PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 31, Fasc. 2 (2011), pp. 331–347

KIEFER'S LAW OF THE ITERATED LOGARITHM FOR THE VECTOR OF UPPER ORDER STATISTICS

BY

RASBAGH VASUDEVA* (MYSORE) AND ALIREZA YOUSEFI MORIDANI (MANJIL)

Abstract. Let $\{X_n\}$ be a sequence of independent identically distributed random variables with a common continuous distribution function and let $M_{j,n}$ denote the jth upper order statistic among $X_1, X_2, \ldots, X_n$, $n \geqslant j$. For a large class of distributions, we obtain the law of the iterated logarithm for $\{M_{1,n}, M_{2,n}\}$, properly normalized. As a consequence, we establish a law of the iterated logarithm for the spacings $\{M_{1,n} - M_{2,n}\}$.

2000 AMS Mathematics Subject Classification: Primary: 60F15.

Key words and phrases: Law of the iterated logarithm, upper order statistics, regularly varying function.

1. INTRODUCTION

Let $\{X_n\}$ be a sequence of independent identically distributed (i.i.d.) random variables (r.v.s) defined over a common probability space $(\Omega, \mathbb{F}, P)$. Suppose that the common distribution function (d.f.) F is continuous. Denote the right extremity of F by $r(F)$ and note that $r(F) = \infty$ if $F(x) < 1$ for all x real. On the same space, define a sequence $\{U_n\}$ of uniform $(0,1)$ r.v.s. Let $M_{j,n}$ stand for the jth largest observation among $(X_1, X_2, \ldots, X_n)$ and $M^*_{j,n}$ for the jth largest observation among $(U_1, U_2, \ldots, U_n)$, $1 \leqslant j \leqslant n$, $n \geqslant 1$. Then $M_{j,n}$ is called the *jth upper order statistic among $X_1, X_2, \ldots, X_n$*, and $M^*_{j,n}$ the *jth upper order statistic of $U_1, U_2, \ldots, U_n$*.

Kiefer [8] has established a law of the iterated logarithm (l.i.l.) for the order statistics, which gives a precise almost sure (a.s.) upper bound for $\{M^*_{j,n}\}$. Barndorff-Nielsen [1] has obtained an l.i.l. for $\{M^*_{1,n}\}$, which in turn gives an a.s. lower bound. From the results in [1] one can easily obtain a similar l.i.l. for $\{M^*_{j,n}\}$, $j > 1$. Under certain conditions on the d.f. F, which are in spirit close to von Mises type conditions, de Haan and Hordijk [6] have established l.i.l. for $\{M_{1,n}\}$. For the setup of [6], the set of a.s. limit points has been given in [5]. Hall [7] has extended

* The research work of the first author has been supported by UGC-DSA program, University of Mysore, Mysore, India.

Kiefer's l.i.l. to a class of distributions, which includes exponential, normal, Gumbel and so on. To be precise, Hall [7] has considered the class of d.f.s F with $-\log \overline{F}(x)$ regularly varying, as $x \to \infty$, where $\overline{F}(x) = 1 - F(x)$. Vasudeva and Savitha [11] have given the l.i.l. for $\{M_{1,n}\}$, assuming that $\overline{F}(x)$ is regularly varying, as $x \to \infty$. In this paper, we obtain the a.s. limit set of $\{M_{1,n}, M_{2,n}\}$ properly normalized, which extends the results in [7], [8] and [11]. Also, for a class of d.f.s with $r(F) < \infty$, we give the l.i.l. for $\{M_{1,n}, M_{2,n}\}$. As a corollary, we obtain the l.i.l. for the spacing $\{M_{1,n} - M_{2,n}\}$. One may note that Devroye [3] has established an l.i.l. for the maximum of spacings between consecutive order statistics in the uniform case.

In the next section, we present the l.i.l. results that are established in this paper along with some remarks. Section 3 contains some lemmas and their proofs for $\{M^*_{1,n}, M^*_{2,n}\}$. The proofs of all the theorems presented in Section 2 are given in a separate section at the end.

Throughout the paper, for any positive x, $[x]$ stands for the greatest integer less than or equal to x. We assume also that c and k (integer), with or without a suffix, denote positive constants. A point (θ_1, θ_2) of real components will be called an *a.s. limit point of a random sequence* $\{Z_n\} = \{Z_{1,n}, Z_{2,n}\}$, $n \geqslant 1$, if for any given $\epsilon_1, \epsilon_2 > 0$,

$$P\big(Z_n \in (\theta_1 - \epsilon_1, \theta_1 + \epsilon_1) \times (\theta_2 - \epsilon_2, \theta_2 + \epsilon_2) \text{ i.o.}\big) = 1.$$

2. MAIN RESULTS

In this section, we present l.i.l. for $\{M_{1,n}, M_{2,n}\}$, properly normalized, for d.f.s F which belong to three major classes in extreme value theory, denoted for convenience by $\mathbf{C_1}$, $\mathbf{C_2}$ and $\mathbf{C_3}$. The class $\mathbf{C_1}$ is that of all F with $-\log \overline{F}(x)$ regularly varying, as $x \to \infty$. This class contains the Gumbel d.f. and the exponential and normal distributions which belong to the domain of attraction of Gumbel law. Further, by [9], p. 1102, one may note that all d.f.s with $-\log \overline{F}$ regularly varying with index γ, $0 < \gamma < 1$, belong to the domain of attraction of a Gumbel law. The class $\mathbf{C_2}$ is that of d.f.s with $\overline{F}(x)$ regularly varying, as $x \to \infty$. It is well known that $\mathbf{C_2}$ is the class of all d.f.s which belong to the domain of attraction of Fréchet law (see, for example, [4]). $\mathbf{C_3}$ is the class of all d.f.s F (with finite right extremity) belonging to the domain of attraction of a Weibull law.

2.1. Law of the iterated logarithm when $F \in \mathbf{C_1}$. Let $U(x) = -\log \overline{F}(x)$ and suppose that it has a unique inverse $V(x)$. Assuming that $V(x)$ satisfies the condition

$$\lim_{x \to \infty} \frac{V\big(x\big(1 + a(x)\big) - V(x)\big)}{a(x)V(x)} = \frac{1}{\gamma}, \tag{2.1}$$

where $\gamma > 0$ is some constant and $a(x)$ is a real-valued function with $a(x) \to 0$ as $x \to \infty$, Hall [7] has extended the Kiefer's l.i.l. (to this class). He also notes that all d.f.s F with $-\log \overline{F}(x)$ regularly varying (as $x \to \infty$) belong to this class. To be precise, for d.f.s F satisfying (2.1) he has established the following

THEOREM A (Hall [7]). *Let $M_{r,n}$ denote the rth upper order statistic. Then*

$$\limsup_{n\to\infty} \frac{\gamma \log n}{\log\log n}\left(\frac{M_{r,n}}{V(\log n)} - 1\right) = \frac{1}{r} \quad \textit{a.s.} \tag{2.2}$$

Our theorem below generalizes Theorem A. Let

$$W_{j,n} = \frac{\gamma \log n}{\log\log n}\left(\frac{M_{j,n}}{V(\log n)} - 1\right), \quad j = 1, 2.$$

Then we have the following

THEOREM 2.1. *The set of all a.s. limit points of $\{W_n\} = \{W_{1,n}, W_{2,n}\}$ is given by*

$$L = \big\{(x, y) : 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1/2, y \leqslant x, x + y \leqslant 1\big\}.$$

REMARK 2.1. In particular, for unit exponential and Gumbel d.f.s, one can see that $\gamma = 1$ and $V(\log n) = \log n$. As such, Theorem 2.1 establishes that the set of all a.s. limit points of

$$\left\{\frac{M_{1,n} - \log n}{\log\log n}, \frac{M_{2,n} - \log n}{\log\log n}\right\}$$

is L.

When F is standard normal, γ is 2 and $V(\log n) = \sqrt{2\log n}$, $n \geqslant 1$. Hence the set of a.s. limit points of

$$\left\{\frac{\sqrt{2\log n}(M_{1,n} - \sqrt{2\log n})}{\log\log n}, \frac{\sqrt{2\log n}(M_{2,n} - \sqrt{2\log n})}{\log\log n}\right\}$$

is L.

As a corollary to Theorem 2.1, one can get the l.i.l. for $\{M_{1,n} - M_{2,n}\}$, as given below.

COROLLARY 2.1. *Let*

$$\eta_n = \frac{\gamma \log n}{\log\log n} \frac{M_{1,n} - M_{2,n}}{V(\log n)}, \quad n \geqslant 3.$$

Then

$$\liminf_{n\to\infty} \eta_n = 0 \ \textit{a.s.}, \quad \limsup_{n\to\infty} \eta_n = 1 \ \textit{a.s.}$$

and all points in $(0, 1)$ are a.s. limit points of $\{\eta_n\}$.

Probability and Mathematical Statistics 31, 2011, z. 2

2.2. Law of the iterated logarithm when $F \in \mathbf{C_2}$. Here, we assume that $\overline{F}(x)$ is regularly varying with index $(-\alpha)$, $\alpha > 0$. Let B_n be a solution of the equation $n\big(1 - F(B_n)\big) = 1$, $n \geqslant 1$. It is well known that $(B_n^{-1} M_{1,n})$ converges to a Fréchet law with parameter α. We have the following laws of the iterated logarithm.

THEOREM 2.2. *For any* $r \geqslant 1$,

$$\limsup_{n\to\infty} \left(\frac{M_{r,n}}{B_n}\right)^{1/\log\log n} = e^{1/(r\alpha)} \quad a.s., \tag{2.3}$$

$$\liminf_{n\to\infty} \left(\frac{M_{r,n}}{B_n}\right)^{1/\log\log n} = 1 \quad a.s. \tag{2.4}$$

REMARK 2.2. For $\{X_n\}$ a sequence of i.i.d. symmetric stable r.v.s with index α, Chover [2] has established an l.i.l. for the partial sum sequence $\big\{S_n\big\}$, by taking $(\log\log n)^{-1}$ in the power, as in the statement of Theorem 2.2. When F is symmetric stable, it is well known that $\overline{F}(x)$ is regularly varying and, as a consequence, F belongs to the domain of attraction of Fréchet law. As such, Theorem 2.2 holds when F is symmetric stable. Hence the l.i.l. in Theorem 2.2 will be called *Chover's form of the l.i.l. for extreme order statistics*.

In order to obtain the l.i.l. for the sequence $\big\{M_{1,n}, M_{2,n}\big\}$, we define $\xi_{i,n} = (M_{i,n}/B_n)^{1/\log\log n}$, $i = 1, 2$, and $\{\xi_n\} = \big\{\xi_{1,n}, \xi_{2,n}\big\}$, $n \geqslant 3$. Then we have the following

THEOREM 2.3. *The set of all a.s. limit points of* $\{\xi_n\}$ *is*

$$L_1 = \{(x, y) : 1 \leqslant x \leqslant e^{1/\alpha}, 1 \leqslant y \leqslant e^{1/(2\alpha)}, y \leqslant x, xy \leqslant e^{1/\alpha}\}.$$

REMARK 2.3. Let F be a Pareto d.f. with $1 - F(x) = 1/x^{\alpha}$ if $x \geqslant 1$ and $1 - F(x) = 1$ if $x < 1$, $\alpha > 0$. Here $B_n = n^{1/\alpha}$. Hence

$$\xi_n = \left(\left(\frac{M_{1,n}}{n^{1/\alpha}}\right)^{1/\log\log n}, \left(\frac{M_{2,n}}{n^{1/\alpha}}\right)^{1/\log\log n}\right), \quad n \geqslant 3,$$

has the a.s. limit set L_1.

The following corollary gives the l.i.l. for the spacing $\{M_{1,n} - M_{2,n}\}$.

COROLLARY 2.2. *Let*

$$\hat{\eta}_n = \left(\frac{M_{1,n} - M_{2,n}}{B_n}\right)^{1/\log\log n}, \quad n \geqslant 3.$$

Then

$$\liminf_{n\to\infty} \hat{\eta}_n = 1, \quad \limsup_{n\to\infty} \hat{\eta}_n = e^{1/\alpha}$$

and all points in $(1, e^{1/\alpha})$ *are a.s. limit points of* $\big\{\hat{\eta}_n\big\}$.

2.3. Law of the iterated logarithm when $F \in \mathbf{C_3}$. Here we consider d.f.s F with $r(F)$ finite and further assume that F belongs to the domain of attraction of a Weibull law. Let $\{X_n\}$ be a sequence of i.i.d. r.v.s with d.f. F and with the jth upper order statistics $M_{j,n}$, $j = 1, 2$. Define

$$Y_n = \frac{1}{r(F) - X_n}, \quad n \geqslant 1,$$

and let $\hat{M}_{j,n}$ denote the jth upper order statistics of $Y_1, Y_2, \ldots, Y_n$, $j = 1, 2$, $n \geqslant 2$. Then the relation

$$\hat{M}_{j,n} = \frac{1}{r(F) - M_{j,n}}, \quad j = 1, 2,$$

is immediate. It is well known that if $\{M_{j,n}\}$, properly normalized, converges to a Weibull law with parameter α, then $\{\hat{M}_{j,n}\}$, properly normalized, converges to a Fréchet law with the same parameter α, $\alpha > 0$. We now have the following theorem:

THEOREM 2.4. *Let F belong to the domain of attraction of a Weibull law with parameter α. Then the set of all a.s. limit points of*

$$\Big\{\Big(B_n\big(r(F) - M_{1,n}\big)\Big)^{1/\log\log n}, \Big(B_n\big(r(F) - M_{2,n}\big)\Big)^{1/\log\log n}\Big\}$$

is $L_2 = \{(x, y) : e^{-1/\alpha} \leqslant x \leqslant 1,\ e^{-1/(2\alpha)} \leqslant y \leqslant 1, x \leqslant y, xy \geqslant e^{-1/\alpha}\}$, where B_n is a solution of the equation $n\big(1 - F\big(r(F) - 1/B_n\big)\big) = 1$.

COROLLARY 2.3. *Let $\zeta_n = \big\{\big(B_n(M_{1,n} - M_{2,n})\big)^{1/\log\log n}\big\}$, $n \geqslant 3$. Then*

$$\liminf_{n\to\infty} \zeta_n = e^{-1/(2\alpha)}, \quad \limsup_{n\to\infty} \zeta_n = 1$$

and all points in $(e^{-1/(2\alpha)}, 1)$ are a.s. limit points of $\{\zeta_n\}$.

REMARK 2.4. Suppose that F is uniform $(0, 1)$. Then $r(F) = 1$. One can see that $B_n = n$. Theorem 2.4 implies that the a.s. limit set of

$$\big\{\big(n(1 - M_{1,n})\big)^{1/\log\log n}, \big(n(1 - M_{2,n})\big)^{1/\log\log n}\big\}$$

is $\{(x, y) : e^{-1} \leqslant x \leqslant 1,\ e^{-1/2} \leqslant y \leqslant 1, x \leqslant y, xy \geqslant e^{-1}\}$. Also, the above corollary shows that $\big(n(M_{1,n} - M_{2,n})\big)^{1/\log\log n}$ has an a.s. limit set $(e^{-1/2}, 1)$.

3. LEMMAS WITH PROOFS

In this section, we confine ourselves to a sequence $\{U_n\}$ of i.i.d. uniform $(0, 1)$ r.v.s with $M^*_{r,n}$ denoting the rth upper order statistic and prove some lemmas

Probability and Mathematical Statistics 31, 2011, z. 2

needed in establishing the limit theorems presented in the previous section. However, the last lemma in this section is on slowly varying (s.v.) function, needed in proving Theorem 2.3. Define

$$T_{r,n} = -\left(\frac{\log(1-M^*_{r,n}) + \log n}{\log\log n}\right), \quad r \geqslant 1, n \geqslant 3.$$

LEMMA 3.1. *For any* $r \geqslant 1$,

$$\limsup_{n\to\infty} T_{r,n} = \frac{1}{r} \text{ a.s.} \quad \text{and} \quad \liminf_{n\to\infty} T_{r,n} = 0 \text{ a.s.}$$

Proof. Let $m^*_{r,n}$ denote the rth lower extreme among $U_1, U_2, \ldots, U_n$. Then, by Theorem 6 of [8], we have

$$\limsup_{n\to\infty} \frac{\log m^*_{r,n} + \log n}{\log\log n} = 0 \text{ a.s.}, \tag{3.1}$$

$$\liminf_{n\to\infty} \frac{\log m^*_{r,n} + \log n}{\log\log n} = -\frac{1}{r} \text{ a.s.} \tag{3.2}$$

Since U_n is uniform $(0,1)$, so is $1-U_n$ and one can see that $M^*_{r,n}$ of $U_1, U_2, \ldots, U_n$ and $m^*_{r,n}$ of $1-U_1, 1-U_2, \ldots, 1-U_n$ are related by $m^*_{r,n} = 1 - M^*_{r,n}$, $r \geqslant 1$. By (3.1) and (3.2), the proof is immediate. ■

From the above lemma one can see that the a.s. limit set of $\{T_{1,n}, T_{2,n}\}$ is included in the rectangle $[0,1]\times[0,\frac{1}{2}]$. Further, $M^*_{1,n} \geqslant M^*_{2,n}$ implies that $T_{1,n} \geqslant T_{2,n}$, $n \geqslant 1$. Consequently, the limit set of $\{T_{1,n}, T_{2,n}\}$ is contained in the set $S = \{(x,y) : 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant \frac{1}{2}, x \geqslant y\}$. The following lemmas show that the limit set is a subset of S.

LEMMA 3.2. *Given any point* $(x,y) \in S$ *with* $x+y>1$, *one can find an* $\epsilon > 0$ *with* $x+y-2\epsilon > 1$ *such that*

$$P(T_{1,n} > x-\epsilon, T_{2,n} > y-\epsilon \text{ i.o.}) = 0. \tag{3.3}$$

Proof. Define

$$\begin{aligned} A_n &= (T_{1,n} > x-\epsilon, T_{2,n} > y-\epsilon) \\ &= \left(M^*_{1,n} > 1 - \frac{1}{n(\log n)^{x-\epsilon}}, M^*_{2,n} > 1 - \frac{1}{n(\log n)^{y-\epsilon}}\right). \end{aligned}$$

For $n_k = [e^k]$, define

$$\alpha_{1,k} = 1 - \frac{1}{n_k(\log n_k)^{x-\epsilon}}, \quad \alpha_{2,k} = 1 - \frac{1}{n_k(\log n_k)^{y-\epsilon}},$$

$$B_k = (M^*_{1,n} > \alpha_{1,k}, M^*_{2,n} > \alpha_{2,k};\ \text{for at least one } n \text{ in } n_k < n \leqslant n_{k+1})$$

and

$$C_k = (M^*_{1,n_{k+1}} > \alpha_{1,k}, M^*_{2,n_{k+1}} > \alpha_{2,k}).$$

Observe that $(A_n \text{ i.o.}) \subseteq (B_k \text{ i.o.}) = (C_k \text{ i.o.})$. Hence the lemma is established, once we show that $P(C_k \text{ i.o.}) = 0$. We have

$$\begin{aligned} P(C_k) &= n_{k+1}(n_{k+1}-1) \int\limits_{\alpha_{1,k}}^{1} \int\limits_{\alpha_{2,k}}^{z_1} z_2^{n_{k+1}-2} dz_2 dz_1 \\ &= 1 - \alpha_{1,k}^{n_{k+1}} - \frac{n_{k+1}}{n_k(\log n_k)^{x-\epsilon}} \alpha_{2,k}^{n_{k+1}-1}. \end{aligned}$$

Expanding both $\alpha_{1,k}^{n_{k+1}}$ and $\alpha_{2,k}^{n_{k+1}-1}$ up to second order, one can find a $k_1 > 0$ such that for all $k \geqslant k_1$

$$P(C_k) \leqslant \frac{2e}{(\log n_k)^{x+y-2\epsilon}}.$$

Recalling that $x + y - 2\epsilon > 1$, one can see that $\sum_k P(C_k) < \infty$. The Borel–Cantelli lemma implies that $P(C_k \text{ i.o.}) = 0$, which completes the proof. ■

LEMMA 3.3. *For any point* $(x, y) \in S$ *with* $x + y < 1$ *and for any constants* ϵ, ϵ' *with* $0 < \epsilon < \epsilon' < 1$,

(3.4) $$P(T_{1,m_k} > x - \epsilon, T_{2,m_k} > y - \epsilon \text{ i.o.}) = 1,$$

(3.5) $$P(T_{1,m_k} > x - \epsilon, T_{2,m_k} > y + \epsilon' \text{ i.o.}) = 0,$$

(3.6) $$P(T_{1,m_k} > x + \epsilon', T_{2,m_k} > y - \epsilon \text{ i.o.}) = 0,$$

where $m_k = [\exp(k^{1/(x+y)})]$, $k \geqslant 1$.

Proof. In order to establish (3.4), define

$$\begin{aligned} M'_{1,m_k} &= \max(X_{m_{k-1}+1}, X_{m_{k-1}+2}, \ldots, X_{m_k}), \\ M'_{2,m_k} &= \text{second max}(X_{m_{k-1}+1}, X_{m_{k-1}+2}, \ldots, X_{m_k}) \end{aligned}$$

and

$$T'_{j,m_k} = -\left(\frac{\log(1 - M'_{j,m_k}) + \log m_k}{\log\log m_k}\right), \quad j = 1, 2.$$

Note that $M_{j,m_k} \geqslant M'_{j,m_k}$, which in turn implies that $T_{j,m_k} \geqslant T'_{j,m_k}$, $j = 1, 2$. Consequently,

$$(T_{1,m_k} > x - \epsilon, T_{2,m_k} > y - \epsilon) \supseteq (T'_{1,m_k} > x - \epsilon, T'_{2,m_k} > y - \epsilon).$$

In order to prove (3.4), it is enough to show that

$$P(T'_{1,m_k} > x - \epsilon, T'_{2,m_k} > y - \epsilon \text{ i.o.}) = 1.$$

The condition $x+y<1$ implies $m_k/m_{k-1} \to \infty$, and hence $(m_k - m_{k-1})/m_k \to 1$ as $k \to \infty$. Define

$$\beta_{1,k} = 1 - \frac{1}{m_k(\log m_k)^{x-\epsilon}} \quad \text{and} \quad \beta_{2,k} = 1 - \frac{1}{m_k(\log m_k)^{y-\epsilon}}, k \geqslant 2.$$

Then

$$\begin{aligned} P(T'_{1,m_k} > x - \epsilon, T'_{2,m_k} > y - \epsilon) &= P(M'_{1,m_k} > \beta_{1,k}, M'_{2,m_k} > \beta_{2,k}) \\ &= \int_{\beta_{1,k}}^{1} \int_{\beta_{2,k}}^{z_1} (m_k - m_{k-1})(m_k - m_{k-1} - 1) z_2^{m_k - m_{k-1} - 2} dz_2 dz_1 \\ &= 1 - \beta_{1,k}^{m_k - m_{k-1}} - \frac{m_k - m_{k-1}}{m_k(\log m_k)^{x-\epsilon}} \beta_{2,k}^{m_k - m_{k-1} - 1}. \end{aligned}$$

Again, by expanding $\beta_{1,k}^{m_k - m_{k-1}}$ and $\beta_{2,k}^{m_k - m_{k-1} - 1}$ up to second order, one can find a $k_2 > 0$ and $c_1 > 0$ such that for all $k \geqslant k_2$,

$$\begin{aligned} P(T'_{1,m_k} > x - \epsilon, T'_{2,m_k} > y - \epsilon) &\geqslant \frac{1}{2(\log m_k)^{x+y-2\epsilon}} \\ &= \frac{c_1}{k^{(1-\delta)}}, \quad \text{where } \delta = \frac{2\epsilon}{x+y}. \end{aligned}$$

Consequently, $\sum_k P(T'_{1,m_k} > x - \epsilon, T'_{2,m_k} > y - \epsilon) = \infty$. From the fact that $\{M'_{1,m_k}, M'_{2,m_k}\}$ is a mutually independent sequence of r.v.s one can immediately see that $\{T'_{1,m_k}, T'_{2,m_k}\}$ is a mutually independent sequence. By the Borel–Cantelli lemma, (3.4) follows.

In order to establish (3.5), recall that

$$\beta_{1,k} = 1 - \frac{1}{m_k(\log m_k)^{x-\epsilon}}$$

and define

$$\gamma_{2,k} = 1 - \frac{1}{m_k(\log m_k)^{y+\epsilon'}}, \quad k \geqslant 1.$$

Then

$$\begin{aligned} P(T_{1,m_k} > x - \epsilon, T_{2,m_k} > y + \epsilon') &= P\big(M_{1,m_k} > \beta_{1,k}, M_{2,m_k} > \gamma_{2,k}\big) \\ &= \int_{\beta_{1,k}}^{1} \int_{\gamma_{2,k}}^{z_1} m_k(m_k - 1) z_2^{m_k - 2} dz_2 dz_1. \end{aligned}$$

Proceeding on the lines of arguments used in getting a bound for $P(C_k)$ in Lemma 3.2, one can find a $c_2 > 0$ and a $k_3 > 0$ such that for all $k \geqslant k_3$,

$$P(T_{1,m_k} > x - \epsilon, T_{2,m_k} > y + \epsilon') \leqslant \frac{c_2}{k^{(1+\delta_1)}}, \quad \text{where } \delta_1 = \frac{\epsilon' - \epsilon}{x + y}.$$

From the choice of $\epsilon' > \epsilon > 0$ we get $\delta_1 > 0$. By the Borel–Cantelli lemma, (3.5) follows. The assertion (3.6) can be established similarly and the details are omitted. ■

The next lemma is a property of slowly varying (s.v.) functions, which is needed in obtaining l.i.l. for $F \in \mathbf{C_2}$.

LEMMA 3.4. *Let $L(x)$ be a function s.v. at ∞ and let (x_n) and (y_n) be sequences of positive constants tending to ∞ as $n \to \infty$. Then for any $\sigma > 0$,*

$$\lim_{n\to\infty} y_n^{-\sigma} \frac{L(x_n y_n)}{L(x_n)} = 0 \quad \textit{and} \quad \lim_{n\to\infty} y_n^{\sigma} \frac{L(x_n y_n)}{L(x_n)} = \infty.$$

For the proof see [10].

4. PROOFS OF THEOREMS

In this section, we give the proofs of all the theorems and corollaries stated in Section 2.

Proof of Theorem 2.1. Recall that

$$W_{j,n} = \frac{\gamma \log n}{\log\log n}\left(\frac{M_{j,n}}{V(\log n)} - 1\right), \quad j = 1, 2.$$

Since $M_{1,n} \geqslant M_{2,n}$, we have $W_{1,n} \geqslant W_{2,n}$. Given that $X_1, X_2, \ldots, X_n$ are i.i.d. with a common d.f. $F \in \mathbf{C_1}$, observe that $F(X_1), F(X_2), \ldots, F(X_n)$ are i.i.d. with a common d.f., uniform over $(0, 1)$. By putting $U_j = F(X_j)$, $j = 1, 2, \ldots, n$, one can see that $M^*_{j,n} = F(M_{j,n})$, $j = 1, 2$. First we show that the limit set of $\{W_n\}$ is contained in $S = \{(x, y) : 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant \frac{1}{2}, y \leqslant x\}$. Using Lemma 3.1, we have for any $\epsilon > 0$

$$P\left(T_{r,n} > \frac{1+\epsilon}{r} \text{ i.o.}\right) = 0, \tag{4.1}$$

and

$$P(T_{r,n} < -\epsilon \text{ i.o.}) = 0. \tag{4.2}$$

Observe that

$$\begin{aligned}\Big(T_{r,n} > \frac{1+\epsilon}{r}\Big) &= \Big(-\log(1-M^*_{r,n}) > \log n + \frac{1+\epsilon}{r}\log\log n\Big)\\ &= \Big(-\log\big(1-F(M_{r,n})\big) > \log n + \frac{1+\epsilon}{r}\log\log n\Big)\\ &= \Big(U(M_{r,n}) > \log n\Big(1+\frac{1+\epsilon}{r}\frac{\log\log n}{\log n}\Big)\Big)\\ &= \Big(M_{r,n} > V\Big(\log n\Big(1+\frac{1+\epsilon}{r}\frac{\log\log n}{\log n}\Big)\Big)\Big)\\ &= \Big(M_{r,n} - V(\log n) > V\Big(\log n\Big(1+\frac{1+\epsilon}{r}\frac{\log\log n}{\log n}\Big)\Big) - V(\log n)\Big).\end{aligned}$$

Consequently, from (2.1) (by putting $a_n = [(1+\epsilon)\log\log n]/[r\log n]$) we have

$$\begin{aligned}P\Big(T_{r,n} > \frac{1+\epsilon}{r}\text{ i.o.}\Big) &= P\Big(M_{r,n} - V(\log n) > \frac{V(\log n)}{\gamma}\frac{(1+\epsilon)\log\log n}{r(\log n)}\text{ i.o.}\Big)\\ &= P\Big(\frac{\gamma\log n}{\log\log n}\Big(\frac{M_{r,n}}{V(\log n)}-1\Big) > \frac{1+\epsilon}{r}\text{ i.o.}\Big)\\ &= P\Big(W_{r,n} > \frac{1+\epsilon}{r}\text{ i.o.}\Big).\end{aligned}$$

From (4.1) we get

$$P\Big(W_{r,n} > \frac{1+\epsilon}{r}\text{ i.o.}\Big) = 0. \tag{4.3}$$

Similarly, for any given $\epsilon > 0$,

$$\begin{aligned}(T_{r,n} < -\epsilon) &= \Big(-\log\big(1-F(M_{r,n})\big) < \log n - \epsilon\log\log n\Big)\\ &= \Big(U(M_{r,n}) < \log n\Big(1+\frac{-\epsilon\log\log n}{\log n}\Big)\Big)\\ &= \Big(M_{r,n} < V\Big(\log n\Big(1+\frac{-\epsilon\log\log n}{\log n}\Big)\Big)\Big).\end{aligned}$$

In (2.1), taking $a_n = (-\epsilon\log\log n)/\log n$ and proceeding on lines similar to those used in obtaining (4.3), one can show that

$$P(T_{r,n} < -\epsilon\text{ i.o.}) = P\Big(\frac{\gamma\log n}{\log\log n}\Big(\frac{M_{r,n}}{V(\log n)}-1\Big) < -\epsilon\text{ i.o.}\Big).$$

Consequently, from (4.2) we have

$$P(W_{r,n} < -\epsilon\text{ i.o.}) = 0. \tag{4.4}$$

Note that $M_{1,n} \geqslant M_{2,n}$ implies $W_{1,n} \geqslant W_{2,n}$. From (4.3), (4.4) and the inequality $W_{1,n} \geqslant W_{2,n}$ we infer that the set of a.s. limit points of $\{W_n\}$ is contained in the set $S = \{(x, y) : 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant \frac{1}{2}, y \leqslant x\}$.

The remaining proof consists of two parts. In part 1, we show that points $(x, y) \in S - L$ fail to be a.s. limit points of $\{W_n\}$ and in part 2 we establish that each $(x, y) \in L$ is an a.s. limit point, and hence complete the proof.

Part 1. $(x, y) \in S - L$. For any point $(x, y) \in S - L$ note that $x \geqslant y$ with $x + y > 1$. Choose $\epsilon > 0$ such that $x + y - 2\epsilon > 1$. For such a point (x, y), by Lemma 3.2 we get

$$P(T_{1,n} \geqslant x - \epsilon, T_{2,n} \geqslant y - \epsilon \text{ i.o.}) = 0. \tag{4.5}$$

Proceeding on the lines of arguments used in establishing (4.3) from (4.1), one can show that

$$P(T_{1,n} \geqslant x - \epsilon, T_{2,n} \geqslant y - \epsilon \text{ i.o.}) = P(W_{1,n} \geqslant x - \epsilon, W_{2,n} \geqslant y - \epsilon \text{ i.o.}).$$

Consequently, (4.5) implies that points $(x, y) \in S - L$ fail to be a.s. limit points of $\{W_n\}$.

Part 2. We now show that all points $(x, y) \in L$ with $x + y < 1$ are a.s. limit points of $\{W_{1,n}, W_{2,n}\}$. Define $m_k = [\exp k^{1/(x+y)}]$, $k \geqslant 1$. For any constants, ϵ and $\epsilon' > 0$ with $\epsilon' > \epsilon$, we show that

$$P\big(\{W_{1,m_k}, W_{2,m_k}\} \in (x - \epsilon, x + \epsilon') \times (y - \epsilon, y + \epsilon') \text{ i.o.}\big) = 1 \tag{4.6}$$

and complete the proof. Note that (4.6) follows, once we show that

$$P(W_{1,m_k} > x - \epsilon, W_{2,m_k} > y - \epsilon \text{ i.o.}) = 1, \tag{4.7}$$

$$P(W_{1,m_k} > x - \epsilon, W_{2,m_k} > y + \epsilon' \text{ i.o.}) = 0, \tag{4.8}$$

$$P(W_{1,m_k} > x + \epsilon', W_{2,m_k} > y - \epsilon \text{ i.o.}) = 0. \tag{4.9}$$

Proceeding as in part 1, one can show that

$$P(W_{1,m_k} > x - \epsilon, W_{2,m_k} > y - \epsilon \text{ i.o.}) = P(T_{1,m_k} > x - \epsilon, T_{2,m_k} > y - \epsilon \text{ i.o.}),$$
$$P(W_{1,m_k} > x - \epsilon, W_{2,m_k} > y + \epsilon' \text{ i.o.}) = P(T_{1,m_k} > x - \epsilon, T_{2,m_k} > y + \epsilon' \text{ i.o.}),$$
$$P(W_{1,m_k} > x + \epsilon', W_{2,m_k} > y - \epsilon \text{ i.o.}) = P(T_{1,m_k} > x + \epsilon', T_{2,m_k} > y - \epsilon \text{ i.o.}).$$

Therefore, (4.7), (4.8) and (4.9) follow from Lemma 3.3, which in turn establishes (4.6). Points $(x, y) \in L$ with $x + y = 1$, being the boundary points, will be a.s. limit points of $\{W_{1,n}, W_{2,n}\}$. ■

We now obtain an l.i.l. for $\{M_{1,n} - M_{2,n}\}$ for $F \in \mathbf{C_1}$.

Proof of Corollary 2.1. Recall that

$$\eta_n = \frac{\gamma \log n}{\log\log n} \frac{M_{1,n} - M_{2,n}}{V(\log n)}, \quad n \geqslant 3.$$

From (4.6) observe that for any $(x, y) \in L$ with $x + y < 1$ and for $\epsilon' > \epsilon > 0$,

$$P\big(\{W_{1,n}, W_{2,n}\} \in (x - \epsilon, x + \epsilon') \times (y - \epsilon, y + \epsilon') \text{ i.o.}\big) = 1. \tag{4.10}$$

Assume that $(x, y) \in L$ with $x > y$ and for such a point suppose that $\epsilon',\ \epsilon > 0$ satisfy the further condition $x - \epsilon > y + \epsilon'$. Then from (4.10) one can see that

$$P\bigg(\frac{\gamma \log n}{\log\log n} \frac{M_{1,n} - M_{2,n}}{V(\log n)} \in (x - y - \epsilon - \epsilon', x - y + \epsilon + \epsilon') \text{ i.o.}\bigg) = 1,$$

which in turn implies that for each $(x, y) \in L$ with $x > y$ the point $(x - y)$ is an a.s. limit point of $\{\eta_n\}$. For x and y close to each other, one may see that $(x - y)$ will be close to zero. Also, for x close to one and y close to zero, one can note that $(x - y)$ is close to one. Similarly, by choice of (x, y) one can get any point $\theta = (x - y) \in (0, 1)$. Thus the proof is complete. ∎

The next two theorems are for $F \in \mathbf{C_2}$. We assume that there exists an $\alpha > 0$ such that $\overline{F}(x)$ is regularly varying with exponent $(-\alpha)$, as $x \to \infty$. One may recall that F belongs to the domain of attraction of a Fréchet law if and only if $\overline{F}(x)$ is regularly varying, as $x \to \infty$.

Proof of Theorem 2.2. Given that $\{X_n\}$ is an i.i.d. sequence with a d.f. F, we know that $\{U_n = F(X_n)\}$ is a sequence of i.i.d. uniform $(0, 1)$ r.v.s and that $M^*_{r,n} = F(M_{r,n})$, $r \geqslant 1$. By Lemma 3.1, for any given $\epsilon \in (0, 1)$ we have

$$P\bigg(T_{r,n} > \frac{1 + \epsilon}{r} \text{ i.o.}\bigg) = 0, \tag{4.11}$$

$$P\bigg(T_{r,n} > \frac{1 - \epsilon}{r} \text{ i.o.}\bigg) = 1, \tag{4.12}$$

where

$$T_{r,n} = -\frac{\log(1 - M^*_{r,n}) + \log n}{\log\log n}, \quad r \geqslant 1.$$

For any $c > 0$, observe that

$$\begin{aligned}\bigg(T_{r,n} > \frac{c}{r}\bigg) &= \bigg(\log(1 - M^*_{r,n}) < -\log n - \frac{c \log\log n}{r}\bigg) \\ &= \bigg(1 - F(M_{r,n}) < \frac{1}{n(\log n)^{c/r}}\bigg).\end{aligned}$$

Let $U_1(x) = 1 - F(X)$ and $V_1(\cdot)$ be its inverse. Then

$$\left(T_{r,n} > \frac{c}{r}\right) = \left(U_1(M_{r,n}) > \frac{1}{n(\log n)^{c/r}}\right) = \left(M_{r,n} > V_1\left(\frac{1}{n(\log n)^{c/r}}\right)\right).$$

Let $U_1(x) = x^{-\alpha}L(x)$ with L an s.v. function, as $x \to \infty$. Then $V_1 = x^{-1/\alpha}l(x^{-1})$, where l is an s.v. function (for details, see [10]). Hence

$$(4.13)\quad P\left(T_{r,n} > \frac{c}{r} \text{ i.o.}\right) = P\left(M_{r,n} > n^{1/\alpha}(\log n)^{c/(r\alpha)} l\big(n(\log n)^{c/r}\big) \text{ i.o.}\right).$$

By Lemma 3.4, taking $x_n = n$, $y_n = (\log n)^{c/r}$ with $c = 1 + \epsilon$, one can show that for n large, $l\big(n(\log n)^{(1+\epsilon)/r}\big) < (\log n)^{\epsilon/(r\alpha)} l(n)$ by choosing $\delta = \epsilon/\big((1+\epsilon)\alpha\big)$. Consequently, from (4.13) one can infer that

$$(4.14)\quad P\left(T_{r,n} > \frac{1+\epsilon}{r} \text{ i.o.}\right) \geqslant P\big(M_{r,n} > n^{1/\alpha} l(n)(\log n)^{(1+2\epsilon)/(r\alpha)} \text{ i.o.}\big).$$

Since F belongs to the domain of attraction of a Fréchet law, $\{M_{1,n}/B_n\}$ converges to a Fréchet r.v., where B_n is a solution of the equation $n\big(1 - F(B_n)\big) = 1$. Also, it is well known that $B_n = n^{1/\alpha} l(n)$. From (4.14) we get

$$P\left(T_{r,n} > \frac{1+\epsilon}{r} \text{ i.o.}\right) \geqslant P\big(M_{r,n} > B_n(\log n)^{(1+2\epsilon)/(r\alpha)} \text{ i.o.}\big).$$

Now (4.11) implies that

$$(4.15)\qquad P\left(\left(\frac{M_{r,n}}{B_n}\right)^{1/\log\log n} > e^{(1+2\epsilon)/(r\alpha)} \text{ i.o.}\right) = 0.$$

Similarly, by Lemma 3.4, taking $x_n = n$, $y_n = (\log n)^{c/r}$ with $c = 1 - \epsilon$, one can show that for n large, $l\big(n(\log n)^{(1-\epsilon)/r}\big) > (\log n)^{-\epsilon/(\mu\alpha)} l(x)$. Consequently, from (4.13) we get

$$P\left(T_{r,n} > \frac{1-\epsilon}{r} \text{ i.o.}\right) = P\big(M_{r,n} > B_n(\log n)^{(1-2\epsilon)/(r\alpha)} \text{ i.o.}\big).$$

Now (4.12) implies that

$$(4.16)\quad P\big(M_{r,n} > B_n(\log n)^{(1-2\epsilon)/(r\alpha)} \text{ i.o.}\big) = 1 \quad \text{or}$$
$$P\left(\left(\frac{M_{r,n}}{B_n}\right)^{1/\log\log n} > e^{(1-2\epsilon)/(r\alpha)} \text{ i.o.}\right) = 1.$$

In view of (4.15) and (4.16) the proof of (2.3) is complete. In order to prove (2.4), one can proceed on similar lines and show that $\liminf_{n\to\infty} T_{r,n} = 0$ a.s. Then Lemma 3.1 implies the required result. ∎

Proof of Theorem 2.3. From Theorem 2.2 we can readily see that the a.s. limit set of $\{\xi_n\}$ is included in $\left[1, e^{1/\alpha}\right] \times \left[1, e^{1/(2\alpha)}\right]$. Further, since $M_{1,n} \geqslant M_{2,n}$, one can see that $\xi_{1,n} \geqslant \xi_{2,n}$. Hence the limit set is included in

$$S_1 = \{(x, y) : 1 \leqslant x \leqslant e^{1/\alpha},\ 1 \leqslant y \leqslant e^{1/(2\alpha)},\ y \leqslant x\}.$$

We now proceed to show that the a.s. limit set is

$$L_1 = \{(x, y) : 1 \leqslant x \leqslant e^{1/\alpha},\ 1 \leqslant y \leqslant e^{1/(2\alpha)},\ y \leqslant x,\ xy \leqslant e^{1/\alpha}\}.$$

First we show that $(x, y) \in S_1 - L_1$ fail to be a.s. limit points. Put $x = e^{a/\alpha}$, $y = e^{b/\alpha}$, $0 < a < 1$, $0 < b < \frac{1}{2}$. The condition $xy \leqslant e^{1/\alpha}$ is equivalent to $a + b \leqslant 1$. Hence for $(x, y) \in S_1 - L_1$ one will have $a + b > 1$. Choose $\epsilon > 0$ such that $a + b - 2\epsilon > 1$. In order to show that $(x, y) \in S_1 - L_1$ is not a limit point of $\{\xi_n\}$ one need establish that

$$P\big(\xi_{1,n} > e^{(a-\epsilon)/\alpha}, \xi_{2,n} > e^{(b-\epsilon)/\alpha} \text{ i.o.}\big) = 0$$

or, equivalently, that when $0 < a < 1$, $0 < b < \frac{1}{2}$ with $a + b > 1$,

$$P\big(M_{1,n} > B_n(\log n)^{(a-\epsilon)/\alpha}, M_{2,n} > B_n(\log n)^{(b-\epsilon)/\alpha} \text{ i.o.}\big) = 0. \tag{4.17}$$

Arguing as in (4.13), one can show that

$$P\big(M_{1,n} > B_n(\log n)^{(a-\epsilon)/\alpha}, M_{2,n} > B_n(\log n)^{(b-\epsilon)/\alpha} \text{ i.o.}\big) \leqslant P(T_{1,n} > x - \delta, T_{2,n} > y - \delta \text{ i.o.}),$$

where $x = e^{a/\alpha}$, $y = e^{b/\alpha}$ and $\delta > 0$ is such that $x + y - 2\delta > 1$. By Lemma 3.2, (4.17) is immediate.

A point $(x = e^{a/\alpha}, y = e^{b/\alpha}) \in L_1$ will be an a.s. limit point of $\{\xi_n\}$ if for any ϵ, $\epsilon' > 0$ with $\epsilon' > \epsilon$,

$$P\big(\xi_{1,n} \in (e^{(a-\epsilon)/\alpha}, e^{(a+\epsilon')/\alpha}), \xi_{2,n} \in (e^{(b-\epsilon)/\alpha}, e^{(b+\epsilon')/\alpha}) \text{ i.o.}\big) = 1. \tag{4.18}$$

We now establish that any point $(x, y) \in L_1$ with $a + b < 1$ will be an a.s. limit point of $\{\xi_n\}$. Note that (4.18) holds whenever

$$P\big(\xi_{1,n} > e^{(a-\epsilon)/\alpha}, \xi_{2,n} > e^{(b-\epsilon)/\alpha} \text{ i.o.}\big) = 1,$$
$$P\big(\xi_{1,n} > e^{(a-\epsilon)/\alpha}, \xi_{2,n} > e^{(b+\epsilon')/\alpha} \text{ i.o.}\big) = 0,$$
$$P\big(\xi_{1,n} > e^{(a+\epsilon')/\alpha}, \xi_{2,n} > e^{(b-\epsilon)/\alpha} \text{ i.o.}\big) = 0$$

or, equivalently, if

$$P\big(M_{1,n} > B_n(\log n)^{(a-\epsilon)/\alpha}, M_{2,n} > B_n(\log n)^{(b-\epsilon)/\alpha} \text{ i.o.}\big) = 1, \tag{4.19}$$

$$P\big(M_{1,n} > B_n(\log n)^{(a-\epsilon)/\alpha}, M_{2,n} > B_n(\log n)^{(b+\epsilon')/\alpha} \text{ i.o.}\big) = 0, \tag{4.20}$$

$$P\big(M_{1,n} > B_n(\log n)^{(a+\epsilon')/\alpha}, M_{2,n} > B_n(\log n)^{(b-\epsilon)/\alpha} \text{ i.o.}\big) = 0. \tag{4.21}$$

The equalities (4.19)–(4.21) can be established by proceeding on the lines of the proof of Theorem 2.1 and the details are omitted. Points $(x, y) \in L_1$ with $xy = e^{1/\alpha}$ (or $a + b = 1$) are limit points of $\{\xi_n\}$, being boundary points of L_1. ∎

Proof of Corollary 2.3. Using Theorem 2.3 and proceeding as in the proof of Corollary 2.1 one can establish the corollary. ∎

The following results are for $F \in \mathbf{C_3}$. Note that such distribution functions belong to the domain of attraction of a Weibull law.

Proof of Theorem 2.4. For $F \in \mathbf{C_3}$, note that $r(F) < \infty$. Let $\hat{M}_{j,n} = \big(r(F) - M_{j,n}\big)^{-1}$, $j = 1, 2$, and let B_n be a solution of the equation

$$n\left(1 - F\left(r(F) - \frac{1}{B_n}\right)\right) = 1.$$

Then it is known that $\{\hat{M}_{1,n}/B_n\}$ converges in distribution to a Fréchet law (see Galambos [4]). Hence by Theorem 2.3 the sequence

$$\left\{\left(\frac{\hat{M}_{1,n}}{B_n}\right)^{1/\log\log n}, \left(\frac{\hat{M}_{2,n}}{B_n}\right)^{1/\log\log n}\right\}$$

has L_1 as the set of its a.s. limit points. In other words, the sequence

$$\left\{\Big(B_n\big(r(F) - M_{1,n}\big)\Big)^{-1/\log\log n}, \Big(B_n\big(r(F) - M_{2,n}\big)\Big)^{-1/\log\log n}\right\}$$

has L_1 as the set of its a.s. limit points. Consequently, the a.s. limit set of

$$\tau_n = \left\{\Big(B_n\big(r(F) - M_{1,n}\big)\Big)^{1/\log\log n}, \Big(B_n\big(r(F) - M_{2,n}\big)\Big)^{1/\log\log n}\right\}$$

is $L_2 = \{(x, y) : e^{-1/\alpha} \leqslant x \leqslant 1,\ e^{-1/(2\alpha)} \leqslant y \leqslant 1,\ x \leqslant y,\ xy \geqslant e^{-1/\alpha}\}$. ∎

Proof of Corollary 2.3. Assume that $\big(x = e^{-a/\alpha}, y = e^{-b/(2\alpha)}\big)$ is a point with $x \leqslant y$ and $xy \geqslant e^{-1/\alpha}$. Then from Theorem 2.4 we know that (x, y) is an a.s. limit point of $\{\tau_n\}$. Consequently, for any $\epsilon > 0$,

$$P\bigg(\frac{1}{(\log n)^{(a+\epsilon)/\alpha}} \leqslant B_n\big(r(F) - M_{1,n}\big) \leqslant \frac{1}{(\log n)^{(a-\epsilon)/\alpha}}, \\ \frac{1}{(\log n)^{(b+\epsilon)/(2\alpha)}} \leqslant B_n\big(r(F) - M_{2,n}\big) \leqslant \frac{1}{(\log n)^{(b-\epsilon)/(2\alpha)}} \text{ i.o.}\bigg) = 1,$$

which in turn implies that

$$(4.22)\quad P\Big(\frac{1}{(\log n)^{(b+\epsilon)/(2\alpha)}} - \frac{1}{(\log n)^{(a-\epsilon)/\alpha}} \leqslant B_n\big(M_{1,n} - M_{2,n}\big) \leqslant \frac{1}{(\log n)^{(b-\epsilon)/(2\alpha)}} - \frac{1}{(\log n)^{(a+\epsilon)/\alpha}} \text{ i.o.}\Big) = 1.$$

Consider an arbitrary point $(x, y) \in L_2$ with $x < y$ or $a > b/2$ and choose $\epsilon > 0$ such that $a - \epsilon > (b + \epsilon)/2$. Then one can see that as $n \to \infty$,

$$\frac{1}{(\log n)^{(b+\epsilon)/(2\alpha)}} - \frac{1}{(\log n)^{(a-\epsilon)/\alpha}} \backsim \frac{1}{(\log n)^{(b+\epsilon)/(2\alpha)}}$$

and

$$\frac{1}{(\log n)^{(b-\epsilon)/(2\alpha)}} - \frac{1}{(\log n)^{(a+\epsilon)/\alpha}} \backsim \frac{1}{(\log n)^{(b-\epsilon)/(2\alpha)}}.$$

Now from (4.22) it follows that

$$P\Big(\frac{1}{(\log n)^{(b+\epsilon)/(2\alpha)}} \leqslant B_n(M_{1,n} - M_{2,n}) \leqslant \frac{1}{(\log n)^{(b-\epsilon)/(2\alpha)}} \text{ i.o.}\Big) = 1$$

or

$$P\Big(e^{-(b+\epsilon)/(2\alpha)} \leqslant \big(B_n(M_{1,n} - M_{2,n})\big)^{1/\log\log n} \leqslant e^{-(b-\epsilon)/(2\alpha)} \text{ i.o.}\Big) = 1.$$

Hence, we have shown that any point $e^{-b/(2\alpha)}$ with $0 < b < 1$ is an a.s. limit point of $\{\zeta_n\}$. Consequently, $\liminf_{n\to\infty} \zeta_n = e^{-1/(2\alpha)}$ a.s. and $\limsup_{n\to\infty} \zeta_n = 1$ a.s. and that all points in $(e^{-1/(2\alpha)}, 1)$ are a.s. limit points of $\{\zeta_n\}$. ∎

Acknowledgments. The authors thank the referees for many constructive suggestions and for bringing to their notice the paper [3].

REFERENCES

[1] O. Barndorff-Nielsen, *On the rate of growth of the partial maxima of a sequence of i.i.d. random variables*, Math. Scand. 9 (1961), pp. 383–394.
[2] J. Chover, *A law of iterated logarithm for stable summands*, Proc. Amer. Math. Soc. 17 (1967), pp. 441–443.
[3] L. Devroye, *Laws of the iterated logarithm for order statistics of uniform spacings*, Ann. Probab. 9 (5) (1981), pp. 860–867.
[4] J. Galambos, *The Asymptotic Theory of Extreme Order Statistics*, first edition, Wiley, New York 1978.
[5] A. Gut, *Limit points of sample maxima*, Statist. Probab. Lett. 9 (1990), pp. 331–336.
[6] L. de Haan and A. Hordijk, *The rate of growth of sample maxima*, Ann. Math. Statist. 43 (1972), pp. 1185–1196.
[7] P. Hall, *On the relative stability of large order statistics*, Math. Proc. Cambridge Philos. Soc. 86 (1979), pp. 467–475.

Probability and Mathematical Statistics 31, 2011, z. 2

[8] J. Kiefer, *Iterated logarithm analogues for sample quantiles when* $p \downarrow 0$, in: *Procedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability*, Vol. 1, 1971, pp. 227–244.

[9] A. G. Pakes, *The number and sum of near-maxima for thin tailed populations*, Adv. in Appl. Probab. 32 (2000), pp. 1100–1116.

[10] E. Seneta, *Regularly Varying Functions*, Lecture Notes in Math., Vol. 508, Springer, Berlin 1976.

[11] R. Vasudeva and S. Savitha, *Law of the iterated logarithm for random subsequences*, J. Indian Statist. Assoc. 30 (1992), pp. 13–24.

Department of Statistics
University of Mysore
Mysore-570006, India
E-mail: vasudeva.rasbagh@gmail.com

Department of Statistics
Payame Noor University
Manjil, Iran
E-mail: moridani@gmail.com

Received on 23.9.2010;
revised version on 8.10.2011