \mathbb{R}_+$, defined on a common probability space, where J_k, $k \geqslant 1$, are positive random variables. For each k the spatial vector Y_k represents the k-th jump of a particle and J_k is the k-th waiting time for the next jump to occur. A CTRW process given by $\{(Y_k, J_k), k \geqslant 1\}$ is defined as follows:

$$X(t) = \sum_{k=1}^{N(t)} Y_k, \quad t \geqslant 0,$$

where $N(t) = \max\left\{k : \sum_{j=1}^{k} J_j \leqslant t\right\}$ is the number of jumps of the particle up to time t. With such a definition, $X(t)$ represents the position of a particle at time t which is also the position of a particle just after the last jump before time t. Closely related to the process X is a process $\tilde{X}$ defined as $\tilde{X}(t) = \sum_{k=1}^{N(t)+1} Y_k$, which can be interpreted as the position of a particle just after the first jump after time t. Both X and $\tilde{X}$ describe simple diffusion mechanisms. The process X is a location of the diffusing particle which first waits for a jump by the random time J_k and then jumps according to the spatial vector Y_k, while $\tilde{X}$ corresponds to the situation in which the particle first jumps along Y_k and then remains in the reached position for the random time J_k. These diffusion models have numerous applications in physics, financial mathematics and many other fields. For an overview of applications see [10], [8], [4] and [6].

It is convenient to consider an asymptotic behavior of CTRW in terms of a weak convergence of CTRW sequences. Let $\{(Y_{n,k}, J_{n,k}), n, k \geqslant 1\}$ be an array of random vectors in $\mathbb{R}^d \times \mathbb{R}_+$, $d \geqslant 1$, defined on a common probability space where $J_{n,k}$ are positive random variables. Define sequences of processes

$$S_n(t) = \sum_{k=1}^{[nt]} Y_{n,k},\ T_n(t) = \sum_{j=k}^{[nt]} J_{n,k} \text{ for } t > 0, \quad S_n(0) = 0,\ T_n(0) = 0,$$

a sequence of renewal processes

$$N_n(t) = \max\big\{k \geqslant 1 : \sum_{j=1}^{k} J_{n,j} \leqslant t\big\}$$

and sequences of CTRWs

$$X_n(t) = \sum_{k=1}^{N_n(t)} Y_{n,k} = S_n\big(N_n(t)/n\big)$$

and

$$\tilde{X}_n(t) = \sum_{k=1}^{N_n(t)+1} Y_{n,k} = S_n\big(N_n(t)/n + 1/n\big).$$

If $J_{n,k}$ have finite means, then $X_n(t)$ and $\tilde{X}_n(t)$ behave like $S_n(t)$ as $n \to \infty$, so we consider only situation when $J_{n,k}$ have infinite means. The aim of the paper is to find conditions for weak convergence of $\{X_n(t)\}$ and $\{\tilde{X}_n(t)\}$ and the form of distributions of their limits $M(t)$ and $\tilde{M}(t)$, respectively, in the case of chain dependence of array $\{(Y_{n,k}, J_{n,k})\}$.

The problem of convergence was considered extensively by Meerschaert and Scheffler in [10] (see also [1] and [9]) in the case when the pairs $(Y_{n,k}, J_{n,k})$, $k \geqslant 1$, are iid in the rows. They have shown that if processes (S_n, T_n), $n \geqslant 1$, converge weakly to a Lévy process (A, D), in J_1 topology, where A and D have no simultaneous jumps, then $X_n \Rightarrow A(D^{-1})$ in J_1 topology. Moreover, for Lévy measure ν^D of D such that $\nu^D(0,\infty) = \infty$ and $\int_0^1 u|\ln u|\nu^D(du) < \infty$, Meerschaert and Scheffler have found the form of the distribution of $A\big(D^{-1}(t)\big)$. In [7] Meerschaert et al. obtained similar results dealing with an asymptotic of X generated by a sequence $\{(Y_k, J_k)\}$ when $\{J_k\}$ is an iid sequence of random variables independent of $\{Y_k\}$ being a stationary time series of type *moving average*: $MA(\infty)$. In [14] Tejedor and Metzler investigated an asymptotic of CTRW generated by a sequence $\{(Y_k, J_k)\}$ with structure similar to the chain-dependence one, considered in this paper. The recent papers [5] and [13] reveal that distributions of $M(t)$ and $\tilde{M}(t)$ may be different. The work [5] was indicated to us by the Referee.

The structure of the paper is the following. In Section 2 the notion of chain dependence is introduced and some auxiliary results related to it are given. They are used to get the main result of the paper, stated in Theorem 3.1 in Section 3. It gives conditions for the weak convergences $X_n(t) \Rightarrow M(t)$ and $\tilde{X}_n(t) \Rightarrow \tilde{M}(t)$ when array $\{(Y_{n,k}, J_{n,k})\}$ is chain dependent and it also states the formulas for the distributions of $M(t)$ and $\tilde{M}(t)$. To get those distributions we do not need the assumption $\int_0^1 u|\ln u|\nu^D(du) < \infty$. In Sections 4 and 5 we discuss conditions under which the distributions of $M(t)$ and $\tilde{M}(t)$ are equal or different along with some examples.

2. PRELIMINARIES

2.1. Structure of chain dependence. In the paper we use the following notation: $\mathbb{R} = (-\infty, \infty)$ denotes the real line, $\mathbb{R}_+ = [0, \infty)$, $\mathbb{R}^m$ the Cartesian product of m copies of $\mathbb{R}$ and $\mathcal{B}(\mathbb{R}^m)$ stands for the Borel σ-field in $\mathbb{R}^m$. Furthermore, we use the notation $\Rightarrow$ for weak convergence of distributions and for convergence in distribution of random elements.

Let $\{L_k, k \geqslant 0\} \equiv \{L_k\}$ be a homogeneous Markov chain on a probability space $(\Omega, \mathcal{F}, P)$ with a countable state space $L \subset \{1, 2, \ldots\}$ and with transition matrix $\mathbb{P} = (p_{i,j}, i, j \in L)$. An array $\{\zeta_{n,k}, k, n \geqslant 1\}$ of random vectors in $\mathbb{R}^m$ on probability space $(\Omega, \mathcal{F}, P)$ is called *chain dependent with respect to Markov chain* $\{L_k\}$ if for all $j \in L, k \geqslant 1$ and all Borel sets B in $\mathbb{R}^m$ we have

$$(2.1)\quad P(\zeta_{n,k} \in B, L_k = j|\mathcal{J}_{n,k-1}) \overset{P.1}{=} P(\zeta_{n,k} \in B, L_k = j|L_{k-1}) \\ = P(\zeta_{n,k} \in B|L_{k-1}) p_{L_{k-1},j} = \mu_{n,L_{k-1}}(B) p_{L_{k-1},j},$$

where $\mathcal{J}_{n,k} = \sigma(\zeta_{n,i}, L_i, i \leqslant k)$ and $\mu_{n,j}(B) = P(\zeta_{n,k} \in B|L_{k-1} = j)$.

Hence we also have

$$(2.2)\qquad P(\zeta_{n,k} \in B|\mathcal{J}_{n,k-1}) \overset{P.1}{=} P(\zeta_{n,k} \in B|L_{k-1}) = \mu_{n,L_{k-1}}(B).$$

Furthermore, for any Borel sets $B_i \in \mathcal{B}(\mathbb{R}^m),\ 1 \leqslant i \leqslant k$, we have

$$(2.3)\qquad P\Big(\bigcap_{i=1}^{k} \{\zeta_{n,i} \in B_i\}|L_0, L_1, \ldots, L_{k-1}\Big) \overset{P.1}{=} \prod_{i=1}^{k} P(\zeta_{n,i} \in B_i|L_{i-1}).$$

The notion of chain dependence and limit theorems for chain dependence were considered by many authors; see, for example, [11] and [3].

Now we give a construction of a chain dependent array $\{\eta_{n,k}, n, k \geqslant 1\} \equiv \{\eta_{n,k}\}$ such that for each $n \geqslant 1$ the process $\{\eta_{n,k}, k \geqslant 1\}$ has the same finite-dimensional distributions as the process $\{\zeta_{n,k}, k \geqslant 1\}$. Let $\{\tilde{L}_k, k \geqslant 0\}$ be a homogeneous Markov chain on a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ with the same transition probability matrix $\mathbb{P} = (p_{i,j}, i, j \in L)$ as for the Markov chain $\{L_k\}$. Let

$\{\xi_{n,k,i}, n, k, i \geqslant 1\} \equiv \{\xi_{n,k,i}\}$ be an array of mutually independent random vectors in $\mathbb{R}^m$ on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ such that $\xi_{n,k,i}, i \geqslant 1$, have distribution $\mu_{n,k}$ and let this array be independent of a Markov chain $\{\tilde{L}_k\}$. Define an array $\{\eta_{n,k}, n, k \geqslant 1\}$ as follows:

$$\eta_{n,k} = \xi_{n,\tilde{L}_{k-1},k}, \quad k, n \geqslant 1. \tag{2.4}$$

PROPOSITION 2.1. (i) *The array* $\{\eta_{n,k}, n, k \geqslant 1\}$ *is chain dependent with respect to* $\{\tilde{L}_k\}$.

(ii) *If* $P(\zeta_{n,k} \in B | L_{k-1} = j) = \mu_{n,j}(B)$, *then for all* $n \geqslant 1$ *the processes* $\{\zeta_{n,k}, k \geqslant 1\}$ *and* $\{\eta_{n,k}, k \geqslant 1\}$ *have the same finite-dimensional distributions.*

The array $\{\eta_{n,k}\}$ defined in (2.4) is called the *canonical representation of chain dependent array* $\{\zeta_{n,k}\}$.

Proof. Let $\tilde{\mathcal{J}}_{n,k} = \sigma(\tilde{L}_0, \tilde{L}_i, \eta_{n,i}, i \leqslant k)$. Then for any $B \in \mathcal{B}(\mathbb{R}^m)$ and any $j \in L$ we have

$$\begin{aligned}
&\tilde{P}(\eta_{n,k} \in B, \tilde{L}_k = j | \tilde{\mathcal{J}}_{n,k-1}) = \tilde{P}(\xi_{n,\tilde{L}_{k-1},k} \in B, \tilde{L}_k = j | \tilde{\mathcal{J}}_{n,k-1}) \\
&\quad \overset{P.1}{=} \tilde{P}(\xi_{n,\tilde{L}_{k-1},k} \in B, \tilde{L}_k = j | \tilde{L}_{k-1}) = \tilde{P}(\xi_{n,\tilde{L}_{k-1},k} \in B | \tilde{L}_{k-1}) p_{\tilde{L}_{k-1},j} \\
&\quad = \mu_{n,\tilde{L}_{k-1}}(B) p_{\tilde{L}_{k-1},j}.
\end{aligned}$$

This proves the assertion (i). To prove (ii) let us notice that for any Borel sets $B_i \in \mathcal{B}(\mathbb{R}^m)$ and all $j_0, j_1, \ldots, j_{k-1}$ from the state space L we have

$$\begin{aligned}
&\tilde{P}\big(\bigcap_{i=1}^{k} \{\eta_{n,i} \in B_i\} | \tilde{L}_0 = j_0, \tilde{L}_1 = j_1, \ldots, \tilde{L}_{k-1} = j_{k-1}\big) \\
&\qquad = \prod_{i=1}^{k} \tilde{P}(\eta_{n,i} \in B_i | \tilde{L}_{i-1} = j_{i-1}) = \prod_{i=1}^{k} \mu_{n,j_{i-1}}(B_i) \\
&\qquad = \prod_{i=1}^{k} P(\zeta_{n,i} \in B_i | L_{i-1} = j_{i-1}).
\end{aligned}$$

Hence we get (ii), which completes the proof of the proposition. ■

2.2. Convergence to Lévy process under chain dependence. Let $Z = \{Z(t), t \geqslant 0\}$ be an m-dimensional Lévy process. Then its characteristic function has the form $E \exp\{i(\theta, Z(t))\} = \exp(t\psi(\theta))$ with $\theta \in \mathbb{R}^m$ and

$$\psi(\theta) = i\langle a, \theta\rangle + \frac{1}{2}\langle \theta, \mathbb{Q}\theta\rangle + \int_{x \neq 0} \big(e^{i\langle \theta, x\rangle} - 1 - i\langle \theta, x\rangle \mathbf{1}(\|x\| < \gamma)\big)\nu(dx),$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^m$, $\|x\| = \sqrt{\langle x, x\rangle}$, $a \in \mathbb{R}^m$, $\mathbb{Q}$ is a nonnegative definite matrix of size $m \times m$, ν is a Lévy measure on $\mathbb{R}_0^m = \mathbb{R}^m \setminus 0$

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and γ is a constant such that $\nu(\partial\{x \in \mathbb{R}_0^m : \|x\| < \gamma\}) = 0$, where $\partial(A)$ stands for the edge of the set A. Vector a is called a *drift parameter of process Z* and matrix $\mathbb{Q}$ is said to be a *Gaussian component of Lévy process*. Lévy process is characterized by the triple $(a, \mathbb{Q}, \nu)$.

In the sequel we give conditions for convergence in distribution of processes $\sum_{k=1}^{[nt]} \zeta_{n,k}$ to a Lévy process when $\{\zeta_{n,k}, k, n \geqslant 1\}$ is a chain dependent array. In view of Proposition 2.1 we can assume, without loss of generality, that

$$\zeta_{n,k} = \xi_{n,L_{k-1},k}, \quad k \geqslant 1, n \geqslant 1, \tag{2.5}$$

where, for simplicity of the notation, we put L_k instead of $\tilde{L}_k$ and $\{\xi_{n,k,i}, n, k, i \geqslant 1\}$ is an array of mutually independent random vectors such that for each $n, k \geqslant 1$ the random vectors $\xi_{n,k,1}, \xi_{n,k,2}, \ldots$ have a common distribution, say $\mu_{n,k}$, and the array $\{\xi_{n,k,i}, n, k, i \geqslant 1\}$ is independent of $\{L_k\}$. Define the processes

$$S_n(t) \equiv \sum_{k=1}^{[nt]} \zeta_{n,k}, \quad t \geqslant 0, n \geqslant 1,$$

$$S_{n,j}(t) = \sum_{i=1}^{[nt]} \xi_{n,j,i}, \quad t \geqslant 0, n \geqslant 1, j \in L.$$

Notice that, for each $n \geqslant 1$, the processes $S_{n,j},\ j \in L$, are mutually independent. Let $C_j(t)$ be a number of visits to state j of the process $\{L_k\}$ up to time t.

PROPOSITION 2.2. *Let $\{L_k\}$ be an irreducible, homogeneous, aperiodic, positive recurrent Markov chain with state space L and with stationary distribution $\pi = \{\pi_j, j \in L\}$. Let $\{\zeta_{n,k}, k, n \geqslant 1\}$ be chain dependent with respect to $\{L_k\}$ and with representation* (2.5) *and let for each $j \in L$*

$$S_{n,j} \Rightarrow Z_j \quad \text{in } D\big([0,\infty), \mathbb{R}^m\big) \text{ with } J_1 \text{ topology}, \tag{2.6}$$

where $Z_j, j \in L$, are Lévy processes with triples $(a_j, \mathbb{Q}_j, \nu_j),\ j \in L$, respectively. Furthermore, assume that for any $t \geqslant 0$ and for any $\varepsilon > 0,\ \delta > 0$ there exist $j_0 \geqslant 1$ and $n_0 \in \mathbb{N}$ such that

$$P\Big(\sum_{j \in L, j \geqslant j_0} \sup_{0 \leqslant s \leqslant t} \Big\| \sum_{i=1}^{C_j(ns)} \xi_{n,j,i} \Big\| > \delta \Big) \leqslant \varepsilon \tag{2.7}$$

for all $n \geqslant n_0$ and $j \geqslant j_0$. Then

$$S_n \Rightarrow Z \quad \text{in } D\big([0,\infty), \mathbb{R}^m\big) \text{ with } J_1 \text{ topology}, \tag{2.8}$$

where Z is a Lévy process with triple $\big(\sum_{j\in L} \pi_j a_j, \sum_{j \in L} \pi_j \mathbb{Q}_j, \sum_{j\in L} \pi_j \nu_j\big)$.

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Proof. Since processes $S_{n,j}, j \in L$, are mutually independent, we can assume, without loss of generality, that processes $Z_j, j \in L$, are also independent, which jointly with assumption (2.6) gives the following convergence:

$$(S_{n,j}, j \in L) \Rightarrow (Z_j, j \in L) \quad \text{in } D\big([0,\infty),\mathbb{R}^m\big)^\infty. \tag{2.9}$$

Notice that

$$\begin{aligned} S_n(t) &\equiv \sum_{k=1}^{[nt]} \zeta_{n,k} = \sum_{k=1}^{[nt]} \zeta_{n,k} \sum_{j\in L} \mathbf{1}(L_{k-1}=j) \\ &= \sum_{j\in L}\sum_{k=1}^{[nt]} \zeta_{n,k}\mathbf{1}(L_{k-1}=j) \stackrel{D}{=} \sum_{j\in L}\sum_{i=1}^{C_j([nt])} \xi_{n,j,i}, \end{aligned} \tag{2.10}$$

where $\mathbf{1}(A)$ is the indicator of the set A. Moreover, for any $t_1 < t_2 < \ldots < t_r$ we get

$$\begin{aligned} &\big(S_n(t_1), S_n(t_2), \ldots, S_n(t_r)\big) \\ &\quad \stackrel{D}{=} \big(\sum_{j\in L}\sum_{i=1}^{C_j([nt_1])} \xi_{n,j,i}, \sum_{j\in L}\sum_{i=1}^{C_j([nt_2])} \xi_{n,j,i}, \ldots, \sum_{j\in L}\sum_{i=1}^{C_j([nt_r])} \xi_{n,j,i}\big). \end{aligned} \tag{2.11}$$

For simplicity, we show that (2.11) holds true only for $r=2$. In this case we have

$$\begin{aligned} &P\big(S_n(t_1)\in B_1, S_n(t_2)\in B_2 | \bigcap_{i=0}^{[nt_2]} L_i = j_i\big) \\ &= P\big(\sum_{j\in L}\sum_{k=1}^{[nt_1]} \zeta_{n,k}\mathbf{1}(L_{k-1}=j)\in B_1, \sum_{j\in L}\sum_{k=1}^{[nt_2]} \zeta_{n,k}\mathbf{1}(L_{k-1}=j)\in B_2 | \bigcap_{i=0}^{[nt_2]} L_i = j_i\big) \\ &= P\big(\sum_{j\in L}\sum_{i=1}^{C_j([nt_1])} \xi_{n,j,i}\in B_1, \sum_{j\in L}\sum_{i=1}^{C_j([nt_2])} \xi_{n,j,i}\in B_2 | \bigcap_{i=0}^{[nt_2]} L_i = j_i\big). \end{aligned}$$

Now summing the above over all $j_i \in L$ we get (2.11) for $r=2$.

Hence, to show the convergence (2.8) it is enough to prove $\tilde{S}_n \Rightarrow Z$ for processes $\tilde{S}_n$ defined as

$$\tilde{S}_n(t) = \sum_{j\in L}\sum_{i=1}^{C_j([nt])} \xi_{n,j,i}. \tag{2.12}$$

By the ergodic theorem for Markov chains we have the following convergences:

$$\frac{C_j([nt])}{n} \to \pi_j t \quad \text{a.e. as } n\to\infty,\ j\in L. \tag{2.13}$$

Hence, by (2.9) and by using the method of the random change of time, we infer that

$$\Big(\sum_{i=1}^{C_j([n\cdot])} \xi_{n,j,i}, j \in L\Big) \Rightarrow \big(Z_j(\pi_j \cdot), j \in L\big) \quad \text{in } D\big([0,\infty), \mathbb{R}^m\big)^{\infty}.$$

Since processes $Z_j(\pi_j \cdot), j \in L$, are mutually independent, by Theorem 4.1 in [15] we get

$$(2.14) \qquad \sum_{j \leqslant j_0} \sum_{i=1}^{C_j([n\cdot])} \xi_{n,j,i} \Rightarrow \sum_{j \leqslant j_0} Z_j(\pi_j \cdot) \quad \text{in } D\big([0,\infty), \mathbb{R}^m\big).$$

Now, using the assumption (2.7) and $P\big(\sum_{j\in L, j\geqslant j_0} \sup_{0\leqslant s\leqslant t} \|Z_j(\pi_j s)\| > \delta\big) \leqslant \varepsilon$ for sufficiently large j_0, we obtain the convergence

$$(2.15) \quad \tilde{S}_n(\cdot) = \sum_{j\in L} \sum_{i=1}^{C_j([n\cdot])} \xi_{n,j,i} \Rightarrow \sum_{j\in L} Z_j(\pi_j \cdot) = Z(\cdot) \quad \text{in } D\big([0,\infty), \mathbb{R}^m\big),$$

which completes the proof of the assertion. ■

REMARK 2.1. *If the state space L of Markov chain $\{L_k\}$ is finite, then condition* (2.7) *holds.*

Later on we use the following result.

PROPOSITION 2.3. *Under the conditions of Proposition* 2.2 *we have the convergences*

$$\Big(\sum_{i=1}^{[ns]} \zeta_{n,i} | L_{[ns]} = j\Big) \Rightarrow Z(s) \quad \textit{for all } s \geqslant 0 \textit{ and } j \in L,$$

where Z is a Lévy process as in Proposition 2.2.

P r o o f. Notice that

$$\begin{aligned}
&P\Big(\sum_{i=1}^{[ns]} \zeta_{n,i} \in B | L_{[ns]} = j\Big) \\
&= P\Big(\sum_{l\in L} \sum_{i=1}^{C_l([ns]-1)} \xi_{n,l,i} + \xi_{n,j,[ns]} \in B | L_{[ns]} = j\Big) \\
&= \int_{\mathbb{R}^m} P\Big(\sum_{l\in L} \sum_{i=1}^{C_l([ns]-1)} \xi_{n,l,i} \in B - x | L_{[ns]} = j\Big) P(\xi_{n,j,[ns]} \in dx | L_{[ns]} = j) \\
&= \int_{\mathbb{R}^m} P\Big(\sum_{i=1}^{[ns]-1} \zeta_{n,i} \in B - x | L_{[ns]} = j\Big) \mu_{n,j}(dx).
\end{aligned}$$

Since

$$\frac{C_l([ns])}{n} \to \pi_l s \text{ a.e.},$$

we infer that $C_l([ns])$ are asymptotically independent of the process $\{L_k\}$. Therefore, as in the proof of Proposition 2.2 we get

$$\Big(\sum_{l\in L}\sum_{i=1}^{C_l([ns]-1)} \xi_{n,l,i} | L_{[ns]} = j\Big) \Rightarrow \sum_{l\in L} Z_l(\pi_l s) \stackrel{D}{=} Z(s).$$

This and the convergence $\mu_{n,j} \Rightarrow \delta_0$ for all $j \in L$, where δ_0 is the probability measure at $0 \in \mathbb{R}^m$, give the assertion. ■

3. CONVERGENCE OF SEQUENCES OF CHAIN DEPENDENT CTRWs

Let us recall that processes S_n, T_n, N_n, X_n and $\tilde{X}_n$ are defined, as in Section 1, by an array $\{\zeta_{n,k} = (Y_{n,k}, J_{n,k}), k, n \geqslant 1\} \equiv \{\zeta_{n,k}\}$ of random vectors in $\mathbb{R}^d \times \mathbb{R}_+$. Let us introduce the notation (C1)–(C6) for conditions on the array $\{\zeta_{n,k} = (Y_{n,k}, J_{n,k})\}$ and on the process (A, D).

(C1) Array $\{\zeta_{n,k} = (Y_{n,k}, J_{n,k})\}$, where $Y_{n,k}$ are random vectors in $\mathbb{R}^d$ and $J_{n,k} > 0$ with $EJ_{n,k} = \infty$, is chain dependent with respect to an irreducible, aperiodic, positive recurrent homogeneous Markov chain $\{L_k, k \geqslant 0\} \equiv \{L_k\}$ with state space $L \subset \{1, 2, \dots\}$ and stationary distribution $\pi = \{\pi_j,\ j \in L\}$.

In view of Proposition 2.1 we may assume, without loss of generality, that

$$\zeta_{n,k} = \xi_{n,L_{k-1},k}, \quad k, n \geqslant 1, \tag{3.1}$$

where $\{\xi_{n,k,i}\} \equiv \{\xi_{n,k,i} = (Y_{n,k,i}, J_{n,k,i}), n, k, i \geqslant 1\}$ is an array of mutually independent random vectors in $\mathbb{R}^d \times \mathbb{R}_+$, independent of Markov chain $\{L_k\}$ and such that random vectors $\xi_{n,k,i} = (Y_{n,k,i}, J_{n,k,i}), i \geqslant 1$, have common distribution $\mu_{n,k}$. Let

$$S_{n,j}(t) = \sum_{i=1}^{[nt]} Y_{n,j,i}, \quad T_{n,j}(t) = \sum_{i=1}^{[nt]} J_{n,j,i}, \quad t \geqslant 0, n \geqslant 1, j \in L.$$

For each $n \geqslant 1$ the processes $(S_{n,j}, T_{n,j}), j \geqslant 1$, are mutually independent.

(C2) For all $j \in L$ the following convergences hold:

$$nP\big((Y_{n,j,1}, J_{n,j,1}) \in B\big) \to \nu_j(B)$$

for all $B \in \mathcal{B}(\mathbb{R}^d \times \mathbb{R}_+)$ such that $0 \notin B$ and $\nu_j(\partial B) = 0$, where ν_j are Lévy measures on $\mathbb{R}^d \times \mathbb{R}_+ \setminus \{0\}$.

(C3) Process (A, D) is such that the process A has a Gaussian component or its Lévy measure $\nu^A = \sum_{j\in L} \pi_j \nu_j^A$ is such that $\nu^A(\mathbb{R}^d) = \infty$, where $\nu_j^A(\cdot) = \nu_j(\cdot \times \mathbb{R}_+)$, and the process D has Lévy measure $\nu^D = \sum_{j\in L} \pi_j \nu_j^D$ such that $\nu^D(0, \infty) = \infty$ and $\int_0^\infty (1 \wedge x)\nu^D(dx) < \infty$, where $\nu_j^D(\cdot) = \nu_j(\mathbb{R}^d \times \cdot)$.

(C4) For any $\delta > 0$,

$$\sup_{j\in L}\sup_{n\geqslant 1} nP(J_{n,j,1} > \delta) \equiv c(\delta) < \infty.$$

(C5) For each $t \geqslant 0$ and $j \in L$ the following convergences hold:

$$P\big(S_n(t) \in B_1, T_n(t) \in B_2 | L_{[nt]} = j\big) \to P\big(A(t) \in B_1, D(t) \in B_2\big)$$

for all sets B_1 and B_2 being continuity sets of distributions of $A(t)$ and $D(t)$, respectively, where (A, D) is a Lévy process with triplet

$$\left((a,0), \begin{bmatrix} \mathbb{Q} & 0 \\ 0 & 0 \end{bmatrix}, \nu^{(A,D)}\right),$$

and $\nu^{(A,D)} = \sum_{j\in L} \pi_j \nu_j$, $a \in \mathbb{R}^d$, $\mathbb{Q}$ is a nonnegative definite $d \times d$ matrix.

(C6) For all $j \in L$ the following convergences hold:

$$(S_{n,j}, T_{n,j}) \Rightarrow (A_j, D_j) \quad \text{in } D\big([0,\infty), \mathbb{R}^d \times \mathbb{R}_+\big) \text{ with } J_1 \text{ topology},$$

where $(A_j, D_j), j \in L$, are Lévy processes with Lévy triplets

$$\left((a_j,0), \begin{bmatrix} \mathbb{Q}_j & 0 \\ 0 & 0 \end{bmatrix}, \nu_j\right),$$

respectively, $a_j \in \mathbb{R}^d$, $\mathbb{Q}_j$ are nonnegative definite $d \times d$ matrices and ν_j are Lévy measures on $\mathbb{R}^d \times \mathbb{R}_+$.

REMARK 3.1. *If an array $\{(Y_{n,k}, J_{n,k})\}$ satisfies the conditions* (C1) *and* (C6), *then it satisfies* (C2). *Moreover, if it satisfies* (2.7) *with $\xi_{n,j,i} = (Y_{n,j,i}, J_{n,j,i})$, then by Proposition* 2.3 *it satisfies* (C5) *and, additionally, $(S_n, T_n) \Rightarrow (A, D)$ in the space $D\big([0,\infty), \mathbb{R}^d \times \mathbb{R}_+\big)$ with J_1 topology and with Lévy triplets*

$$\left(\sum_{j\in L} \pi_j(a_j,0), \begin{bmatrix} \sum_{j\in L} \pi_j \mathbb{Q}_j & 0 \\ 0 & 0 \end{bmatrix}, \sum_{j\in L} \pi_j \nu_j\right).$$

THEOREM 3.1. *Let the array $\{(Y_{n,k}, J_{n,k})\}$ satisfy the conditions* (C1)–(C5) *and let, for fixed $t > 0$, $P\big(\sum_{k=1}^{N_n(t)} J_{n,k} = t\big) = 0$ for sufficiently large n. Then the convergences $X_n(t) \Rightarrow M(t)$ and $\tilde{X}_n(t) \Rightarrow \tilde{M}(t)$ hold and for any set $B \in \mathcal{B}(\mathbb{R}^d)$ we have*

$$P\big(M(t) \in B\big) = \int_0^\infty \int_0^t \nu^D(t-u, \infty) P\big(A(s) \in B, D(s) \in du\big) ds, \tag{3.2}$$

$$P\big(\tilde{M}(t) \in B\big) = \int_0^\infty \int_0^t \int_{\mathbb{R}^d} \nu^{(A,D)}\big((B-v) \times (t-u, \infty)\big) P\big(A(s) \in dv, D(s) \in du\big) ds. \tag{3.3}$$

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P r o o f. First we prove the convergence $X_n(t) \Rightarrow M(t)$. Let us observe that $\nu^D(0,\infty) = \infty$ and, by Theorem 3.1 in [10], it follows that the integral

$$\int_0^\infty \int_0^t \nu^D(t-u,\infty) P\big(D(s) \in du\big) ds$$

is finite. Therefore for any $\varepsilon > 0$ there exists a positive integer $b_1(\varepsilon) \equiv b_1$ such that

$$\int_{b_1}^\infty \int_0^t \nu^D(t-u,\infty) P\big(A(s) \in B, D(s) \in du\big) ds \leqslant \varepsilon.$$

Notice that the weak convergence of $\{T_n\}$ yields the weak convergence of the sequence $\{N_n(t)/n\}$, so the sequence $\{N_n(t)/n\}$ is tight. Therefore, for $\varepsilon > 0$ let b be a positive integer such that $b > b_1$ and $P\big(N_n(t)/n > b\big) < \varepsilon$ for all $n \geqslant 1$ and let $R_n = \{r = k/n : k \leqslant bn\}$.

By the weak convergence $T_n \Rightarrow D$ and by Theorem 1 in [13] it follows that $\sum_{k=1}^{N_n(t)} J_{n,k} \Rightarrow (D \circ D^{-1})^{-1}(t) \equiv Z(t)$. Thus the sequence $\big\{\sum_{k=1}^{N_n(t)} J_{n,k}, n \geqslant 1\big\}$ is tight. The inequality $\sum_{k=1}^{N_n(t)} J_{n,k} \leqslant t$ and the assumption $P\big(\sum_{k=1}^{N_n(t)} J_{n,k} = t\big) = 0$ allow us to restrict the state space of $\sum_{k=1}^{N_n(t)} J_{n,k}$ to the interval $[0,t)$. Therefore, by the tightness of $\big\{\sum_{k=1}^{N_n(t)} J_{n,k}, n \geqslant 1\big\}$ we infer that for ε there exists $\delta > 0$ such that $P\big(\sum_{k=1}^{N_n(t)} J_{n,k} \leqslant t-\delta\big) \geqslant 1-\varepsilon$ for all $n \geqslant 1$. Hence for any fixed set $B \in \mathcal{B}(\mathbb{R}^d)$ we have

$$\begin{aligned}
&P\big(X_n(t) \in B\big) = P\Big(X_n(t) \in B, \sum_{k=1}^{N_n(t)} J_{n,k} \leqslant t\Big) \\
&= P\Big(S_n\big(N_n(t)/n\big) \in B, \big\{T_n\big(N_n(t)/n\big) > t-\delta\big\} \cup \{N_n(t)/n > b\}\Big) \\
&\quad + P\Big(S_n\big(N_n(t)/n\big) \in B, T_n\big(N_n(t)/n\big) \leqslant t-\delta, N_n(t)/n \leqslant b\Big) \\
&\leqslant 2\varepsilon + \sum_{r \in R_n} P\Big(S_n\big(N_n(t)/n\big) \in B,\ T_n\big(N_n(t)/n\big) \leqslant t-\delta,\ N_n(t) = nr\Big) \\
&= 2\varepsilon + \sum_{r \in R_n} P\big(S_n(r) \in B,\ T_n(r) \leqslant t-\delta,\ N_n(t) = nr\big),
\end{aligned}$$

which implies the inequality

$$\Big|P\big(X_n(t) \in B\big) - \sum_{r \in R_n} P\big(S_n(r) \in B,\ T_n(r) \leqslant t-\delta,\ N_n(t) = nr\big)\Big| \leqslant 2\varepsilon. \tag{3.4}$$

But

$$(3.5)\quad P\big(S_n(r)\in B,\ T_n(r)\leqslant t-\delta,\ N_n(t)=nr\big)$$
$$= P\big(S_n(r)\in B,\ T_n(r)\leqslant t-\delta,\ N_n(t)\geqslant nr\big)$$
$$- P\big(S_n(r)\in B,\ T_n(r)\leqslant t-\delta,\ N_n(t)\geqslant nr+1\big)$$
$$= P\big(S_n(r)\in B,\ T_n(r)\leqslant t-\delta,\ T_n(r)\leqslant t\big)$$
$$- P\big(S_n(r)\in B,\ T_n(r)\leqslant t-\delta,\ T_n(r)+J_{n,L_{nr},nr+1}\leqslant t\big).$$

Since $\{\zeta_{n,k}\}$ is chain dependent with respect to the Markov chain $\{L_k\}$, we have

$$P\big(S_n(r)\in B,\ T_n(r)\leqslant t-\delta,\ T_n(r)+J_{n,L_{nr},nr+1}\leqslant t\big)$$
$$=\sum_{j\in L} P\big(S_n(r)\in B,\ T_n(r)\leqslant t-\delta,\ T_n(r)+J_{n,j,nr+1}\leqslant t|L_{nr}=j\big)P(L_{nr}=j)$$
$$=\sum_{j\in L}\int_0^{t-\delta} P(J_{n,j,nr+1}\leqslant t-u|L_{nr}=j)$$
$$\times P\big(S_n(r)\in B, T_n(r)\in du|L_{nr}=j\big)P(L_{nr}=j)$$
$$=\sum_{j\in L}\int_0^{t-\delta} P(J_{n,j,1}\leqslant t-u)P\big(S_n(r)\in B, T_n(r)\in du|L_{nr}=j\big)P(L_{nr}=j).$$

Notice also that

$$P\big(S_n(r)\in B, T_n(r)\leqslant t-\delta\big)$$
$$=\sum_{j\in L}\int_0^{t-\delta} P\big(S_n(r)\in B, T_n(r)\in du|L_{nr}=j\big)P(L_{nr}=j).$$

Therefore, by (3.5) and the above equality, we have

$$(3.6)\quad P\big(S_n(r)\in B,\ T_n(r)\leqslant t-\delta,\ N_n(t)=nr\big)=\frac{1}{n}\sum_{j\in L} H_{n,j}(B,r),$$

where for any $s\geqslant 0$

$$H_{n,j}(B,s)\stackrel{\text{df}}{=}$$
$$\int_0^{t-\delta} n\big(1-P(J_{n,j,1}\leqslant t-u)\big)P\big(S_n(s)\in B, T_n(s)\in du|L_{[ns]}=j\big)P(L_{[ns]}=j).$$

Since sample paths of S_n and T_n are step functions, for any nonnegative integer k we have

$$\int_{k/n}^{(k+1)/n} P\big(S_n(s)\in B, T_n(s)\in du|L_{[ns]}=j\big)ds$$
$$=\frac{1}{n}P\big(S_n(k/n)\in B, T_n(k/n)\in du|L_k=j\big).$$

Hence by (3.6) we get

$$(3.7)\quad \sum_{r\in R_n} P\big(S_n(r)\in B, T_n(r)\leqslant t-\delta, N_n(t)=nr\big)=\int_0^b \sum_{j\in L} H_{n,j}(B,s)ds.$$

Notice that

$$H_{n,j}(B,s)=\int_0^{t-\delta} nP(J_{n,j,1}>t-u)P(L_{[ns]}=j)\kappa_{n,j}(du,s),$$

where

$$\kappa_{n,j}(G,s)\overset{\mathrm{df}}{=}P\big(S_n(s)\in B, T_n(s)\in G|L_{[ns]}=j\big),\qquad G\in\mathcal{B}(\mathbb{R}_+).$$

Condition (C3) and Theorem 27.4 in [12], p. 175, imply that $A(s)$ and $D(s)$ have continuous distributions for all $s>0$. Hence and by condition (C2) we infer that for all Borel sets $B\in\mathcal{B}(\mathbb{R}^d)$ and $G\in\mathcal{B}(\mathbb{R}_+)$ and all $j\in L$ the following convergences hold:

$$(3.8)\quad \kappa_{n,j}(G,s)\to P\big(A(s)\in B, D(s)\in G\big)\equiv\kappa(G,s)\quad \text{as } n\to\infty.$$

For any fixed $s>0$ define probability measures $\tilde{\kappa}$ and $\tilde{\kappa}_{n,j}$ by $\tilde{\kappa}(G,s)=\kappa(G,s)/\kappa(I_\delta,s)$ and $\tilde{\kappa}_{n,j}(G,s)=\kappa_{n,j}(G,s)/\kappa_{n,j}(I_\delta,s)$, where $I_\delta=[0,t-\delta]$ and $G\in\mathcal{B}(I_\delta)$. Let η and $\eta_{n,j}, n\geqslant 1, j\in L$, be nonnegative random variables with distributions $\tilde{\kappa}$ and $\tilde{\kappa}_{n,j}$, respectively. Since for all $j\in L$, $\tilde{\kappa}_{n,j}\Rightarrow\tilde{\kappa}$ as $n\to\infty$, we have $\eta_{n,j}\Rightarrow\eta$ as $n\to\infty$ for all $j\in L$. On the interval I_δ define functions

$$h_{n,j}(u)=nP(J_{n,j,1}>t-u)\quad\text{and}\quad h_j(u)=\nu_j^D(t-u,\infty).$$

Then

$$\kappa_{n,j}(I_\delta,s)Eh_{n,j}(\eta_{n,j})=\int_0^{t-\delta} nP(J_{n,j,1}>t-u)\kappa_{n,j}(du,s)$$

and

$$H_{n,j}(B,s)=P(L_{[ns]}=j)\kappa_{n,j}(I_\delta,s)Eh_{n,j}(\eta_{n,j}).$$

Moreover,

$$\kappa(I_\delta,s)Eh_j(\eta)=\int_0^{t-\delta}\nu_j^D(t-u,\infty)\kappa(du,s).$$

Define the sets

$$F_j\overset{\mathrm{df}}{=}\{x\in I_\delta:\ \text{there exist } x_n\to x \text{ such that } h_{n,j}(x_n)\nrightarrow h_j(x)\}.$$

All functions $h_{n,j}$ are nondecreasing and $h_{n,j}(u)\to h_j(u)$ as $n\to\infty$ for all $j\in L$ and all points $u\in[0,t-\delta]$ such that $\nu_j^D(\{t-u\})=0$. Therefore, F_j contains only the points u for which points $t-u$ are the atoms of measure ν_j^D. Note that the distributions of $A(s)$ and $D(s)$ do not have atoms, so neither has $\tilde{\kappa}$. Thus $\tilde{\kappa}(F_j) = 0$. From Theorem 5.5 in [2] we get the convergences $h_{n,j}(\eta_{n,j}) \Rightarrow h_j(\eta)$ as $n \to \infty$ for all $j \in L$. Condition (C4) implies the uniform integrability of the sequences $\{h_{n,j}(\eta_{n,j}), n \geqslant 1\}$. Then, by Theorem 5.4 in [2], we get the convergence

$$Eh_{n,j}(\eta_{n,j}) \to Eh_j(\eta) \quad \text{as } n \to \infty \text{ for all } j \in L. \tag{3.9}$$

Thus, (3.9) together with the convergence $P(L_{[ns]} = j) \to \pi_j$ as $n \to \infty$ imply

$$\begin{aligned} H_{n,j}(B,s) &\to \pi_j \kappa(I_\delta, s) Eh_j(\eta) \\ &= \pi_j \int_0^{t-\delta} \nu_j^D(t-u,\infty) P\big(A(s) \in B, D(s) \in du\big) \equiv H_j(B,s). \end{aligned} \tag{3.10}$$

Now notice that

$$\begin{aligned} \sum_{j\in L} &|H_{n,j}(B,s) - H_j(B,s)| \\ &\leqslant \sum_{j\in L} c_{n,j} |P(L_{[ns]} = j) - \pi_j| + \sum_{j\in L} \pi_j |c_{n,j} - c_j|, \end{aligned} \tag{3.11}$$

where

$$c_{n,j} = \kappa_{n,j}(I_\delta, s) Eh_{n,j}(\eta_{n,j}), \quad c_j = \kappa(I_\delta, s) Eh_j(\eta).$$

By (3.8) and (3.9) we get $c_{n,j} \to c_j$ and also $\sup_{n,j} c_{n,j} < \infty$. Hence and by the convergence $\sum_{j\in L} |P(L_{[ns]} = j) - \pi_j| \to 0$ as $n \to \infty$, we can take limit in n under the sum in (3.11). Consequently, we obtain

$$\sum_{j\in L} H_{n,j}(B,s) \to \sum_{j\in L} H_j(B,s)$$

which, in turn, implies the convergence

$$\begin{aligned} \int_0^b \sum_{j\in L} H_{n,j}(B,s)\,ds &\to \int_0^b \sum_{j\in L} H_j(B,s)\,ds \\ &= \int_0^b \sum_{j\in L} \pi_j \int_0^{t-\delta} \nu_j^D(t-u,\infty) P\big(A(s) \in B, D(s) \in du\big)\,ds \\ &= \int_0^b \int_0^{t-\delta} \nu^D(t-u,\infty) P\big(A(s) \in B, D(s) \in du\big)\,ds. \end{aligned}$$

Hence and by (3.4) we get

$$\Big| P\big(X_n(t) \in B\big) - \int_0^\infty \int_0^t \nu^D(t-u,\infty) P\big(A(s) \in B, D(s) \in du\big)\,ds \Big| \leqslant 3\varepsilon.$$

Since ε was arbitrary small, we get $X_n(t) \Rightarrow M(t)$ and equality (3.2).

The proof of the convergence $\tilde{X}_n(t) \Rightarrow \tilde{M}(t)$ and formula (3.3) proceeds in a similar way to the proof of $X_n(t) \Rightarrow M(t)$. Here we point out only its main steps. We use the same constants ε, b, δ and the set R_n. Notice that for any Borel set B in $\mathcal{B}(\mathbb{R}^d)$ we have

$$(3.12)\quad P\big(\tilde{X}_n(t) \in B\big) = P\Big(\tilde{X}_n(t) \in B,\ \sum_{k=1}^{N_n(t)} J_{n,k} \leqslant t\Big)$$

$$= P\Big(S_n\big(N_n(t)/n + 1/n\big) \in B, \Big\{\sum_{k=1}^{N_n(t)} J_{n,k} > t-\delta \text{ or } N_n(t)/n > b\Big\}\Big)$$

$$+ P\Big(S_n\big(N_n(t)/n + 1/n\big) \in B,\ \sum_{k=1}^{N_n(t)} J_{n,k} \leqslant t-\delta,\ N_n(t) \leqslant nb\Big)$$

$$\leqslant 2\varepsilon + \sum_{r\in R_n} P\big(S_n(r') \in B,\ T_n(r) \leqslant t-\delta,\ N_n(t) = nr\big), \quad \text{where } r' = r + 1/n.$$

But

$$(3.13)\quad P\big(S_n(r') \in B,\ T_n(r) \leqslant t-\delta,\ N_n(t) = nr\big)$$
$$= P\big(S_n(r') \in B,\ T_n(r) \leqslant t-\delta,\ N_n(t) \geqslant nr\big)$$
$$- P\big(S_n(r') \in B,\ T_n(r) \leqslant t-\delta,\ N_n(t) \geqslant nr+1\big)$$
$$= P\big(S_n(r') \in B,\ T_n(r) \leqslant t-\delta,\ T_n(r) \leqslant t\big)$$
$$- P\big(S_n(r') \in B,\ T_n(r) \leqslant t-\delta,\ T_n(r) + J_{n,L_{nr},nr+1} \leqslant t\big).$$

Since $\{\zeta_{n,k}\}$ is chain dependent with respect to the Markov chain $\{L_k\}$, we have

$$P\big(S_n(r') \in B,\ T_n(r) \leqslant t-\delta,\ T_n(r) + J_{n,L_{nr},nr+1} \leqslant t\big)$$

$$= \sum_{j\in L} P\big(S_n(r) + Y_{n,j,nr+1} \in B,\ T_n(r) \leqslant t-\delta,\ T_n(r) + J_{n,j,nr+1} \leqslant t | L_{nr} = j\big)$$
$$\times P(L_{nr} = j)$$

$$= \sum_{j\in L} \int_0^{t-\delta} \int_{\mathbb{R}^m} P(Y_{n,j,nr+1} \in B - v, J_{n,j,nr+1} \leqslant t-u, | L_{nr} = j)$$
$$\times P\big(S_n(r) \in dv, T_n(r) \in du | L_{nr} = j\big) P(L_{nr} = j)$$

$$= \sum_{j\in L} \int_0^{t-\delta} \int_{\mathbb{R}^m} P(Y_{n,j,1} \in B - v, J_{n,j,1} \leqslant t-u) P\big(S_n(r) \in dv, T_n(r) \in du | L_{nr} = j\big)$$
$$\times P(L_{nr} = j).$$

Notice also that

$$
\begin{aligned}
&P\big(S_n(r^{'}) \in B, T_n(r) \leqslant t-\delta\big)\\
&= \sum_{j\in L} \int_0^{t-\delta} \int_{\mathbb{R}^m} P(Y_{n,j,1} \in B-v) P\big(S_n(r) \in dv, T_n(r) \in du | L_{nr} = j\big) P(L_{nr} = j).
\end{aligned}
$$

Consequently,

$$
(3.14)\quad \sum_{r\in R_n} P\big(S_n(r^{'}) \in B, T_n(r) \leqslant t-\delta, N_n(t) = nr\big) = \int_0^b \sum_{j\in L} \tilde{H}_{n,j}(B,s)ds,
$$

where

$$
\begin{aligned}
\tilde{H}_{n,j}(B,s) &\stackrel{\text{df}}{=} \int_0^{t-\delta} \int_{\mathbb{R}^m} n\big(P(Y_{n,j,1} \in B-v) - P(Y_{n,j,1} \in B-v, J_{n,j,1} \leqslant t-u)\big)\\
&\qquad \times P\big(S_n(s) \in dv, T_n(s) \in du | L_{[ns]} = j\big) P(L_{[ns]} = j)\\
&= \int_0^{t-\delta} \int_{\mathbb{R}^m} nP(Y_{n,j,1} \in B-v, J_{n,j,1} > t-u) \hat{\kappa}_{n,j}(du,s)
\end{aligned}
$$

with

$$
\hat{\kappa}_{n,j}(G,s) \stackrel{\text{df}}{=} P\big(S_n(s) \in B, T_n(s) \in G | L_{[ns]} = j\big), \qquad G \in \mathcal{B}(\mathbb{R}_+).
$$

Hence, arguing as before, we get the convergences

$$
\begin{aligned}
\tilde{H}_{n,j}(B,s) &\to \pi_j \int_0^{t-\delta} \int_{\mathbb{R}^m} \nu_j^{(A_j,D_j)}\big((B-v)\times(t-u,\infty)\big) P\big(A(s)\in dv, D(s)\in du\big)\\
&\equiv \tilde{H}_j(B,s),
\end{aligned}
$$

$$
\sum_{j\in L} \tilde{H}_{n,j}(B,s) \to \sum_{j\in L} \tilde{H}_j(B,s),
$$

$$
\int_0^b \sum_{j\in L} \tilde{H}_{n,j}(B,s)ds \to \int_0^b \sum_{j\in L} \tilde{H}_j(B,s)ds.
$$

Finally, by reasoning as in the proof of the first convergence of the theorem we get the second convergence of the theorem and equality (3.3). This completes the proof of the theorem. ■

Meerschaert et al. obtained formula (3.2) in Theorem 3.6 of [10] under the assumptions that rows of the array $\{\zeta_{n,k}\}$ form iid sequences of random vectors and that the Lévy measure ν^D of D satisfies the additional condition

$$
\int_0^1 u|\ln u|\nu^D(du) < \infty.
$$

4. DISCUSSION ON CONDITIONS FOR $M(T) \stackrel{D}{=} \tilde{M}(T)$

Formulas (3.2) and (3.3) reveal that, in general, the distributions of $M(t)$ and $\tilde{M}(t)$ are different. One may ask under which conditions the equality $M(t) \stackrel{D}{=} \tilde{M}(t)$ holds true. An answer to this question is given by the following corollary.

COROLLARY 4.1. *Let the conditions of Theorem* 3.1 *be satisfied. Assume*, *additionally*, *that for all sets $B_1 \in \mathcal{B}(\mathbb{R}^d)$ and $B_2 \in \mathcal{B}(\mathbb{R}_+)$*

$$\nu_j(B_1 \times B_2) = \nu_j^A(B_1)\delta_0^{(2)}(B_2) + \delta_0^{(1)}(B_1)\nu_j^D(B_2), \quad j \in L, \tag{4.1}$$

where $\delta_0^{(1)}, \delta_0^{(2)}$ denote the probability measures concentrated on $0 \in \mathbb{R}^d$ and on $0 \in \mathbb{R}_+$, respectively. Then $\tilde{M}(t)$ has the same distribution as $M(t)$.

P r o o f. By (4.1) it follows that

$$\begin{aligned}\nu^{(A,D)}(B_1 \times B_2) &= \sum_{j\in L} \pi_j \nu_j^A(B_1)\delta_0^{(2)}(B_2) + \delta_0^{(1)}(B_1) \sum_{j\in L} \nu_j^D(B_2)\\ &= \nu^A(B_1)\delta_0^{(2)}(B_2) + \delta_0^{(1)}(B_1)\nu^D(B_2).\end{aligned}$$

Then the most internal integral in formula (3.3) has the form

$$\begin{aligned}&\int_{\mathbb{R}^d} \delta_0^{(1)}(B - v)\nu^D(t - u, \infty)P\big(A(s) \in dv, D(s) \in du\big)\\ &\qquad\qquad = \nu^D(t - u, \infty)P\big(A(s) \in B, D(s) \in du\big).\end{aligned}$$

Hence

$$P\big(\tilde{M}(t)\in B\big) = \int_0^\infty \int_0^t \nu^D(t - u, \infty)P\big(A(s)\in B, D(s) \in du\big)ds = P\big(M(t)\in B\big),$$

which completes the proof of the corollary. ∎

Condition (4.1) is sufficient for the equality $M(t) \stackrel{D}{=} \tilde{M}(t)$ to hold. Moreover, it means an asymptotic independence of $S_n(t)$ and $T_n(t)$, i.e.

$$\big(S_n(t), T_n(t)\big) \Rightarrow \big(A(t), D(t)\big),$$

where $A(t)$ and $D(t)$ are independent. Below we give two examples of chain dependent arrays $\{(Y_{n,k}, J_{n,k})\}$ for which $S_n(t)$ and $T_n(t)$ are asymptotically independent.

EXAMPLE 4.1. Let an array $\{(Y_{n,k}, J_{n,k}), n, k \geqslant 1\}$ satisfy condition (C1) and let the array $\{(Y_{n,k,i}, J_{n,k,i}), n, k, i \geqslant 1\}$ from the canonical representation of $\{(Y_{n,k}, J_{n,k}), n, k \geqslant 1\}$ defined in (3.1) satisfy conditions (C6), (C4) and (C3). Furthermore, let $Y_{n,k,i}$ and $J_{n,k,i}$ be mutually independent with distributions $\mu^{(1)}_{n,k}$ and $\mu^{(2)}_{n,k}$, respectively, and let condition (3.1) hold true. Then for any $B_1 \in \mathcal{B}(\mathbb{R}^d)$ and $B_2 \in \mathcal{B}(\mathbb{R}_+)$ we have

$$
\begin{aligned}
P(Y_{n,k} \in B_1, J_{n,k} \in B_2)& \\
&= \sum_{j \in L} P(Y_{n,j,k} \in B_1, J_{n,j,k} \in B_2 | L_{k-1} = j) P(L_{k-1} = j) \\
&= \sum_{j \in L} \mu^{(1)}_{n,j}(B_1) \mu^{(2)}_{n,j}(B_2) P(L_{k-1} = j).
\end{aligned}
$$

This means that sequences $\{Y_{n,k}, k \geqslant 1\}$ and $\{J_{n,k}, k \geqslant 1\}$ are not independent, because they are driven by the same Markov chain $\{L_k\}$. Nevertheless, $S_n(t)$ and $T_n(t)$ are asymptotically independent. This follows from the fact that processes $\{S_{n,j}, j \geqslant 1\}$ and $\{T_{n,j}, j \geqslant 1\}$ are mutually independent. Hence, by Remark 3.1 and Proposition 2.2 we get the convergence $\big(S_n(t), T_n(t)\big) \Rightarrow \big(A(t), D(t)\big)$, where the Lévy process (A, D) has Lévy measure of the form

$$
\begin{aligned}
\nu^{(A,D)} &= \sum_{j \in L} \pi_j \nu_j = \sum_{j \in L} \pi_j (\nu^A_j \delta^{(2)}_0 + \delta^{(1)}_0 \nu^D_j) \\
&= \big(\sum_{j \in L} \pi_j \nu^A_j\big) \delta^{(2)}_0 + \delta^{(1)}_0 \big(\sum_{j \in L} \pi_j \nu^D_j\big) = \nu^A \delta^{(2)}_0 + \delta^{(1)}_0 \nu^D.
\end{aligned}
$$

Hence $S_n(t)$ and $T_n(t)$ are asymptotically independent and (4.1) holds. ■

Note that conclusions of the example above remain true if for all $j \in L$ we replace independence of processes $S_{n,j}(t)$ and $T_{n,j}(t)$ by their asymptotic independence. Below we present an example of this case.

EXAMPLE 4.2. Let us assume that the arrays $\{(Y_{n,k}, J_{n,k}), n, k \geqslant 1\}$ and $\{(Y_{n,k,i}, J_{n,k,i}), n, k, i \geqslant 1\}$ be as in the previous example. Let $\{M_i, i \geqslant 1\}$ be an iid sequence of random variables taking values in $\{1, 2, 3, \ldots\}$ and with finite mean μ and defined on the same probability space as $\{(Y_{n,k,i}, J_{n,k,i}), n, k, i \geqslant 1\}$. Let $K(n) = \sum_{i=1}^n M_i$. Define an array $\{(\bar{Y}_{n,k,i}, \bar{J}_{n,k,i}), n, k, i \geqslant 1\}$ as follows:

$$
(\bar{Y}_{n,k,i}, \bar{J}_{n,k,i}) = \big(\sum_{l=K(i-1)+1}^{K(i)} Y_{n,k,l}, \sum_{l=K(i-1)+1}^{K(i)} J_{n,k,l}\big).
$$

It is easy to see that $\bar{Y}_{n,k,i}$ and $\bar{J}_{n,k,i}$ are not independent. Notice that

$$
\begin{aligned}
\big(\bar{S}_{n,k}(t), \bar{T}_{n,k}(t)\big) &= \big(\sum_{i=1}^{[nt]} \bar{Y}_{n,k,i}, \sum_{i=1}^{[nt]} \bar{J}_{n,k,i}\big) = \big(\sum_{l=1}^{K([nt])} Y_{n,k,l}, \sum_{l=1}^{K([nt])} J_{n,k,l}\big) \\
&= \Big(S_{n,k}\big(K([nt])/n\big), T_{n,k}\big(K([nt])/n\big)\Big).
\end{aligned}
$$

Since $K([nt])/n \xrightarrow{\text{a.s.}} \mu t$, we have

$$\big(\bar{S}_{n,k}(t), \bar{T}_{n,k}(t)\big) \Rightarrow \big(A_k(\mu t), D_k(\mu t)\big).$$

Since processes A_k and D_k are independent, we infer that processes $\bar{S}_{n,k}(t)$ and $\bar{T}_{n,k}(t)$ are asymptotically independent. Arguing as in the previous example one can show that $\bar{S}_n$ and $\bar{T}_n$, defined as

$$\bar{S}_n(t) = \sum_{k=1}^{[nt]} \bar{Y}_{n,k} \quad \text{and} \quad \bar{T}_n(t) = \sum_{k=1}^{[nt]} \bar{J}_{n,k},$$

are asymptotically independent processes and assumption (4.1) is satisfied for the array $\{(\bar{Y}_{n,k}, \bar{J}_{n,k}), n, k \geqslant 1\}$. ■

Sequences of CTRWs generated by the array $\{(\bar{Y}_{n,k}, \bar{J}_{n,k}), n, k \geqslant 1\}$ defined above are called *cluster CTRWs* or *randomly coarse grained CTRWs*. An extensive survey of such processes was given by Jurlewicz in [4]; see also [6].

We conclude this section with the following remark:

REMARK 4.1. *Assume that an array* $\{(Y_{n,k}, J_{n,k}), n, k \geqslant 1\}$ *satisfies the conditions* (C1), (C6), (C3), (C4) *and* (2.7). *Then*

$$(S_n, T_n) \Rightarrow (A, D) \quad \textit{in } D\big([0,\infty), \mathbb{R}^d \times \mathbb{R}_+\big) \textit{ with } J_1 \textit{ topology.} \tag{4.2}$$

If, additionally, (4.1) *holds, then processes* A *and* D *are mutually independent.*

Mutual independence of A and D implies that

$$P\big(\mathrm{Disc}(A) \cap \mathrm{Disc}(D) = \emptyset\big) = 1, \tag{4.3}$$

where $\mathrm{Disc}(x)$ denotes the set of discontinuity points of $x \in D\big([0,\infty), \mathbb{R}^d \times \mathbb{R}_+\big)$. Convergence (4.2) and condition (4.3) imply, by Theorem 3.2.4 in [16] and Theorems 5.1 and 5.5 in [2] (see also Theorem 2.1 in [10]), the following functional convergences:

$$X_n \Rightarrow M \quad \text{and} \quad \tilde{X}_n \Rightarrow \tilde{M},$$

and $M = \tilde{M} = A \circ D^{-1}$, where $D^{-1}(t) = \inf\{s : D(s) > t\}$.

Observe also that when the conditions of Remark 4.1 are satisfied, then

$$\big(S_n(\cdot), T_n(\cdot)\big) \Rightarrow \sum_{j \in L} \big(A_j(\pi_j \cdot), D_j(\pi_j \cdot)\big).$$

5. DISCUSSION ON CONDITIONS FOR $M(T) \overset{D}{\neq} \tilde{M}(T)$

The recent papers [5] and [13] reveal that distributions of $M(t)$ and $\tilde{M}(t)$ may be different if $Y_{n,k}$ and $J_{n,k}$ are dependent. In this section we discuss two exemplary situations in which distributions of $M(t)$ and $\tilde{M}(t)$ are essentially different.

The fact that distribution of $M(t)$ may be different from that of $\tilde{M}(t)$ is especially clear when we take $Y_{n,k} = J_{n,k}$. In Theorem 2 in [13] we give limit distributions of sequences $X_n(t)$ and $\tilde{X}_n(t)$ generated by an array $\{J_{n,k}\}$ such that its rows are iid sequences of positive random variables with infinite means. The corollary given below extends this result to the chain dependent case.

COROLLARY 5.1. *Let an array* $\{(Y_{n,k}, J_{n,k}), n, k \geqslant 1\}$ *with* $Y_{n,k} = J_{n,k}$ *satisfy the conditions of Theorem* 3.1. *Then for any fixed* $t > 0$

$$X_n(t) = \sum_{k=1}^{N_n(t)} J_{n,k} \Rightarrow M(t) \quad \textit{and} \quad \tilde{X}_n(t) = \sum_{k=1}^{N_n(t)+1} J_{n,k} \Rightarrow \tilde{M}(t),$$

where

$$(5.1) \qquad P\big(M(t) \leqslant x\big) = \int_0^\infty \int_0^x \nu^D(t-u, \infty) P\big(D(s) \in du\big) ds, \quad x \leqslant t,$$

and

$$(5.2) \quad P\big(\tilde{M}(t) - t \geqslant x\big) = \int_0^\infty \int_0^t \nu^D(x+t-u, \infty) P\big(D(s) \in du\big) ds, \quad x \geqslant 0.$$

Proof. Since $Y_{n,k} = J_{n,k}$, we have $A = D$ in the condition (C5). The assertions of Theorem 3.1 imply the convergences $X_n(t) \Rightarrow M(t)$, $\tilde{X}_n(t) \Rightarrow \tilde{M}(t)$ and equality (5.1). To see that equality (5.2) also holds true, put $B = (x+t, \infty)$ in (3.3). Then the most inner integral in (3.3) has the form

$$\int_{\mathbb{R}_+} \nu^D\big((x+t-v, \infty) \cap (t-u, \infty)\big) P\big(D(s) \in dv, D(s) \in du\big)$$
$$= \nu^D(x+t-u, \infty) P\big(D(s) \in du\big).$$

This completes the proof of the corollary. ∎

EXAMPLE 5.1. Let an array $\{(Y_{n,k}, J_{n,k}), n, k \geqslant 1\}$ with $Y_{n,k} = J_{n,k}$ be such as in Corollary 5.1, i.e. it satisfies the conditions of Theorem 3.1. Assume that it satisfies condition (C6) and limiting processes $D_j, j \in L$, are strictly increasing stable subordinators such that distributions of $D_j(1)$ are stable with parameters $\mathcal{S}_\alpha(1, \sigma_j, 0)$, $\sigma_j > 0$ and some $\alpha \in (0,1)$. Then formulas (5.1) and (5.2) together with the well-known properties of stable processes yield

$$(5.3) \qquad P\big(X_n(t) \leqslant x\big) \to \frac{\sin(\pi\alpha)}{\pi} \int_0^{x/t} (1-u)^{-\alpha} u^{\alpha-1} du \quad \text{for all } x \leqslant t,$$

$$(5.4) \qquad P\big(\tilde{X}_n(t) - t \leqslant x\big) \to \frac{\sin(\pi\alpha)}{\pi} \int_1^{1+x/t} (v-1)^{-\alpha} v^{-1} dv \quad \text{for all } x > 0.$$

Detailed computations are given in Corollary 1 in [13]. ∎

In the above example, the Lévy measure $\nu^{(A,D)}$ of the process (A,D) is concentrated on the set $\{(x,y)\in\mathbb{R}_+\times\mathbb{R}_+ :\ x=y\}$, so it was not full on $\mathbb{R}_+\times\mathbb{R}_+$. One may ask if in the case when the Lévy measure $\nu^{(A,D)}$ is full on $\mathbb{R}^d\times\mathbb{R}_+$ (i.e. not concentrated on any curve or hyperplane in $\mathbb{R}^d\times\mathbb{R}_+$) the distributions of $M(t)$ and $\tilde{M}(t)$ are the same. The following example shows that it is not true.

EXAMPLE 5.2. Let $\{(\hat{Y}_{n,k},J_{n,k}),n,k\geqslant 1\}$ be an array of pairs of random variables $\hat{Y}_{n,k}$ and $J_{n,k}$ such that $J_{n,k}>0,\ EJ_{n,k}=\infty$, sequences $\{\hat{Y}_{n,k},k\geqslant 1\}$ and $\{J_{n,k},k\geqslant 1\}$ are mutually independent for each $n\geqslant 1$ and each of them is a sequence of iid random variables. Let $\gamma_n, n\geqslant 1$, be random variables such that $P(\gamma_n=1)=p,\ P(\gamma_n=0)=1-p,\ 0<p<1$, and γ_n is independent of $\{(\hat{Y}_{n,k},J_{n,k}),k\geqslant 1\}$. Let $Y_{n,k}=\gamma_n\hat{Y}_{n,k}+(1-\gamma_n)J_{n,k}$. Let us notice that, for each $n\geqslant 1$, $\{(Y_{n,k},J_{n,k}),k\geqslant 1\}$ is a sequence of iid distributed random vectors. Consider processes

$$\hat{S}_n(t)=\sum_{k=1}^{[nt]}\hat{Y}_{n,k},\quad T_n(t)=\sum_{k=1}^{[nt]}J_{n,k},\quad S_n(t)=\sum_{k=1}^{[nt]}Y_{n,k}.$$

Then $S_n=\gamma_n\hat{S}_n+(1-\gamma_n)T_n$. Let

$$(\hat{S}_n,T_n)\Rightarrow(\hat{A},D), \tag{5.5}$$

where $\hat{A}$ and D are independent Lévy processes and D is a subordinator. Then

$$(S_n,T_n)\Rightarrow(A,D)=\big(\gamma\hat{A}+(1-\gamma)D,D\big), \tag{5.6}$$

where γ is a random variable independent of $(\hat{A},D)$ and $P(\gamma=1)=p$, $P(\gamma=0)=1-p$. Notice that for sets $B_1\in\mathcal{B}(\mathbb{R})$ and $B_2\in\mathcal{B}(\mathbb{R}_+)$ it follows that

(5.7)
$$\begin{aligned}nP(Y_{n,1}\in B_1,J_{n,1}\in B_2)&=nP\big(\gamma_n\hat{Y}_{n,1}+(1-\gamma_n)J_{n,1}\in B_1,J_{n,1}\in B_2\big)\\&=pnP(\hat{Y}_{n,1}\in B_1,J_{n,1}\in B_2)+(1-p)nP(J_{n,1}\in B_1\cap B_2)\\&=pnP(\hat{Y}_{n,1}\in B_1)P(J_{n,1}\in B_2)+(1-p)nP(J_{n,1}\in B_1\cap B_2).\end{aligned}$$

Since for each $n\geqslant 1$, $\{(Y_{n,k},J_{n,k}),k\geqslant 1\}$ is a sequence of iid random vectors, we infer by (5.5) and (5.6) that the Lévy measure of (A,D) has the form

$$\begin{aligned}\nu^{(A,D)}(B_1\times B_2)=p\big(\delta_0^{(1)}(B_1)\nu^D(B_2)+\nu^{\hat{A}}(B_1)\delta_0^{(2)}(B_2)\big)\\+(1-p)\nu^D(B_1\cap B_2).\end{aligned} \tag{5.8}$$

This measure is full on $\mathbb{R}\times\mathbb{R}_+$. Notice that the array $\{(Y_{n,k},J_{n,k}),n,k\geqslant 1\}$ satisfies the conditions of Theorem 3.1, so the convergences $X_n(t)\Rightarrow M(t)$ and $\tilde{X}(t)\Rightarrow\tilde{M}(t)$ as well as equalities (3.2) and (3.3) hold true. Now we show that

$$P\big(M(t)\in B\big)\neq P\big(\tilde{M}(t)\in B\big)\quad\text{for any set } B\in\mathcal{B}(\mathbb{R}).$$

Let

$$I \stackrel{\mathrm{df}}{=} \int_0^t \int_{R_+} \nu^{(A,D)}\big((B-v)\times(t-u,\infty)\big)P\big(A(s)\in dv, D(s)\in du\big).$$

By (5.6) we have

$$\begin{aligned}
I &= \\
&= \int_0^t \int_{R_+} \nu^{(A,D)}\big((B-v)\times(t-u,\infty)\big)P\big(\gamma\hat{A}(s)+(1-\gamma)D(s)\in dv, D(s)\in du\big)\\
&= p\int_0^t \int_{R_+} \nu^{(A,D)}\big((B-v)\times(t-u,\infty)\big)P\big(\hat{A}(s)\in dv, D(s)\in du\big)\\
&\quad + (1-p)\int_0^t \int_{R_+} \nu^{(A,D)}\big((B-v)\times(t-u,\infty)\big)P\big(D(s)\in dv, D(s)\in du\big)\\
&= p\int_0^t \int_{R_+} \nu^{(A,D)}\big((B-v)\times(t-u,\infty)\big)P\big(\hat{A}(s)\in dv\big)P\big(D(s)\in du\big)\\
&\quad + (1-p)\int_0^t \int_{R_+} \nu^{(A,D)}\big((B-v)\times(t-u,\infty)\big)P\big(D(s)\in dv, D(s)\in du\big)\\
&\equiv pI_1+(1-p)I_2.
\end{aligned}$$

Using (5.8), we get

$$\begin{aligned}
I_1 = {} & p\int_0^t \nu^D(t-u,\infty)P\big(\hat{A}(s)\in B\big)P\big(D(s)\in du\big)\\
&+ (1-p)\int_0^t \int_{R_+} \nu^D\big((B-v)\cap(t-u,\infty)\big)P\big(\hat{A}(s)\in dv\big)P\big(D(s)\in du\big),
\end{aligned}$$

and

$$\begin{aligned}
I_2 &= \int_0^t \int_{R_+} \nu^{(A,D)}\big((B-v)\times(t-u,\infty)\big)P\big(D(s)\in dv, D(s)\in du\big)\\
&= \int_0^t \nu^{(A,D)}\big((B-u)\times(t-u,\infty)\big)P\big(D(s)\in du\big)\\
&= p\int_{[0,t]\cap B} \nu^D(t-u,\infty)P\big(D(s)\in du\big)\\
&\quad + (1-p)\int_0^t \nu^D\big((B-u)\cap(t-u,\infty)\big)P\big(D(s)\in du\big).
\end{aligned}$$

Put

$$\bar{I} = \int_0^t \nu^D(t-u,\infty)P\big(A(s)\in B, D(s)\in du\big)$$
$$= p\int_0^t \nu^D(t-u,\infty)P\big(\hat{A}(s)\in B\big)P\big(D(s)\in du\big)$$
$$+ (1-p)\int_{[0,t]\cap B} \nu^D(t-u,\infty)P\big(D(s)\in du\big).$$

Then

$$I = p\bar{I} + p(1-p)\int_0^t \int_{R_+} \nu^D\big((B-v)\cap(t-u,\infty)\big)P\big(\hat{A}(s)\in dv\big)P\big(D(s)\in du\big)$$
$$+ (1-p)^2 \int_{[0,t]} \nu^D\big((B-u)\cap(t-u,\infty)\big)P\big(D(s)\in du\big).$$

Observe that the conditions of Theorem 3.1 are satisfied. Then I is equal to inner integrals in formula (3.3) and $\bar{I}$ is the inner integral in (3.2). Hence we have $P\big(M(t)\in B\big) \neq P\big(\tilde{M}(t)\in B\big)$. ■

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