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QUANTILE HEDGING FOR AN INSIDER

BY

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Abstract. In this paper we consider the problem of the quantile hedging from the point of view of a better informed agent acting on the market. The additional knowledge of the agent is modelled by a filtration initially enlarged by some random variable. By using equivalent martingale measures introduced in [1] and [2] we solve the problem for the complete case, by extending the results obtained in [4] to the insider context. Finally, we consider the examples with the explicit calculations within the standard Black–Scholes model.

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1. INTRODUCTION

A trader on the stock market is usually assumed to make his decisions relying on all the information which is generated by the market events. However, it is registered that some people have more detailed information than others, in the sense that they act with the present time knowledge of some future event. This is the so-called insider information and those dealers taking advantage of it are the insiders. The financial markets with economic agents possessing additional knowledge have been studied in a number of papers (see, e.g., [1], [2], [6], [10]). We take approach originated in [3] and [9] assuming that the insider possesses some extra information stored in the random variable G known at the beginning of the trading interval and not available to the regular trader. The typical examples of G are $G = S_{T+\delta}$, $G = \mathbf{1}_{[a,b]}(S_{T+\delta})$ or $G = \sup_{t\in[0,T+\delta]} S_t$ ($\delta > 0$), where S is a semimartingale representing the discounted stock price process and T is a fixed time horizon till which the insider is allowed to trade.

In this paper we show how much better and with which strategies an insider can perform on the market if he uses optimally the extra information he has at his disposal. The problem of pricing and perfect hedging of contingent claims is well understood in the context of arbitrage-free models which are complete. In such models every contingent claim can be replicated by a self-financing trading strategy. The cost of replication equals the discounted expectation of the claim under the unique equivalent martingale measure. Moreover, this cost is the same for the insider and the regular trader. Therefore, instead of this strategy we will employ the quantile hedging strategy of an insider for the replication, following an idea of Föllmer and Leukert [4], [5]. That is, we will seek for the self-financing strategies of two types:

1. maximizing the probability of success of hedge under a given initial capital;
2. minimizing the initial capital under a given lower bound of the probability of the successful hedge.

This is the case when the insider is unwilling to put up the initial amount of capital required by a perfect hedging. This approach might be also seen as a dynamic version of the VaR.

We use powerful technique of *grossissement de filtrations* developed by Jeulin and Yor [7], [8] and utilize the results of Amendinger [1] and Amendinger et al. [2].

The paper is organized as follows. In Section 2 we present the main results. In Section 3 we analyze in detail some examples. Finally, in Section 4 we give the proofs of the main results.

2. MAIN RESULTS

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete probability space and $S = (S_t)_{t \geqslant 0}$ be an $(\mathbb{F}, \mathbb{P})$-semimartingale representing the discounted stock price process. Assume that the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geqslant 0}$ is the natural filtration of S satisfying usual conditions with the trivial σ-algebra $\mathcal{F}_0$. Thus, the regular trader makes his portfolio decisions according to the information flow $\mathbb{F}$. In addition to the regular trader we will consider the insider, whose knowledge will be modelled by the *initial enlargement* of $\mathbb{F}$, that is filtration $\mathbb{G} = (\mathcal{G}_t)_{t \geqslant 0}$ given by

$$\mathcal{G}_t = \mathcal{F}_t \vee \sigma(G),$$

where G is an $\mathcal{F}$-measurable random variable. In particular, G can be an $\mathcal{F}_{T+\delta}$-measurable random variable ($\delta > 0$) for T being a fixed time horizon representing the expiry date of the hedged contingent claim.

We will assume that the market is complete and arbitrage-free for the regular trader, hence there exists a unique equivalent martingale measure $\mathbb{Q}_{\mathbb{F}}$ such that S is an $(\mathbb{F}, \mathbb{Q}_{\mathbb{F}})$-martingale on $[0, T]$. Denote by $(Z_t^{\mathbb{F}})_{t \in [0,T]}$ the density process of $\mathbb{Q}_{\mathbb{F}}$ with respect to $\mathbb{P}$, i.e.

$$Z_t^{\mathbb{F}} = \frac{\mathrm{d}\mathbb{Q}_{\mathbb{F}}}{\mathrm{d}\mathbb{P}}\bigg|_{\mathcal{F}_t}.$$

We will consider the contingent claim H being an $\mathcal{F}_T$-measurable, nonnegative random variable and the replicating investment strategies for insider, which are expressed in terms of the integrals with respect to S. To define them properly we assume that S is a $(\mathbb{G},\mathbb{P})$-semimartingale, which follows from the requirement:

$$\mathbb{P}(G \in \cdot|\mathcal{F}_t) \ll \mathbb{P}(G \in \cdot) \ \ \mathbb{P}\text{-a.s.} \tag{2.1}$$

for all $t \in [0,T]$ (see, e.g., [1] and [8]). In fact, we assume from now on more, that is, that the measure $\mathbb{P}(G \in \cdot|\mathcal{F}_t)$ and the law of G are equivalent for all $t \in [0,T]$:

$$\mathbb{P}(G \in \cdot|\mathcal{F}_t) \sim \mathbb{P}(G \in \cdot) \ \ \mathbb{P}\text{-a.s.} \tag{2.2}$$

Under the condition (2.2) there exists an equivalent $\mathbb{G}$-martinagle measure $\mathbb{Q}_{\mathbb{G}}$ defined by:

$$\mathbb{Q}_{\mathbb{G}}(A) = \int_A \frac{Z_T^{\mathbb{F}}}{p_T^G}(\omega)d\mathbb{P}(\omega), \qquad A \in \mathcal{G}_T, \tag{2.3}$$

where $p_t^x\ \mathbb{P}(G \in dx)$ is a version of $\mathbb{P}(G \in dx|\mathcal{F}_t)$; see [1] and [2] (and also Theorems 4.1 and 4.2).

For $\mathbb{H} \in \{\mathbb{F},\mathbb{G}\}$ we will consider only self-financing admissible trading strategies (V_0,ξ) on $[0,T]$ for which the value process

$$V_t = V_0 + \int_0^t \xi_u \, dS_u, \qquad t \in [0,T],$$

is well defined, where an initial capital $V_0 \geqslant 0$ is $\mathcal{H}_0$-measurable, a process ξ is $\mathbb{H}$-predictable and

$$V_t \geqslant 0 \ \ \mathbb{P}\text{-a.s.}$$

for all $t \in [0,T]$. Denote all admissible strategies associated with the filtration $\mathbb{H} \in \{\mathbb{F},\mathbb{G}\}$ by $\mathcal{A}^{\mathbb{H}}$.

Under the assumption (2.2) the insider can perfectly replicate the contingent claim $H \in L^1(\mathbb{Q}_{\mathbb{F}}) \cap L^1(\mathbb{Q}_{\mathbb{G}})$:

$$\mathbb{E}_{\mathbb{Q}_{\mathbb{G}}}(H|\mathcal{G}_t) = H_0 + \int_0^t \xi_u \, dS_u \ \ \mathbb{P}\text{-a.s.},$$

where $H_0 = \mathbb{E}_{\mathbb{Q}_{\mathbb{G}}}(H|\mathcal{G}_0)$. Moreover, from [1] and [2] it follows that $H_0 = \mathbb{E}_{\mathbb{Q}_{\mathbb{G}}} H$ (see Theorem 4.1). In this paper we will analyze the case when the insider is unwilling to pay the initial capital H_0 required by a perfect hedge. We will consider the following pair of dual problems.

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PROBLEM 2.1. Let α be a given $\mathcal{G}_0$-measurable random variable taking values in $[0,1]$. We are looking for a strategy $(\alpha\mathbb{E}_{\mathbb{Q}_{\mathbb{G}}}H,\xi)\in\mathcal{A}^{\mathbb{G}}$ which maximizes for any realization of G the insider's probability of a successful hedge:

$$\mathbb{P}\big(\alpha\mathbb{E}_{\mathbb{Q}_{\mathbb{G}}}H+\int_0^T\xi_t\,dS_t\geqslant H\big|\mathcal{G}_0\big)\ \ \mathbb{P}\text{-a.s.}$$

PROBLEM 2.2. Let ϵ be a given $\mathcal{G}_0$-measurable random variable taking values in $[0,1]$. We are looking for a minimal $\mathcal{G}_0$-measurable random variable α for which there exists ξ such that $(\alpha\mathbb{E}_{\mathbb{Q}_{\mathbb{G}}}H,\xi)\in\mathcal{A}^{\mathbb{G}}$ and

$$\mathbb{P}\big(\alpha\mathbb{E}_{\mathbb{Q}_{\mathbb{G}}}H+\int_0^T\xi_t\,dS_t\geqslant H\big|\mathcal{G}_0\big)\geqslant 1-\epsilon\ \ \mathbb{P}\text{-a.s.}\tag{2.4}$$

REMARK 2.1. Recall that in the quantile hedging problem for the usual trader we maximize the objective probability $\mathbb{P}\big(\alpha\mathbb{E}_{\mathbb{Q}_{\mathbb{F}}}H+\int_0^T\xi_t\,dS_t\geqslant H\big)$, where α is a number from $[0,1]$. In Problems 2.1 and 2.2 we use conditional probability, since now the insider's perception of the market at time $t=0$ depends on the knowledge described by $\mathcal{G}_0$.

The set

$$\big\{\alpha\mathbb{E}_{\mathbb{Q}_{\mathbb{G}}}H+\int_0^T\xi_t\,dS_t\geqslant H\big\}\tag{2.5}$$

will be called a *success set*. Let us put

$$\mathbb{Q}^*(A)=\frac{\mathbb{E}_{\mathbb{Q}_{\mathbb{G}}}(H\mathbf{1}_A)}{\mathbb{E}_{\mathbb{Q}_{\mathbb{G}}}(H)},\qquad A\in\mathcal{G}_T.$$

The following theorems solve Problems 2.1 and 2.2.

THEOREM 2.1. *Suppose that there exists a $\mathcal{G}_0$-measurable random variable k such that*

$$\mathbb{Q}^*\left(\frac{d\mathbb{Q}^*}{d\mathbb{P}}\bigg|_{\mathcal{G}_T}\leqslant k\bigg|\mathcal{G}_0\right)=\alpha.\tag{2.6}$$

Then the maximal probability of a success set solving Problem 2.1 *equals*:

$$\mathbb{P}\left(\frac{d\mathbb{Q}^*}{d\mathbb{P}}\bigg|_{\mathcal{G}_T}\leqslant k\bigg|\mathcal{G}_0\right)$$

and it is realized by the strategy

$$(\mathbb{E}_{\mathbb{Q}_{\mathbb{G}}}[H\mathbf{1}_{\{(d\mathbb{Q}^*/d\mathbb{P})|_{\mathcal{G}_T}\leqslant k\}}|\mathcal{G}_0],\tilde{\xi}),$$

which replicates the payoff $H\mathbf{1}_{\{(d\mathbb{Q}^*/d\mathbb{P})|_{\mathcal{G}_T}\leqslant k\}}$.

THEOREM 2.2. *Suppose that there exists a $\mathcal{G}_0$-measurable random variable k such that*

$$\mathbb{P}\left(\frac{d\mathbb{Q}^*}{d\mathbb{P}}\bigg|_{\mathcal{G}_T} \leqslant k \bigg| \mathcal{G}_0\right) = 1 - \epsilon. \tag{2.7}$$

Then the minimal $\mathcal{G}_0$-measurable random variable α solving Problem 2.2 *equals*:

$$\mathbb{Q}^*\left(\frac{d\mathbb{Q}^*}{d\mathbb{P}}\bigg|_{\mathcal{G}_T} \leqslant k \bigg| \mathcal{G}_0\right)$$

and it is realized by the strategy

$$(\mathbb{E}_{\mathbb{Q}_{\mathbb{G}}}[H\mathbf{1}_{\{(d\mathbb{Q}^*/d\mathbb{P})|_{\mathcal{G}_T}\leqslant k\}}|\mathcal{G}_0], \tilde{\xi})$$

being the perfect hedge of $H\mathbf{1}_{\{(d\mathbb{Q}^*/d\mathbb{P})|_{\mathcal{G}_T}\leqslant k\}}$.

REMARK 2.2. The assumptions that there exists k satisfying (2.6) and (2.7) are satisfied if $\mathbb{P}(Z_T^{\mathbb{F}}H = 0|\mathcal{G}_0) < \alpha$ and $\mathbb{P}(Z_T^{\mathbb{F}}H = 0|\mathcal{G}_0) < 1 - \epsilon$, respectively, and $Z_T^{\mathbb{F}}H$ has the conditional density on $\mathbb{R}_+$ given $G = g$; see Section 3 for the examples.

The proofs of Theorems 2.1 and 2.2 are given in Section 4.

3. NUMERICAL EXAMPLES

In this section we consider the standard Black–Scholes model in which the price evolution is described by the equation

$$dS_t = \sigma S_t dW_t + \mu S_t dt,$$

where W is a Brownian motion, $\sigma, \mu > 0$. For simplicity we assume that interest rate is zero. We analyze Problem 2.2 for two examples of the insider information and provide numerical results for pricing the vanilla call option, where

$$H = (S_T - K)^+$$

and K is a strike price.

3.1. The case of $G = W_{T+\delta}$. It means that insider knows the stock price $G = S_{T+\delta}$ after the expiry date T. In this case we have

$$\begin{aligned}\mathbb{P}(W_{T+\delta} \in dz|\mathcal{F}_t) &= \mathbb{P}(W_{T+\delta} - W_t + W_t \in dz|\mathcal{F}_t)\\ &= \frac{1}{\sqrt{2\pi(T+\delta-t)}}\exp\left(-\frac{(z-W_t)^2}{2(T+\delta-t)}\right)dz\\ &= p_t^z\mathbb{P}(W_{T+\delta} \in dz),\end{aligned}$$

where

$$p_t^z = \sqrt{\frac{T+\delta}{T+\delta-t}}\exp\Big(-\frac{(z-W_t)^2}{2(T+\delta-t)}+\frac{z^2}{2(T+\delta)}\Big).$$

Therefore

$$\begin{aligned}\frac{d\mathbb{Q}_{\mathbb{G}}}{d\mathbb{P}}\Big|_{\mathcal{G}_T} &= \frac{Z_T^{\mathbb{F}}}{p_T^G} = \frac{(d\mathbb{Q}_{\mathbb{F}}/d\mathbb{P})|_{\mathcal{F}_T}}{p_T^G}\\ &= \frac{\exp\big(-(\mu/\sigma)W_T-\frac{1}{2}(\mu/\sigma)^2T\big)}{\sqrt{(T+\delta)/\delta}\exp\big(-(W_{T+\delta}-W_T)^2/(2\delta)+W_{T+\delta}^2/[2(T+\delta)]\big)}\\ &= \sqrt{\frac{\delta}{T+\delta}}\exp\Big(-\frac{\mu}{\sigma}W_T-\frac{1}{2}\Big(\frac{\mu}{\sigma}\Big)^2T+\frac{(W_{T+\delta}-W_T)^2}{2\delta}-\frac{W_{T+\delta}^2}{2(T+\delta)}\Big)\end{aligned}$$

and

$$\frac{d\mathbb{Q}^*}{d\mathbb{P}}\Big|_{\mathcal{G}_T} = \frac{H}{\mathbb{E}_{\mathbb{Q}_{\mathbb{G}}}H}\,\frac{d\mathbb{Q}_{\mathbb{G}}}{d\mathbb{P}}\Big|_{\mathcal{G}_T}. \tag{3.1}$$

Note that $(d\mathbb{Q}^*/d\mathbb{P})|_{\mathcal{G}_T}$ is a random variable with the conditional density on $\mathbb{R}_+$ given $G=g$ and for given $\epsilon\in[0,1]$ we can find a $\mathcal{G}_0$-measurable random variable k such that

$$\mathbb{P}\Big(\frac{d\mathbb{Q}^*}{d\mathbb{P}}\Big|_{\mathcal{G}_T}\leqslant k\Big|\mathcal{G}_0\Big)=1-\epsilon \quad \text{if } \mathbb{P}(H=0|\mathcal{G}_0)<1-\epsilon.$$

Therefore, by Theorem 2.2 the cost of the quantile hedging for the insider can be reduced in this case by the factor

$$\alpha = \mathbb{Q}^*\Big(\frac{d\mathbb{Q}^*}{d\mathbb{P}}\Big|_{\mathcal{G}_T}\leqslant k\Big|\mathcal{G}_0\Big). \tag{3.2}$$

In Table 1 we provide the values of α for $\mu=0.08$, $\sigma=0.25$, $S_0=100$, $K=110$, $T=0.25$, $\delta=0.02$, and different values of G and ϵ. In the programme we use the simple fact that $\mathbb{E}[f(W_T)|G=g]=f\big(g-W(\delta)\big)$ for a measurable function f.

TABLE 1

		G							
		105	106	107	108	109	110	111	112
ϵ	0.01	0.05	0.09	0.13	0.17	0.22	0.27	0.32	0.37
	0.05	<0.01	0.01	0.04	0.07	0.10	0.14	0.18	0.23
	0.10	<0.01	<0.01	0.01	0.03	0.05	0.08	0.12	0.16
	0.15	<0.01	<0.01	<0.01	0.01	0.03	0.05	0.08	0.12
	0.20	<0.01	<0.01	<0.01	<0.01	0.01	0.03	0.06	0.09
	0.25	<0.01	<0.01	<0.01	<0.01	<0.01	0.02	0.04	0.07

3.2. The case of $G = \mathbf{1}_{\{W_{T+\delta} \in [a,b]\}}$. In this example the insider knows the range of the stock price $S_{T+\delta}$ after the expiry date T. The straightforward calculation yields

$$\begin{aligned}\mathbb{P}(G=1|\mathcal{F}_t) &= \frac{1}{\sqrt{2\pi(T+\delta-t)}} \int\limits_{a-W_t}^{b-W_t} \exp\Big(-\frac{u^2}{2(T+\delta-t)}\Big)\,du \\ &= \Phi\left(\frac{b-W_t}{\sqrt{T+\delta-t}}\right) - \Phi\left(\frac{a-W_t}{\sqrt{T+\delta-t}}\right),\end{aligned}$$

where Φ is c.d.f. of the standard normal distribution. Thus,

$$p_t^1 = \frac{\mathbb{P}(G=1|\mathcal{F}_t)}{\mathbb{P}(G=1)} = \frac{\Phi\big((b-W_t)/\sqrt{T+\delta-t}\big) - \Phi\big((a-W_t)/\sqrt{T+\delta-t}\big)}{\Phi(b/\sqrt{T+\delta}) - \Phi(a/\sqrt{T+\delta})},$$

and similarly

$$p_t^0 = \frac{1+\Phi\big((a-W_t)/\sqrt{T+\delta-t}\big) - \Phi\big((b-W_t)/\sqrt{T+\delta-t}\big)}{1+\Phi(a/\sqrt{T+\delta}) - \Phi(b/\sqrt{T+\delta})}.$$

Hence

$$\begin{aligned}&\frac{d\mathbb{Q}_{\mathbb{G}}}{d\mathbb{P}}\bigg|_{\mathcal{G}_T} = \frac{\exp\big(-(\mu/\sigma)W_T - (1/2)(\mu/\sigma)^2 T\big)}{p_T^0 \mathbf{1}_{\{G=0\}} + p_T^1 \mathbf{1}_{\{G=1\}}} \\ &= \exp\bigg(-\frac{\mu}{\sigma}W_T - \frac{T}{2}\Big(\frac{\mu}{\sigma}\Big)^2\bigg) \\ &\times \Bigg(\mathbf{1}_{\{G=0\}} \frac{1+\Phi\big((a-W_t)/\sqrt{T+\delta-t}\big) - \Phi\big((b-W_t)/\sqrt{T+\delta-t}\big)}{1+\Phi(a/\sqrt{T+\delta}) - \Phi(b/\sqrt{T+\delta})} \\ &+ \mathbf{1}_{\{G=1\}} \frac{\Phi\big((b-W_t)/\sqrt{T+\delta-t}\big) - \Phi\big((a-W_t)/\sqrt{T+\delta-t}\big)}{\Phi(b/\sqrt{T+\delta}) - \Phi(a/\sqrt{T+\delta})}\Bigg)^{-1}\end{aligned}$$

and $\mathbb{Q}^*$ and α are defined in (3.1) and (3.2).

TABLE 2

		$[a,b]$				
		[109,111]	[108,112]	[107,113]	[112,114]	[106,108]
ϵ	0.01	0.272	0.284	0.296	0.413	0.135
	0.05	0.142	0.150	0.157	0.277	0.039
	0.10	0.087	0.088	0.095	0.209	0.010
	0.15	0.053	0.055	0.059	0.164	0.001
	0.20	0.032	0.033	0.034	0.129	< 0.001
	0.25	0.017	0.019	0.020	0.102	< 0.001

Table 2 provides the values of the optimal α for $\mu = 0.08$, $\sigma = 0.25$, $S_0 = 100$, $K = 110$, $T = 0.25$, $\delta = 0.02$, $G = 1$ and different values of ϵ and endpoints of the interval $[a, b]$ for $S_{T+\delta}$. In the programme we use the simple fact that $\mathbb{E}[f(W_T)|G = 1] = \mathbb{E}\big[f(W_T), W_{T+\delta} \in [a, b]\big]/\mathbb{P}(G = 1)$ for a measurable function f and to simulate a numerator we choose only those trajectories for which $W_{T+\delta} \in [a, b]$.

4. PROOFS

Before we give the proofs of the main Theorems 2.1 and 2.2 we present a few introductory lemmas and theorems. We start with the result of [1] and [2] concerning the properties of the equivalent martingale measure $\mathbb{Q}_{\mathbb{G}}$ for the insider. We recall that the condition (2.2) is assumed to be satisfied.

THEOREM 4.1. (i) *The process* $Z_t^{\mathbb{G}} := Z_t^{\mathbb{F}}/p_t^G$ *is a* $(\mathbb{G}, \mathbb{P})$*-martingale.*

(ii) *The measure* $\mathbb{Q}_{\mathbb{G}}$ *defined in* (2.3) *has the following properties*:

(a) $\mathcal{F}_T$ *and* $\sigma(G)$ *are independent under* $\mathbb{Q}_{\mathbb{G}}$;

(b) $\mathbb{Q}_{\mathbb{G}} = \mathbb{Q}_{\mathbb{F}}$ *on* $(\Omega, \mathcal{F}_T)$ *and* $\mathbb{Q}_{\mathbb{G}} = \mathbb{P}$ *on* $\big(\Omega, \sigma(G)\big)$.

We are now in a position to state the theorem which relates the martingale measures of the insider and the regular trader.

THEOREM 4.2. *Let* $X = (X_t)_{t \geqslant 0}$ *be an* $\mathbb{F}$*-adapted process. The following statements are equivalent*:

(i) X *is an* $(\mathbb{F}, \mathbb{Q}_{\mathbb{F}})$*-martingale*;

(ii) X *is an* $(\mathbb{F}, \mathbb{Q}_{\mathbb{G}})$*-martingale*;

(iii) X *is a* $(\mathbb{G}, \mathbb{Q}_{\mathbb{G}})$*-martingale.*

Proof. The equivalence of (i) and (ii) follows from the fact that $\mathbb{Q}_{\mathbb{F}} = \mathbb{Q}_{\mathbb{G}}$ on $\mathcal{F}_T$. The implication (iii) $\Rightarrow$ (ii) is a consequence of the tower property of the conditional expectation. Finally, taking $A = A_s \cap \{\omega \in \Omega : G(\omega) \in B\}$ ($A_s \in \mathcal{F}_s$, B is a Borel set), we obtain the implication (ii) $\Rightarrow$ (iii) from the standard monotone class arguments and the following equalities:

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}_{\mathbb{G}}}(\mathbf{1}_A X_t) &= \mathbb{E}_{\mathbb{Q}_{\mathbb{G}}}(\mathbf{1}_{A_s}\mathbf{1}_{\{G \in B\}} X_t) = \mathbb{Q}_{\mathbb{G}}(G \in B)\mathbb{E}_{\mathbb{Q}_{\mathbb{G}}}(\mathbf{1}_{A_s} X_t)\\ &= \mathbb{Q}_{\mathbb{G}}(G \in B)\mathbb{E}_{\mathbb{Q}_{\mathbb{G}}}(\mathbf{1}_{A_s} X_s) = \mathbb{E}_{\mathbb{Q}_{\mathbb{G}}}(\mathbf{1}_A X_s), \quad s \leqslant t,\end{aligned}$$

where in the second equality we use Theorem 4.1 (ii). ■

REMARK 4.1. We need Theorem 4.2 to guarantee the martingale representation for the insider's replicating strategy. Moreover, from this representation and Theorem 4.1 we can deduce that the cost of perfect hedging for the insider is the same as for the regular trader, that is $\mathbb{E}_{\mathbb{Q}_{\mathbb{G}}}[H|\mathcal{G}_0] = \mathbb{E}_{\mathbb{Q}_{\mathbb{F}}} H$.

REMARK 4.2. In general, Theorem 4.2 is not true for the local martingales, since a localizing sequence (τ_n) of $\mathbb{G}$-stopping times is not a sequence of $\mathbb{F}$-stopping times.

LEMMA 4.1. *Let k be a positive $\mathcal{G}_t$-measurable random variable. For every $A \in \mathcal{F}$ such that*

$$\mathbb{Q}^*(A|\,\mathcal{G}_t) \leqslant \mathbb{Q}^*\left(\frac{d\mathbb{Q}^*}{d\mathbb{P}}\bigg|_{\mathcal{G}_T} \leqslant k \bigg| \mathcal{G}_t\right) \ \mathbb{P}\text{-}a.s.$$

we have

$$\mathbb{P}(A|\,\mathcal{G}_t) \leqslant \mathbb{P}\left(\frac{d\mathbb{Q}^*}{d\mathbb{P}}\bigg|_{\mathcal{G}_T} \leqslant k \bigg| \mathcal{G}_t\right) \ \mathbb{P}\text{-}a.s.$$

Similarly, *if*

$$\mathbb{P}(A|\,\mathcal{G}_t) \geqslant \mathbb{P}\left(\frac{d\mathbb{Q}^*}{d\mathbb{P}}\bigg|_{\mathcal{G}_T} \leqslant k \bigg| \mathcal{G}_t\right) \ \mathbb{P}\text{-}a.s.,$$

then

$$\mathbb{Q}^*(A|\,\mathcal{G}_t) \geqslant \mathbb{Q}^*\left(\frac{d\mathbb{Q}^*}{d\mathbb{P}}\bigg|_{\mathcal{G}_T} \leqslant k \bigg| \mathcal{G}_t\right) \ \mathbb{P}\text{-}a.s.$$

P r o o f. Let us put $\tilde{A} := \{(d\mathbb{Q}^*/d\mathbb{P})|_{\mathcal{G}_T} \leqslant k\}$. Note that

$$(\mathbf{1}_{\tilde{A}} - \mathbf{1}_A)\left(k - \frac{d\mathbb{Q}^*}{d\mathbb{P}}\bigg|_{\mathcal{G}_T}\right) \geqslant 0.$$

Thus

$$\begin{aligned}
\frac{d\mathbb{Q}^*}{d\mathbb{P}}\bigg|_{\mathcal{G}_t} \big(\mathbb{Q}^*(\tilde{A}|\mathcal{G}_t) - \mathbb{Q}^*(A|\mathcal{G}_t)\big) &= \frac{d\mathbb{Q}^*}{d\mathbb{P}}\bigg|_{\mathcal{G}_t} \mathbb{E}_{\mathbb{Q}^*}\left((\mathbf{1}_{\tilde{A}} - \mathbf{1}_A)\,|\,\mathcal{G}_t\right) \\
&= \mathbb{E}_{\mathbb{P}}\left(\frac{d\mathbb{Q}^*}{d\mathbb{P}}\bigg|_{\mathcal{G}_T} (\mathbf{1}_{\tilde{A}} - \mathbf{1}_A)\bigg|\mathcal{G}_t\right) \\
&\leqslant k\big(\mathbb{P}(\tilde{A}|\mathcal{G}_t) - \mathbb{P}(A|\mathcal{G}_t)\big),
\end{aligned}$$

which completes the proof. ■

LEMMA 4.2. *The following holds true:*

$$\frac{d\mathbb{Q}^*}{d\mathbb{Q}_{\mathbb{G}}}\bigg|_{\mathcal{G}_0} = 1.$$

P r o o f. Note that for $A = \{\omega \in \Omega : G \in B\} \in \mathcal{G}_0$ (B is a Borel set) we have

$$\begin{aligned}
\mathbb{Q}^*(A) = \mathbb{E}_{\mathbb{Q}_{\mathbb{G}}}\left[\frac{d\mathbb{Q}^*}{d\mathbb{Q}_{\mathbb{G}}}\mathbf{1}_A\right] &= \frac{\mathbb{E}_{\mathbb{Q}_{\mathbb{G}}}[H\mathbf{1}_A]}{\mathbb{E}_{\mathbb{Q}_{\mathbb{G}}}H} \\
&= \frac{\mathbb{E}_{\mathbb{Q}_{\mathbb{G}}}[H]\,\mathbb{Q}_{\mathbb{G}}(A)}{\mathbb{E}_{\mathbb{Q}_{\mathbb{G}}}H} = \mathbb{Q}_{\mathbb{G}}(A),
\end{aligned}$$

where in the last but one equality we use Theorem 4.1. ■

P r o o f o f T h e o r e m 2.1. Consider the value process $V_t = \alpha \mathbb{E}_{\mathbb{Q}_{\mathbb{G}}} H + \int_0^t \xi_u \, dS_u$ for any strategy $(\alpha \mathbb{E}_{\mathbb{Q}_{\mathbb{G}}} H, \xi) \in \mathcal{A}^{\mathbb{G}}$. Note that for its success set A defined in (2.5) we have

$$V_T \geqslant H \mathbf{1}_A.$$

Moreover, by Theorem 4.2 the process V_t is a nonnegative $(\mathbb{G}, \mathbb{Q}_{\mathbb{G}})$-local martingale, hence it is a $(\mathbb{G}, \mathbb{Q}_{\mathbb{G}})$-supermartingale and

$$\alpha \mathbb{E}_{\mathbb{Q}_{\mathbb{G}}} H = \alpha V_0 \geqslant \mathbb{E}_{\mathbb{Q}_{\mathbb{G}}} (V_T | \mathcal{G}_0) \geqslant \mathbb{E}_{\mathbb{Q}_{\mathbb{G}}} (H \mathbf{1}_A | \mathcal{G}_0).$$

Thus, from Lemmas 4.2 and 4.1 we obtain

$$\mathbb{Q}^*(A|\mathcal{G}_0) \leqslant \frac{\alpha}{(d\mathbb{Q}^*/d\mathbb{Q}_{\mathbb{G}})|_{\mathcal{G}_0}} = \alpha \ \mathbb{P}\text{-a.s.},$$

and therefore

$$\mathbb{P}(A|\mathcal{G}_0) \leqslant \mathbb{P}\left(\frac{d\mathbb{Q}^*}{d\mathbb{P}} \bigg|_{\mathcal{G}_T} \leqslant k \bigg| \mathcal{G}_0 \right) \ \mathbb{P}\text{-a.s.} \tag{4.1}$$

It remains to show that $(\mathbb{E}_{\mathbb{Q}_{\mathbb{G}}}[H \mathbf{1}_{\{(d\mathbb{Q}^*/d\mathbb{P})|_{\mathcal{G}_T} \leqslant k\}} | \mathcal{G}_0], \tilde{\xi}) \in \mathcal{A}^{\mathbb{G}}$, and that this strategy attains the upper bound (4.1). The first statement follows directly from the definition of $\tilde{\xi}$:

$$\mathbb{E}_{\mathbb{Q}_{\mathbb{G}}}(H \mathbf{1}_{\{(d\mathbb{Q}^*/d\mathbb{P})|_{\mathcal{G}_T} \leqslant k\}} | \mathcal{G}_0) + \int_0^t \tilde{\xi}_u \, dS_u = \mathbb{E}_{\mathbb{Q}_{\mathbb{G}}}(H \mathbf{1}_{\{(d\mathbb{Q}^*/d\mathbb{P})|_{\mathcal{G}_T} \leqslant k\}} | \mathcal{G}_t) \geqslant 0.$$

Moreover,

$$\begin{aligned} &\mathbb{P}\big(\mathbb{E}_{\mathbb{Q}_{\mathbb{G}}}[H \mathbf{1}_{\{(d\mathbb{Q}^*/d\mathbb{P})|_{\mathcal{G}_T} \leqslant k\}} | \mathcal{G}_0] + \int_0^T \tilde{\xi}_u \, dS_u \geqslant H \big| \mathcal{G}_0 \big) \\ &\qquad = \mathbb{P}(H \mathbf{1}_{\{(d\mathbb{Q}^*/d\mathbb{P})|_{\mathcal{G}_T} \leqslant k\}} \geqslant H | \mathcal{G}_0) \geqslant \mathbb{P}\left(\frac{d\mathbb{Q}^*}{d\mathbb{P}} \bigg|_{\mathcal{G}_T} \leqslant k \bigg| \mathcal{G}_0 \right), \end{aligned}$$

which completes the proof in view of (4.1). ■

P r o o f o f T h e o r e m 2.2. Observe that for any $(\alpha \mathbb{E}_{\mathbb{Q}_{\mathbb{G}}} H, \xi) \in \mathcal{A}^{\mathbb{G}}$ we have

$$\begin{aligned} &\mathbb{Q}^*\big(\alpha \mathbb{E}_{\mathbb{Q}_{\mathbb{G}}} H + \int_0^T \xi_u dS_u \geqslant H \big| \mathcal{G}_0 \big) \\ &\qquad = \frac{1}{(d\mathbb{Q}^*/\mathrm{d}\mathbb{Q}_{\mathbb{G}})|_{\mathcal{G}_0}} \mathbb{E}_{\mathbb{Q}_{\mathbb{G}}} \left(\frac{d\mathbb{Q}^*}{d\mathbb{Q}_{\mathbb{G}}} \bigg|_{\mathcal{G}_T} \mathbf{1}_{\{\alpha \mathbb{E}_{\mathbb{Q}_{\mathbb{G}}} H + \int_0^T \xi_u dS_u \geqslant H\}} \bigg| \mathcal{G}_0 \right) \\ &\qquad \leqslant \frac{\mathbb{E}_{\mathbb{Q}_{\mathbb{G}}} \big(\alpha \mathbb{E}_{\mathbb{Q}_{\mathbb{G}}} H + \int_0^T \xi_u dS_u \big| \mathcal{G}_0 \big)}{\mathbb{E}_{\mathbb{Q}_{\mathbb{G}}} H} = \alpha. \end{aligned}$$

Applying the second part of Lemma 4.1 for the success set

$$A = \Big\{\alpha \mathbb{E}_{\mathbb{Q}_{\mathbb{G}}} H + \int_0^T \xi_u \, dS_u \geqslant H\Big\}$$

and using the required inequality (2.4) and the definition of k given in (2.7) we derive

$$\alpha \geqslant \mathbb{Q}^*\big(\alpha \mathbb{E}_{\mathbb{Q}_{\mathbb{G}}} H + \int_0^T \xi_u \, dS_u \geqslant H \big| \mathcal{G}_0\big) \geqslant \mathbb{Q}^*\bigg(\frac{d\mathbb{Q}^*}{d\mathbb{P}}\bigg|_{\mathcal{G}_T} \leqslant k \bigg| \mathcal{G}_0\bigg). \tag{4.2}$$

We prove now that for this particular minimal choice of α being the right-hand side of (4.2) the strategy $(\mathbb{E}_{\mathbb{Q}_{\mathbb{G}}}[H\mathbf{1}_{\{(d\mathbb{Q}^*/d\mathbb{P})|_{\mathcal{G}_T} \leqslant k\}}|\mathcal{G}_0], \tilde{\xi})$ satisfies the inequality (2.4) of Problem 2.2:

$$\begin{aligned}
&\mathbb{P}\big(\alpha \mathbb{E}_{\mathbb{Q}_{\mathbb{G}}} H + \int_0^T \tilde{\xi}_u \, dS_u \geqslant H \big| \mathcal{G}_0\big) \\
&\quad = \mathbb{P}\big(\mathbb{E}_{\mathbb{Q}_{\mathbb{G}}}[H\mathbf{1}_{\{(d\mathbb{Q}^*/d\mathbb{P})|_{\mathcal{G}_T} \leqslant k\}}|\mathcal{G}_0] + \int_0^T \tilde{\xi}_u \, dS_u \geqslant H \big| \mathcal{G}_0\big) \\
&\quad = \mathbb{P}(H\mathbf{1}_{\{(d\mathbb{Q}^*/d\mathbb{P})|_{\mathcal{G}_T} \leqslant k\}} \geqslant H|\mathcal{G}_0) \geqslant \mathbb{P}\bigg(\frac{d\mathbb{Q}^*}{d\mathbb{P}}\bigg|_{\mathcal{G}_T} \leqslant k \bigg| \mathcal{G}_0\bigg) = 1 - \epsilon.
\end{aligned}$$

This completes the proof. ■

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