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q-ANALOGS OF ORDER STATISTICS

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Abstract. We introduce the notion of the q-analog of the k-th order statistics. We give a distribution and asymptotic distributions of q-analogs of the k-th order statistics and the intermediate order statistics with $r \to \infty$ and $r - k \to \infty$ in the projective geometry $PG\,(r-1, q)$.

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1. INTRODUCTION

The main results of this paper are asymptotic distributions of the q-analogs of the k-th minimal order statistics (Theorem 2.1) and the intermediate order statistics (Theorem 2.2). Theorem 2.2 generalizes the Theorem from [5] and Fact 3 from [6]. This paper is an extension (with full proofs) of the results announced in [7].

Let $GF\,(q)$ be a Galois field, where q is the power of prime. Let $V\,(r, q)$ be an r-dimensional vector space over $GF\,(q)$. There exists a one-to-one correspondence between k-dimensional subspaces of projective geometry $PG\,(r-1, q)$ and k-dimensional subspaces of the space $V\,(r, q)$. "Directions" in $V\,(r, q)$ are points of the projective geometry $PG\,(r-1, q)$ of dimension $r-1$. The subspace of dimension $k-1$ has the rank k. For example, a line has a dimension one, but it has a rank two. Let $\sigma\,(A)$ denote the subspace spanned by A, i.e. the smallest subspace including A. Let $\rho\,(A)$ denote the rank of $\sigma\,(A)$. The monograph by Hirschfeld [2] gives a detailed exposition of this subject; see also [9] or [11]. Let q be fixed and n be a nonnegative integer. We use the standard notation $[n] = (q^n - 1)/(q-1)$ (see, for example, [3] or [4]). It is well known (see [2]) that the projective geometry $PG\,(r-1, q)$ has $[r]$ elements.

Projective geometries can be defined in an axiomatic way. A *projective geometry* satisfies the following axioms:

(1) Any two distinct points are on exactly one line.

(2) Let x, y, w, z be four distinct points such that no three points are collinear. If xy intersects zw, then xz intersects yw.

(3) Each line contains at least three points.

Every geometry of dimension $r-1>2$ is isomorphic to $PG\,(r-1,q)$ defined as above.

In the area of combinatorics and special functions, a *q-analog* is a theorem or identity in the variable q that gives back a known result in the limit, as $q \to 1$. The earliest q-analog studied in detail is the basic hypergeometric series, which was introduced in the 19th century. q-analog, also called *q-extension* or *q-generalization*, is a mathematical expression parameterized by a quantity q and $[n]$ instead of n, which generalizes a known expression and reduces to a known expression in the limit $q \to 1$. Since $[n] \to n$, if (formally) $q \to 1$, then $[n]$ is the q-analog of a number n (see [1] or [10]). In the case when q is the power of prime, the subspaces of rank k in $PG\,(r-1,q)$ are q-analogs of k-element sets. In such a meaning, our results are the q-analogs of known ones in the theory of extremal order statistics.

Let a sequence of random variables $X_1, X_2, \ldots, X_n$ be given. We define the order statistics $Z_k^{(n)}$, $k=1,2,\ldots,n$, as random variables which are functions of random vector $(X_1, X_2, \ldots, X_n)$ defined as follows. For any event ω, we arrange a sequence of realizations $X_1\,(\omega)=x_1$, $X_2\,(\omega)=x_2, \ldots, X_n\,(\omega)=x_n$ in a nondecreasing sequence $z_1 \leqslant z_2 \leqslant \ldots \leqslant z_n$. In this sequence, z_k is the realization of the random variable $Z_k^{(n)}$, i.e. $Z_k^{(n)}\,(\omega)=z_k$. For a fixed integer k, the random variables $Z_k^{(n)}$ are the *k-th minimal order statistics* and the random variables $Z_{n-k+1}^{(n)}$ are the *k-th maximal order statistics*.

For the case of the projective geometry $PG\,(r-1,q)$ we shall take $n=[r]$. Let $\{X_e\}$ be independent, identically distributed random variables with distribution function $F\,(x)$ and assigned to elements of $PG\,(r-1,q)$. Let us order the elements $e_1, e_2, \ldots, e_{[r]}$ of $PG\,(r-1,q)$ so that e_i has weight Z_i. Let $(Y_1, Y_2, \ldots, Y_r)$ be a subsequence of the sequence $(Z_1, Z_2, \ldots, Z_n)$ such that $Y_i^{(n)} = Z_{k_i}^{(n)}$ (for simplicity of the notation we sometimes write Y_i or Z_{k_i}) and k_i is the least index with $e_{k_i} \notin \sigma\{e_{k_1}, e_{k_2}, \ldots, e_{k_{i-1}}\}$. Note that $k_1=1, k_2=2$, i.e. $Y_1=Z_1$, $Y_2=Z_2$ and $k_i \geqslant i$ for $i \geqslant 3$. The random variables $Y_1, Y_2, \ldots, Y_r$ will be called the *q-analogs of the order statistics.*

For the better clarity of further formulas, we consider $(k+1)$-st order statistics, $k=0,1,\ldots$, instead of k-th one, $k=1,2,\ldots$

PROPOSITION 1.1. *Let $F\,(x)$ be a distribution function of a random variable X_k. Then q-analog of the $(k+1)$-st order statistics, $k \geqslant 0$, has distribution function*

$$\Pr(Y_{k+1}^{(n)} > x) = \sum_{m=k}^{[k]} \left(\left(\prod_{l=k}^{m-1} \frac{[k]-l}{[r]-l} \right) \frac{[r]-[k]}{[r]-m} \sum_{t=0}^{m} \binom{n}{t} \big(F\,(x)\big)^t \big(1-F\,(x)\big)^{n-t} \right).$$

P r o o f. Note that

$$p_1 = \frac{[k]-l}{[r]-l} \tag{1.1}$$

is a probability that a point belongs to the space spanned by l earlier points, because $[k]$ denotes the number of elements of rank-k space, $[r]$ is a number of all elements and l means the number of earlier chosen elements. Then

$$p_2 = \prod_{l=k}^{m-1} \frac{[k]-l}{[r]-l} \tag{1.2}$$

is a probability that successively chosen points belong to a space determined by earlier chosen points, so we have to choose a next point, and

$$p_3 = \frac{[r]-[k]}{[r]-m} \tag{1.3}$$

is a probability that an m-th point does not belong to a space determined by earlier points, i.e. it spans a space of higher dimension, and

$$p_{4,t} = \binom{n}{t} \big(F(x)\big)^t \big(1-F(x)\big)^{n-t} \tag{1.4}$$

are probabilities that exactly t points have weights smaller than x. Combining (1.1), (1.2), (1.3) and (1.4) we conclude that

$$p_1 p_2 p_3 \sum_{t=1}^{m} p_{4,t} = \left(\prod_{l=k}^{m-1} \frac{[k]-l}{[r]-l}\right) \frac{[r]-[k]}{[r]-m} \sum_{t=0}^{m} \binom{n}{t} \big(F(x)\big)^t \big(1-F(x)\big)^{n-t}$$

is a probability that m is an index of a point with the smallest weight, which does not belong to rank-k space, spanned by earlier chosen points. ∎

2. LIMIT DISTRIBUTIONS

In this section, using known results concerning simple order statistics and limit distributions of random subsets of finite projective spaces, we will find limit distribution of q-analogs of order statistics.

We standardize random variables $Z_k^{(n)}$ as follows:

$$\widetilde{Z}_k^{(n)} = \frac{Z_k^{(n)} - b_n}{a_n}$$

with constants $a_n > 0$, b_n appropriately chosen, k fixed, and n increasing infinitely. N. W. Smirnov (see, for example, [8]) has shown that nondegenerate asymptotic

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distributions of the normalized k-th minimal order statistics $\widetilde{Z}_k^{(n)}$ can be of three types only:

$$\Psi_1^{(k)}(x) = 1 - P\big(k, \exp(x)\big), \quad -\infty < x < \infty, \tag{2.1}$$

$$\Psi_2^{(k)}(x) = \begin{cases} 0, & x \leqslant 0, \\ 1 - P(k, x^{\alpha}), & x \geqslant 0, \alpha > 0, \end{cases} \tag{2.2}$$

$$\Psi_3^{(k)}(x) = \begin{cases} 1 - P\big(k, (-x)^{-\alpha}\big), & x < 0, \alpha > 0, \\ 1, & x \geqslant 0, \end{cases} \tag{2.3}$$

where

$$P(k, \lambda) = \sum_{j=0}^{k-1} \frac{\lambda^j}{j!} \exp(-\lambda), \quad \lambda > 0. \tag{2.4}$$

Now we investigate a limit behaviour of a q-analog of the k-th minimal order statistics.

THEOREM 2.1. *For independent random variables with a distribution $F(x)$ a distribution of a q-analog of the k-th order statistics when $n \to \infty$ is given by*

$$\Pr\left(\frac{Y_{k+1}^{(n)} - b_n}{a_n} < x\right) \to \Psi_i(x), \quad i = 1, 2, 3, \tag{2.5}$$

where a function Ψ is defined by formulas (2.1)–(2.3).

P r o o f. Replacing x by $a_n x + b_n$ in Proposition 1.1 we get

$$\begin{aligned} &\Pr\left(\frac{Y_{k+1}^{(n)} - b_n}{a_n} > x\right) \\ &= \sum_{m=k}^{[k]} \left(\left(\prod_{l=k}^{m-1} \frac{[k]-l}{[r]-l}\right) \frac{[r]-[k]}{[r]-m} \right. \\ &\quad \left. \times \sum_{t=0}^{m} \binom{n}{t} \big(F(a_n x + b_n)\big)^t \big(1 - F(a_n x + b_n)\big)^{n-t}\right) \\ &= \sum_{m=k+1}^{[k]} \left(\left(\prod_{l=k}^{m-1} \frac{[k]-l}{[r]-l}\right) \frac{[r]-[k]}{[r]-m} + \left(\prod_{l=k}^{m-1} \frac{[k]-l}{[r]-l}\right) \frac{[r]-[k]}{[r]-k}\right) \\ &\quad \times \sum_{t=0}^{m} \binom{n}{t} \big(F(a_n x + b_n)\big)^t \big(1 - F(a_n x + b_n)\big)^{n-t}. \end{aligned} \tag{2.6}$$

Using an asymptotic distribution (see formulas (2.1)–(2.3)) and the fact that when $n = [r] \to \infty$

$$\prod_{l=k}^{m-1} \frac{[k]-l}{[r]-l} \to 0, \qquad \frac{[r]-[k]}{[r]-m} \to 1, \qquad \prod_{l=k}^{k-1} \frac{[k]-l}{[r]-l} \to 1, \qquad \frac{[r]-[k]}{[r]-k} \to 1,$$

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we get

$$\Pr\left(\frac{Y_{k+1}^{(n)} - b_n}{a_n} < x\right) \to \Psi_i(x), \quad i = 1, 2, 3. \ \blacksquare$$

For fixed k, as $n = [r] \to \infty$ the asymptotic distribution of the q-analog of the k-th order statistics coincides with the distribution of the simple k-th order statistics. This is because the number $n = [r]$ of points of the projective geometry $PG\,(r-1, q)$ is exponentially growing in $r \to \infty$ (q is fixed) so that, for $i \ll r$, the points $e_1, e_2, \ldots, e_i$ are such that each e_i is, with probability tending to one, independent of $e_1, e_2, \ldots, e_{i-1}$. Thus, for k fixed, the k-th minimal order statistics Y_k is asymptotically equal to the k-th order statistics Z_k.

It is also interesting to consider the cases when $k = k_n \to \infty$ as $n = [r] \to \infty$, which can be called the cases of *increasing ranks* (see [8]). Two particular rates of increase are of special interest:

(1) $k_n \to \infty$ and $k_n/n \to 0$, which is called the case of *intermediate ranks* (the *intermediate order statistics*);

(2) $k_n/n \sim \Theta\,(0 < \Theta < 1)$, which is called the case of *central ranks* (the *central order statistics*).

If $\{k_n\}$ is a non-decreasing intermediate order statistics sequence and there are constants $a_n > 0$ and b_n such that $\Pr\big(a_n(Z_n^{(k_n)} - b_n) \leqslant x\big) \to L(x)$ for a nondegenerate distribution L, then L has one of the three forms:

$$L_1\,(x) = \begin{cases} \Phi\big(-a\log(-x)\big), & x < 0, a > 0, \\ 1, & x \geqslant 0, \end{cases} \tag{2.7}$$

$$L_2\,(x) = \begin{cases} 0, & x \leqslant 0, a > 0, \\ \Phi(a\log x), & x > 0, \end{cases} \tag{2.8}$$

$$L_3\,(x) = \Phi(x), \quad -\infty < x < \infty, \tag{2.9}$$

where $\Phi\,(\tau)$ is a Gaussian distribution function with zero mean and variance one.

Define a discrete random process $\omega_r\,(k)$ as a Markov chain of subsets of elements of the $PG\,(r-1, q)$, which starts with empty set and for $k = 1, 2, \ldots, n = [r]$, $\omega_r\,(k)$ is obtained by addition to $\omega_r\,(k-1)$ a new, randomly chosen element of $PG\,(r-1, q)$. In [5] (see also [6]) Kordecki and Łuczak have shown that for $n = [r]$ if $k - r \to \infty$, then $\rho\big(\omega_r(k)\big) = r$ almost surely, whereas for $r - k \to \infty$ we have $\rho\big(\omega_r(k)\big) = k$ almost surely. q-analogs of the intermediate order statistics and the central order statistics ($k/r \to 0$ or $k/r \to \theta$, $0 < \theta < 1$) are expressed by the intermediate ("normal") order statistics, because then $k/n \to 0$ for $n = [r]$.

Now we define q-analogs of order statistics when $k \to \infty$. Let $Y_k^{(n)}$, where $n \to \infty$, $k \to \infty$, but $k/n \to 0$, be a *q-analog of an intermediate order statistics*. Let $Y_k^{(n)}$, where $k \to \infty$, $n \to \infty$, $k/n \sim \Theta\,(0 < \Theta < 1)$, be a *$q$-analog of a central order statistics*.

THEOREM 2.2. *For independent random variables with a distribution* $F(x)$*, a distribution of a q-analog of an intermediate order statistics*, *where* $r-k\to\infty$ *when* $n=[r]\to\infty$ *and* $k\to\infty$, *is expressed as*

$$\Pr\left(\frac{Y_{k+1}^{(n)}-b_n}{a_n}<x\right)\to L_i(x),\qquad i=1,2,3, \tag{2.10}$$

where the functions L_i *are defined by formulas* (2.7)–(2.9).

P r o o f. Consider once again the equation (2.6) in the proof of Theorem 2.1. Because first factors of the products

$$\prod_{l=k}^{m-1}\frac{[k]-l}{[r]-l}$$

are the greatest and

$$0<\prod_{l=k}^{m-1}\frac{[k]-l}{[r]-l}<1,$$

we have

$$\prod_{l=k}^{m-1}\frac{[k]-l}{[r]-l}<\frac{[k]-k}{[r]-k}.$$

Similarly, because

$$0<\frac{[r]-[k]}{[r]-m}<1,$$

we get

$$\begin{aligned}
0&<\sum_{m=k+1}^{[k]}\left(\prod_{l=k}^{m-1}\frac{[k]-l}{[r]-l}\right)\frac{[r]-[k]}{[r]-m}\\
&=\left(\prod_{l=k}^{k}\frac{[k]-l}{[r]-l}\right)\frac{[r]-[k]}{[r]-(k+1)}+\left(\prod_{l=k}^{k+1}\frac{[k]-l}{[r]-l}\right)\frac{[r]-[k]}{[r]-(k+2)}\\
&\quad+\left(\prod_{l=k}^{k+2}\frac{[k]-l}{[r]-l}\right)\frac{[r]-[k]}{[r]-(k+3)}+\ldots+\left(\prod_{l=k}^{[k]-1}\frac{[k]-l}{[r]-l}\right)\frac{[r]-[k]}{[r]-[k]}\\
&=\frac{[k]-k}{[r]-k}\,\frac{[r]-[k]}{[r]-(k+1)}\left(1+\frac{[k]-(k+1)}{[r]-(k+2)}+\frac{[k]-(k+1)}{[r]-(k+2)}\,\frac{[k]-(k+2)}{[r]-(k+3)}\right.\\
&\quad\left.+\ldots+\frac{[k]-(k+1)}{[r]-(k+2)}\,\frac{[k]-(k+2)}{[r]-(k+3)}\cdots\frac{[k]-([k]-1)}{[r]-[k]}\right).
\end{aligned}$$

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Moreover,

$$\sum_{m=k+1}^{[k]} \left(\prod_{l=k}^{m-1} \frac{[k]-l}{[r]-l} \right) \frac{[r]-[k]}{[r]-m}$$
$$< \frac{[k]-k}{[r]-k} \frac{[r]-[k]}{[r]-(k+1)} \left(1 + \frac{[k]-k}{[r]-[k]} + \left(\frac{[k]-k}{[r]-[k]} \right)^2 + \ldots + \left(\frac{[k]-k}{[r]-[k]} \right)^{[k]-k-1} \right)$$
$$= \frac{[k]-k}{[r]-k} \frac{[r]-[k]}{[r]-(k+1)} \frac{1 - \big(([k]-k)/([r]-[k]) \big)^{[k]-k-1}}{1 - ([k]-k)/([r]-[k])} \to 0$$

when $r-k \to \infty$, $r \to \infty$, $k \to \infty$, because

$$\frac{[k]-k}{[r]-k} = \frac{(q^k-1)/(q-1)-k}{(q^r-1)/(q-1)-k} \approx \frac{q^k}{q^r} = q^{k-r} \to 0,$$

and we obtain

$$\frac{[r]-[k]}{[r]-(k+1)} = \frac{1-[k]/[r]}{1-(k+1)/[r]} \to 1$$

because

$$\frac{[k]}{[r]} = \frac{(q^k-1)/(q-1)}{(q^r-1)/(q-1)} \approx \frac{q^k}{q^r} = q^{k-r} \to 0,$$
$$\frac{k+1}{[r]} = \frac{k+1}{(q^r-1)/(q-1)} \approx \frac{k}{q^{r-1}} \to 0.$$

When $n = [r] \to \infty$ we have

$$\frac{[r]-[k]}{[r]-m} \to 1, \qquad \prod_{l=k}^{[k]-1} \frac{[k]-l}{[r]-l} \to 1, \qquad \frac{[r]-[k]}{[r]-k} \to 1.$$

Then, using an asymptotic distribution (see formulas (2.7)–(2.9)), we get

$$\Pr\left(\frac{Y_{k+1}^{(n)} - b_n}{a_n} < x \right) \to L_i(x), \quad i = 1, 2, 3. \quad \blacksquare$$

Note that from the assumption that $r-k \to \infty$, $r \to \infty$, $k \to \infty$ we infer that

$$\frac{k}{n} = \frac{k}{[r]} = \frac{k}{(q^r-1)/(q-1)} \approx \frac{k}{q^{r-1}} \to 0.$$

By Theorem 2.2 we obtain another proof of Fact 3 from [6].

Theorem 2.2 solves a problem of an asymptotic distribution of a q-analog of an intermediate order statistics when $r-k \to \infty$. Problems of q-analog of asymptotic distributions for central and maximal order statistics remain unsolved.

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