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GENERALIZED BACKWARD DOUBLY STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY LÉVY PROCESSES WITH NON-LIPSCHITZ COEFFICIENTS*

BY

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Abstract. We prove an existence and uniqueness result for generalized backward doubly stochastic differential equations driven by Lévy processes with non-Lipschitz assumptions.

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1. INTRODUCTION

Nonlinear backward stochastic differential equations (BSDEs in short) have been introduced by Pardoux and Peng [10]. The original motivation for the study of this kind of equations was to provide probabilistic interpretation for solutions of both parabolic and elliptic semilinear partial differential equations (see Pardoux and Peng [11], Peng [14]). Thanks to its link with finance [3], the stochastic control and stochastic game theory (see [5] and references therein), the theory of BSDEs has quickly taken a real enthusiasm since 1990.

Moreover, in order to give a probabilistic representation for a class of quasi-linear stochastic partial differential equations (SPDEs in short), Pardoux and Peng [12] considered a new kind of BSDEs, called backward doubly stochastic differential equations (BDSDEs in short). There exist two different kinds of stochastic integrals driven respectively by two independent Brownian motions. The first integral is the well-known backward Itô integral and the second is the forward one. Following it, Bally and Matoussi [1] gave the probabilistic representation of the weak solutions to parabolic semilinear SPDEs in Sobolev spaces by means of BDSDEs. Furthermore, Boufoussi et al. [2] recommended a class of generalized BDSDEs (GBDSDEs in short) which involved another integral with respect to an adapted,

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continuous and increasing process and gave the probabilistic representation for stochastic viscosity solutions of semilinear SPDEs with a Neumann boundary condition. Recently, Hu et al. [6] showed the existence and uniqueness of solutions to GBDSDEs driven by Teugel's martingales associated with Lévy process and gave probabilistic interpretation for solutions to a class of stochastic partial differential integral equations (SPDIEs in short) with a nonlinear Neumann boundary condition. These results are obtained with strong conditions on the coefficients such as Lipschitz conditions and monotony ones. Recently, N'zi and Owo [9] proved an existence and uniqueness result of solutions to BDSDEs with non-Lipschitz conditions.

Inspired by [9], the aim of this paper is to extend the study of GBDSDEs driven by Lévy processes introduced in Hu et al. [6]. We prove an existence and uniqueness result in the non-Lipschitz case.

The rest of the paper is organized as follows. In Section 2, we introduce some preliminaries and notation. Section 3 is devoted to the proof of the existence and uniqueness of the solution to GBDSDEs driven by Lévy processes with non-Lipschitz coefficients.

2. PRELIMINARIES AND NOTATION

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space on which all the processes considered in this paper are defined and let T be a fixed final time. Let $\{B_t; 0 \leqslant t \leqslant T\}$ be a standard Brownian motion with values in $\mathbb{R}$ and $\{L_t; 0 \leqslant t \leqslant T\}$ be an $\mathbb{R}$-valued Lévy process independent of $\{B_t; 0 \leqslant t \leqslant T\}$ and corresponding to a standard Lévy measure ν such that $\int_{\mathbb{R}}(1 \wedge y)\nu(dy) < \infty$. Let $\mathcal{N}$ denote the class of P-null sets of $\mathcal{F}$. For each $t \in [0,T]$, we define

$$\mathcal{F}_t \triangleq \mathcal{F}_t^L \vee \mathcal{F}_{t,T}^B,$$

where, for any process $\{\eta_t\}$, $\mathcal{F}_{s,t}^\eta = \sigma\{\eta_r - \eta_s; s \leqslant r \leqslant t\} \vee \mathcal{N}$, $\mathcal{F}_t^\eta = \mathcal{F}_{0,t}^\eta$.

Note that $\{\mathcal{F}_t^L, t \in [0,T]\}$ is an increasing filtration and $\{\mathcal{F}_{t,T}^B, t \in [0,T]\}$ is a decreasing filtration, and the collection $\{\mathcal{F}_t, t \in [0,T]\}$ is neither increasing nor decreasing so that it does not constitute a filtration.

Let ℓ^2 denote the set of real-valued sequences $x = (x^{(i)})_{i \geqslant 1}$ such that $\|x\|^2 = \sum_{i=1}^{\infty} |x^{(i)}|^2 < \infty$.

We will denote by $\mathcal{M}^2(0,T,\ell^2)$ the set of (class of $dP \otimes dt$ a.e. equal) ℓ^2-valued processes which satisfy

(i) $\|\varphi\|^2_{\mathcal{M}^2(\ell^2)} = \mathbb{E}\big(\int_0^T \|\varphi_t\|^2 dt\big) < \infty$;

(ii) φ_t is $\mathcal{F}_t$-measurable for a.e. $t \in [0,T]$.

Similarly, $\mathcal{S}^2(0,T)$ stands for the set of real-valued random processes which satisfy:

(i) $\|\varphi\|^2_{\mathcal{S}^2} = \mathbb{E}(\sup_{0 \leqslant t \leqslant T} |\varphi_t|^2) < \infty$;

(ii) φ_t is $\mathcal{F}_t$-measurable for any $t \in [0,T]$.

In the sequel, let $\{A_t; 0 \leqslant t \leqslant T\}$ be a continuous and increasing real-valued process such that A_t is $\mathcal{F}_t$-measurable for any $t \in [0,T]$ and $A_0 = 0$.

Let $\mathcal{A}^2(0,T)$ denote the set of (class of $dP \otimes dA_t$ a.e. equal) real-valued measurable random processes $\{\varphi_t; 0 \leqslant t \leqslant T\}$ such that

$$\mathbb{E}\big(\int_0^T |\varphi_t|^2 dA_t\big) < \infty.$$

We will denote by $\mathcal{E}(0,T) = \big(\mathcal{S}^2(0,T) \cap \mathcal{A}^2(0,T)\big) \times \mathcal{M}^2(0,T,\ell^2)$ the set of $(\mathbb{R} \times \ell^2)$-valued processes (Y,Z) defined on $\Omega \times [0,T]$ which satisfy the condition (ii) as above and such that

$$\|(Y,Z)\|_{\mathcal{E}}^2 = \mathbb{E}\big(\sup_{0 \leqslant t \leqslant T} |Y_t|^2 + \int_0^T |Y_s|^2 dA_s + \int_0^T \|Z_s\|^2 ds\big) < \infty.$$

$\mathcal{E}(0,T)$ endowed with the norm $\|\cdot\|_{\mathcal{E}}$ is a Banach space.

Let us denote by $(H^{(i)})_{i \geqslant 1}$ the Teugel martingale associated with the Lévy process $\{L_t; 0 \leqslant t \leqslant T\}$. More precisely,

$$H_t^{(i)} = c_{i,i} T_t^{(i)} + c_{i,i-1} T_t^{(i-1)} + \ldots + c_{i,1} T_t^{(1)},$$

where $T_t^{(i)} = L_t^{(i)} - \mathbb{E}(L_t^{(i)}) = L_t^{(i)} - t\mathbb{E}(L_1^{(i)})$ for all $i \geqslant 1$ and $L_t^{(i)}$ are power jump processes such that

$$L_t^{(1)} = L_t \quad \text{and} \quad L_t^{(i)} = \sum_{0 < s \leqslant t} (\Delta L_s)^i \text{ for } i \geqslant 2,$$

with $L_{t^-} = \lim_{s \nearrow t} L_s$ and $\Delta L_s = L_s - L_{s^-}$. Nualart and Schoutens have proved in [8] that the coefficients $c_{i,k}$ correspond to the orthonormalization of the polynomials $1,\ x,\ x^2, \ldots$ with respect to the measure $\mu(dx) = x^2 \nu(dx) + \sigma^2 \delta_0(dx)$:

$$q_i(x) = c_{i,i} x^{i-1} + c_{i,i-1} x^{i-2} + \ldots + c_{i,1}.$$

The martingale $(H^{(i)})_{i \geqslant 1}$ can be chosen to be a pairwise strongly orthonormal martingale such that, for all $i, j,\ \langle H^{(i)}, H^{(j)} \rangle_t = \delta_{ij} t$.

REMARK 2.1. If μ only has mass at 1, we are in the Poisson case N_t with parameter $\lambda > 0$; here $H_t^{(1)} = (N_t - \lambda t)/\lambda$ and $H^{(i)} = 0$, $i = 2,\ 3, \ldots$ This case is degenerate in this Lévy framework.

DEFINITION 2.1. A pair $(Y,Z) : \Omega \times [0,T] \to \mathbb{R} \times \ell^2$ of processes is called a *solution* of GBDSDE(ξ, f, g, h, A) driven by Lévy processes if $(Y,Z) \in \mathcal{E}(0,T)$ so that

$$\begin{aligned} (2.1) \qquad Y_t = \xi &+ \int_t^T f(s, Y_{s^-}, Z_s) ds + \int_t^T h(s, Y_{s^-}) dA_s + \int_t^T g(s, Y_{s^-}, Z_s) \overleftarrow{dB_s} \\ &- \sum_{i=1}^{\infty} \int_t^T Z_s^{(i)} dH_s^{(i)}, \quad t \in [0,T]. \end{aligned}$$

Here the integral with respect to $\{B_t\}$ is the classical backward Itô integral (see Kunita [7]) and the integral with respect to $\{(H_t^{(i)})_{i\geqslant 1}\}$ is a standard forward Itô-type semimartingale integral (see Gong [4]).

First, let us recall the extension of the well-known Itô formula on which our results depend strongly. Its proof goes the same lines as that of Lemma 2.5 in [2] or Lemma 1.3 in [12].

LEMMA 2.1. *Let α, β and γ in $\mathcal{S}^2(0,T)$, $\eta \in \mathcal{A}^2(0,T)$ and $\zeta \in \mathcal{M}^2(0,T,\ell^2)$ satisfy*

$$\alpha_t = \alpha_T + \int_t^T \beta_s ds + \int_t^T \eta_s dA_s + \int_t^T \gamma_s dB_s - \sum_{i=1}^{\infty} \int_t^T \zeta_s^{(i)} dH_s^{(i)}, \quad t \in [0,T].$$

Then

$$\begin{aligned} |\alpha_t|^2 = |\alpha_T|^2 &+ 2\int_t^T \alpha_s\beta_s ds + 2\int_t^T \alpha_s\eta_s dA_s + 2\int_t^T \alpha_s\gamma_s dB_s \\ &- 2\sum_{i=1}^{\infty}\int_t^T \alpha_s\zeta_s^i dH_s^{(i)} + \int_t^T |\gamma_s|^2 ds - \sum_{i,j=1}^{\infty}\int_t^T \zeta_s^i\zeta_s^j d[H_s^{(i)}, H_s^{(j)}]. \end{aligned}$$

Note that $\big(\int_t^T \alpha_s\gamma_s dB_s\big)_{0\leqslant t\leqslant T}$, $\big(\int_0^t \alpha_s\zeta_s^{(i)} dH_s^{(i)}\big)_{0\leqslant t\leqslant T}$ for all $i \geqslant 1$ and $\big(\int_0^t \zeta_s^{(i)}\zeta_s^{(j)} d[H_s^{(i)}, H_s^{(j)}]\big)_{0\leqslant t\leqslant T}$ for $i \neq j$ are uniformly integrable martingales and $\langle H^{(i)}, H^{(j)}\rangle_t = \delta_{ij}t$. We have

$$\begin{aligned} \mathbb{E}|\alpha_t|^2 = \mathbb{E}|\alpha_T|^2 &+ 2\mathbb{E}\int_t^T \alpha_s\beta_s ds + 2\mathbb{E}\int_t^T \alpha_s\eta_s dA_s + \mathbb{E}\int_t^T |\gamma_s|^2 ds \\ &- \mathbb{E}\Big(\int_t^T \sum_{i=1}^{\infty} |\zeta_s^{(i)}|^2 ds\Big), \quad t \in [0,T]. \end{aligned}$$

Next, we recall the existence and uniqueness result on GBDSDE(ξ, f, g, h, A) in the Lipschitz and monotony context. This result is due to Hu et al. in [6], where the following assumptions are used:

(A1) The terminal value $\xi \in \mathrm{L}^2(\Omega, \mathcal{F}_T, \mathbb{P}, \mathbb{R})$ is such that for all $\lambda > 0$

$$\mathbb{E}\big(\exp(\lambda A_T)|\xi|^2\big) < \infty.$$

(A2) The coefficients $f, g\colon \Omega \times [0,T] \times \mathbb{R} \times \ell^2 \to \mathbb{R}$ and $h\colon \Omega \times [0,T] \times \times\mathbb{R} \to \mathbb{R}$ satisfy, for some $\beta_1 \in \mathbb{R}$, $K > 0$, $0 < \alpha < 1$ and $\beta_2 < 0$, three $\mathcal{F}_t$-adapted processes $\{f_t, g_t, h_t : 0 \leqslant t \leqslant T\}$ with values in $[1,\infty[$ and for all $(t,y,z) \in [0,T] \times \mathbb{R} \times \ell^2$, $\lambda > 0$:

(i) $f(\cdot,y,z), g(\cdot,y,z)$ and $h(\cdot,y)$ are jointly measurable;

(ii) $|f(t,y,z)| \leqslant f_t + K(|y| + \|z\|)$, $|g(t,y,z)| \leqslant g_t + K(|y| + \|z\|)$ and $|h(t,y)| \leqslant h_t + K|y|$;

(iii) $\mathbb{E}\big(\int_0^T \exp(\lambda A_t) f_t^2 dt + \int_0^T \exp(\lambda A_t) g_t^2 dt + \int_0^T \exp(\lambda A_t) h_t^2 dt\big) < \infty$;

(iv) $\langle y - y', f(t,y,z) - f(t,y',z)\rangle \leqslant \beta_1 |y - y'|^2$;

(v) $|f(t,y,z) - f(t,y,z')|^2 \leqslant K\|z - z'\|^2$;

(vi) $\langle y - y', h(t,y) - h(t,y')\rangle \leqslant \beta_2 |y - y'|^2$;

(vii) $|g(t,y,z) - g(t,y',z')|^2 \leqslant K|y - y'|^2 + \alpha\|z - z'\|^2$;

(viii) $y \mapsto \big(f(t,y,z), g(t,y,z), h(t,y)\big)$ is continuous for all z, (ω, t).

(A3) $|f(t,y,z) - f(t,y',z)|^2 + |h(t,y) - h(t,y')|^2 \leqslant K|y - y'|^2$.

LEMMA 2.2 (Hu et al. [6]). *Under the assumptions* (A1), (A2) *and* (A3), *the GBDSDE*(ξ, f, g, h, A) *has a unique solution.*

3. EXISTENCE AND UNIQUENESS RESULT IN THE NON-LIPSCHITZ CASE

In order to attain the solution of GBDSDE(ξ, f, g, h, A), we assume the following assumptions. The coefficients $f, g : \Omega \times [0,T] \times \mathbb{R} \times \ell^2 \to \mathbb{R}$, $h : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ and the terminal value ξ satisfy:

(H1) $f(\cdot, y, z), g(\cdot, y, z)$ and $h(\cdot, y)$ are jointly measurable such that

$$0 < \mathbb{E}\Big(\int_0^T |f(s,0,0)|^2\, ds + \int_0^T |h(s,0)|^2\, dA_s + \int_0^T |g(s,0,0)|^2\, ds\Big) < \infty.$$

(H2) For some $K > 0$ and three $\mathcal{F}_t$-measurable processes $\{f_t, g_t, h_t \colon 0 \leqslant t \leqslant T\}$ with values in $[1, \infty[$ and for all $(t, y, z) \in [0,T] \times \mathbb{R} \times \ell^2$, $\lambda > 0$:

$$|f(t,y,z)| \leqslant f_t + K(|y| + \|z\|), \quad |g(t,y,z)| \leqslant g_t + K(|y| + \|z\|),$$
$$|h(t,y)| \leqslant h_t + K|y|,$$
$$\mathbb{E}\Big(\int_0^T \exp(\lambda A_t) f_t^2 dt + \int_0^T \exp(\lambda A_t) g_t^2 dt + \int_0^T \exp(\lambda A_t) h_t^2 dt\Big) < \infty.$$

(H3) For some $\beta < 0$ and for all $y_1, y_2 \in \mathbb{R}$ and $t \in [0,T]$,

$$\langle y_1 - y_2, h(t,y_1) - h(t,y_2)\rangle \leqslant \beta |y_1 - y_2|^2.$$

(H4) For all $(y_1, z_1), (y_2, z_2) \in \mathbb{R} \times \ell^2$ and $t \in [0,T]$,

$$|h(t,y_1) - h(t,y_2)| \leqslant K|y_1 - y_2|,$$
$$|f(t,y_1,z_1) - f(t,y_2,z_2)|^2 \leqslant \rho(t, |y_1 - y_2|^2) + C\|z_1 - z_2\|^2,$$
$$|g(t,y_1,z_1) - g(t,y_2,z_2)|^2 \leqslant \rho(t, |y_1 - y_2|^2) + \alpha\|z_1 - z_2\|^2,$$

where $C > 0$ and $0 < \alpha < 1$ are two constants and $\rho : [0,T] \times \mathbb{R}^+ \to \mathbb{R}^+$ satisfies:

(i) for fixed $t \in [0,T]$, $\rho(t,\cdot)$ is a concave and nondecreasing function such that $\rho(t,0)=0$;

(ii) for fixed u, $\int_0^T \rho(t,u)dt < +\infty$;

(iii) for any $M>0$, the following ODE

$$u' = -M\rho(t,u),$$
$$u(T)=0$$

has a unique solution $u(t)\equiv 0$, $t\in[0,T]$.

(H5) $\xi \in \mathrm{L}^2(\Omega, \mathcal{F}_T, \mathbb{P}, \mathbb{R})$ is such that for all $\lambda>0$

$$\mathbb{E}\big(\exp(\lambda A_T)|\xi|^2\big) < \infty.$$

Under the above assumptions, we now construct an approximate sequence using a Picard-type iteration with the help of Lemma 2.2. Let $Y_t^0 = 0$, $(Y^n, Z^n)_{n\geqslant 1}$ be a sequence in $\mathcal{E}^2(0,T)$ defined recursively by

$$\text{(3.1)} \qquad Y_t^n = \xi + \int_t^T f(s, Y_s^{n-1}, Z_s^n)ds + \int_t^T h(s, Y_s^n)dA_s + \int_t^T g(s, Y_s^{n-1}, Z_s^n)\overleftarrow{dB_s} - \sum_{i=1}^{\infty}\int_t^T Z_s^{n(i)} dH_s^{(i)}.$$

Indeed, for each $n\geqslant 1$ and fixed Y^{n-1} in $\mathcal{S}^2(0,T)$, BDSDE (3.1) satisfies the assumptions (A1), (A2) and (A3). So, by Lemma 2.2, the BDSDE (3.1) has a unique solution $(Y^n, Z^n)\in\mathcal{E}^2(0,T)$.

Our purpose is to prove that the sequence $(Y^n, Z^n)_{n\geqslant 0}$ converges in $\mathcal{E}^2(0,T)$ to (Y,Z) which is the unique solution of BDSDEs (2.1). We begin with some preliminaries results.

LEMMA 3.1. *Let the assumptions* (H1), (H3) *and* (H4) *be satisfied. Then for all* $0\leqslant t\leqslant T$, $n,m\geqslant 1$, *we have*

$$\mathbb{E}|Y_t^{n+m} - Y_t^n|^2 \leqslant \exp\left(\frac{CT}{1-\alpha}\right)\left(\frac{1-\alpha}{C}+1\right)\int_t^T \rho(s, \mathbb{E}|Y_s^{n+m-1} - Y_s^{n-1}|^2)ds.$$

Proof. By Itô's formula, we have

$$\begin{aligned}
&\mathbb{E}|Y_t^{n+m} - Y_t^n|^2 + \mathbb{E}\int_t^T \|Z_s^{n+m} - Z_s^n\|^2 ds \\
&= 2\mathbb{E}\int_t^T \langle Y_s^{n+m} - Y_s^n, f(s, Y_s^{n+m-1}, Z_s^{n+m}) - f(s, Y_s^{n-1}, Z_s^n)\rangle ds \\
&\quad + 2\mathbb{E}\int_t^T \langle Y_s^{n+m} - Y_s^n, h(s, Y_s^{n+m}) - h(s, Y_s^n)\rangle dA_s \\
&\quad + \mathbb{E}\int_t^T |g(s, Y_s^{n+m-1}, Z_s^{n+m}) - g(s, Y_s^{n-1}, Z_s^n)|^2 ds.
\end{aligned}$$

Using (H3) and Young's inequality $2ab \leqslant \theta^{-1}a^2 + \theta b^2$ for any $\theta > 0$, we have

$$\begin{aligned}
&\mathbb{E}|Y_t^{n+m} - Y_t^n|^2 + \mathbb{E}\int_t^T \|Z_s^{n+m} - Z_s^n\|^2 ds + 2|\beta|\mathbb{E}\int_t^T |Y_s^{n+m} - Y_s^n|^2 dA_s \\
&\quad\leqslant \frac{1}{\theta}\mathbb{E}\int_t^T |Y_s^{n+m} - Y_s^n|^2 ds + (\theta+1)\mathbb{E}\int_t^T \rho(s, |Y_s^{n+m-1} - Y_s^{n-1}|^2) ds \\
&\qquad + (\theta C + \alpha)\mathbb{E}\int_t^T \|Z_s^{n+m} - Z_s^n\|^2 ds.
\end{aligned}$$

Choosing $\theta = (1-\alpha)/C > 0$, we infer from Gronwall's inequality and Jensen's inequality that

$$\begin{aligned}
&\mathbb{E}|Y_t^{n+m} - Y_t^n|^2 \\
&\qquad \leqslant \exp\Big(\frac{CT}{1-\alpha}\Big)\Big(\frac{1-\alpha}{C} + 1\Big)\int_t^T \rho(s, \mathbb{E}|Y_s^{n+m-1} - Y_s^{n-1}|^2) ds. \quad \blacksquare
\end{aligned}$$

LEMMA 3.2. *Let the assumptions* (H1), (H3) *and* (H4) *be satisfied. Then there exists* $T_1 \in [0, T[$ *and a constant* $M_1 \geqslant 0$ *such that, for all* $t \in [T_1, T]$ *and each* $n \geqslant 1$, $\mathbb{E}\,|Y_t^n|^2 \leqslant M_1$.

Proof. By Itô's formula, we have

$$\begin{aligned}
&\mathbb{E}\,|Y_t^n|^2 + \mathbb{E}\int_t^T \|Z_s^n\|^2 ds \\
&\quad = \mathbb{E}\,|\xi|^2 + 2\mathbb{E}\int_t^T \langle Y_s^n, f(s, Y_s^{n-1}, Z_s^n)\rangle ds + 2\mathbb{E}\int_t^T \langle Y_s^n, h(s, Y_s^n)\rangle\, dA_s \\
&\qquad + \mathbb{E}\int_t^T |g(s, Y_s^{n-1}, Z_s^n)|^2 ds.
\end{aligned}$$

Using (H3), (H4) and Young's inequality $2ab \leqslant \theta^{-1}a^2 + \theta b^2$ for any $\theta > 0$, we have

$$\begin{aligned}
2\langle Y_s^n, f(s, Y_s^{n-1}, Z_s^n)\rangle &\leqslant \frac{1}{\theta}|Y_s^n|^2 + \theta|f(s, Y_s^{n-1}, Z_s^n)|^2 \\
&\leqslant \frac{1}{\theta}|Y_s^n|^2 + 2\theta\rho(s, |Y_s^{n-1}|^2) + 2\theta C\|Z_s^n\|^2 + 2\theta|f(s,0,0)|^2,
\end{aligned}$$

$$2\,\langle Y_s^n, h(s, Y_s^n)\rangle \leqslant 2\beta\,|Y_s^n|^2 + 2\,\langle Y_s^n, h(s,0)\rangle \leqslant -|\beta|\,|Y_s^n|^2 + \frac{1}{|\beta|}\,|h(s,0)|^2,$$

and

$$|g(s,Y_s^{n-1},Z_s^n)|^2 \leqslant (1+\theta)\rho(s,|Y_s^{n-1}|^2)+(1+\theta)\alpha\|Z_s^n\|^2+\left(1+\frac{1}{\theta}\right)|g(s,0,0)|^2.$$

Therefore,

$$\mathbb{E}|Y_t^n|^2+[1-2\theta C-(1+\theta)\alpha]\mathbb{E}\int_t^T \|Z_s^n\|^2ds+|\beta|\mathbb{E}\int_t^T|Y_s^n|^2dA_s$$
$$\leqslant \mathbb{E}|\xi|^2+\frac{1}{\theta}\mathbb{E}\int_t^T|Y_s^n|^2ds+(3\theta+1)\int_t^T\rho(s,\mathbb{E}|Y_s^{n-1}|^2)ds$$
$$+\mathbb{E}\int_t^T\left[2\theta|f(s,0,0)|^2+\left(1+\frac{1}{\theta}\right)|g(s,0,0)|^2\right]ds+\frac{1}{|\beta|}\mathbb{E}\int_t^T|h(s,0)|^2dA_s.$$

We choose $\theta=(1-\alpha)/(2C+\alpha)>0$; then

$$\mathbb{E}\,|Y_t^n|^2\leqslant \mathbb{E}|\xi|^2+\frac{2C+\alpha}{1-\alpha}\mathbb{E}\int_t^T|Y_s^n|^2\,ds$$
$$+\left(3\frac{1-\alpha}{2C+\alpha}+1\right)\int_t^T\rho(s,\mathbb{E}|Y_s^{n-1}|^2)ds$$
$$+\mathbb{E}\int_t^T\left[\frac{2(1-\alpha)}{2C+\alpha}\,|f(s,0,0)|^2+\left(\frac{1+2C}{1-\alpha}\right)|g(s,0,0)|^2\right]ds$$
$$+\frac{1}{|\beta|}\mathbb{E}\int_t^T|h(s,0)|^2\,dA_s.$$

Now, in view of Gronwall's inequality, we derive

$$\mathbb{E}\,|Y_t^n|^2\leqslant \mu_t^1+\left(3\frac{1-\alpha}{2C+\alpha}+1\right)\exp\left(\frac{(2C+\alpha)T}{1-\alpha}\right)\int_t^T\rho(s,\mathbb{E}|Y_s^{n-1}|^2)ds, \tag{3.2}$$

where

$$\mu_t^1=\exp\left(\frac{(2C+\alpha)T}{1-\alpha}\right)$$
$$\times\left(\mathbb{E}\,|\xi|^2+\mathbb{E}\int_t^T\left[\frac{2(1-\alpha)}{2C+\alpha}\,|f(s,0,0)|^2+\left(\frac{1+2C}{1-\alpha}\right)|g(s,0,0)|^2\right]ds\right.$$
$$\left.+\frac{1}{|\beta|}\mathbb{E}\int_t^T|h(s,0)|^2\,dA_s\right).$$

Let

$$M= \max\left\{\left(3\frac{1-\alpha}{2C+\alpha}+1\right)\exp\left(\frac{(2C+\alpha)T}{1-\alpha}\right),\left(\frac{1-\alpha}{C}+1\right)\exp\left(\frac{CT}{1-\alpha}\right)\right\}>0 \tag{3.3}$$

and

$$M_1 = 2\mu_0^1 = 2\exp\Big(\frac{(2C+\alpha)T}{1-\alpha}\Big)\Big(\mathbb{E}\,|\xi|^2 + \mathbb{E}\int_0^T \Big[\frac{2(1-\alpha)}{2C+\alpha}\,|f(s,0,0)|^2 + \Big(\frac{1+2C}{1-\alpha}\Big)\,|g(s,0,0)|^2\Big]ds + \frac{1}{|\beta|}\mathbb{E}\int_0^T |h(s,0)|^2\,dA_s\Big) \geqslant 0.$$

From (H4) we obtain $\int_0^T \rho(s, M_1)ds < +\infty$, so we can find T_1 such that

$$\int_{T_1}^T \rho(s, M_1)ds \leqslant \frac{\mu_0^1}{M}.$$

Now, we complete the proof as in N'zi and Owo [9]. ∎

Using the above lemmas, we can now prove the existence and uniqueness, which is our main result.

THEOREM 3.1. *Let the assumptions* (H1)–(H5) *be satisfied. Then the equation* (2.1) *has a unique solution* $(Y, Z) \in \mathcal{E}^2(0, T)$.

Proof. Existence. For all $n \geqslant 1$, and $t \in [0, T]$, we let

$$\phi_0(t) = M\int_t^T \rho(s, M_1)ds \quad \text{and} \quad \phi_{n+1}(t) = M\int_t^T \rho\big(s, \phi_n(s)\big)ds.$$

N'zi and Owo proved in [9] that $\big(\phi_n(t)\big)_{n\geqslant 0}$ is nonincreasing and converges uniformly to 0 for all $t \in [T_1, T]$. From Lemmas 3.1 and 3.2 we conclude as in [9] that for all $t \in [T_1, T]$, $n, m \geqslant 1$,

$$\mathbb{E}|Y_t^{n+m} - Y_t^n|^2 \leqslant \phi_{n-1}(t) \leqslant M_1. \tag{3.4}$$

Using Itô's formula, we deduce from the assumptions (H3) and (H4) and Young's inequality $2ab \leqslant \theta^{-1}a^2 + \theta b^2,\ \theta > 0$, that for all $t \in [T_1, T]$

$$(\star)\quad |Y_t^{n+m} - Y_t^n|^2 - (\theta C + \alpha)\int_t^T \|Z_s^{n+m} - Z_s^n\|^2 ds + 2|\beta|\int_t^T |Y_s^{n+m} - Y_s^n|^2 dA_s$$
$$\leqslant \frac{1}{\theta}\int_t^T |Y_s^{n+m} - Y_s^n|^2 ds + (\theta+1)\int_t^T \rho(s, |Y_s^{n+m-1} - Y_s^{n-1}|^2)ds$$
$$+ 2\int_t^T \big\langle Y_s^{n+m} - Y_s^n, \big(g(s, Y_s^{n+m-1}, Z_s^{n+m}) - g(s, Y_s^{n-1}, Z_s^n)\big)\overleftarrow{dB_s}\big\rangle$$
$$- 2\sum_{i,j=1}^{\infty}\int_t^T \langle Y_s^{n+m} - Y_s^n, Z_s^{(n+m)(i)} - Z_s^{n(i)}\rangle dH_s^{(i)} - \sum_{i,j=1}^{\infty}\int_t^T Z_s^{n(i)} Z_s^{n(j)} d[H_s^i, H_s^j].$$

Note that the local martingales

$$\Big(\int_t^T \langle Y_s^{n+m} - Y_s^n, \big(g(s, Y_s^{n+m-1}, Z_s^{n+m}) - g(s, Y_s^{n-1}, Z_s^n)\big)\overleftarrow{dB_s}\rangle\Big)_{0\leqslant t\leqslant T},$$

$$\big(\int_t^T \langle Y_s^{n+m} - Y_s^n, Z_s^{(n+m)(i)} - Z_s^{n(i)}\rangle dH_s^{(i)}\big)_{0\leqslant t\leqslant T} \qquad \text{for all } i \geqslant 1$$

and

$$\big(\int_t^T Z_s^{n(i)} Z_s^{n(j)} d[H_s^i, H_s^j]\big)_{0\leqslant t\leqslant T} \qquad \text{for } i \neq j$$

are uniformly integrable, so that they are martingales. Therefore, taking expectation in ($\star$), it follows from Jensen's inequality and inequality (3.4) that

$$\begin{aligned}
&\mathbb{E}|Y_t^{n+m} - Y_t^n|^2 + (1-\theta C-\alpha)\mathbb{E}\int_t^T \|Z_s^{n+m} - Z_s^n\|^2 ds \\
&\quad + 2|\beta|\mathbb{E}\int_t^T |Y_s^{n+m} - Y_s^n|^2 dA_s \\
&\quad \leqslant \frac{1}{\theta}\mathbb{E}\int_t^T |Y_s^{n+m} - Y_s^n|^2 ds + (\theta+1)\int_t^T \rho(s, \mathbb{E}|Y_s^{n+m-1} - Y_s^{n-1}|^2) ds \\
&\quad \leqslant \frac{1}{\theta}\int_t^T \phi_{n-1}(s) ds + \frac{\theta+1}{M}\phi_{n-1}(t).
\end{aligned}$$

Thus, choosing $\theta = (1-\alpha)/2C$, we get

$$\begin{aligned}
&\sup_{T_1\leqslant t\leqslant T}(\mathbb{E}|Y_t^{n+m} - Y_t^n|^2) + \frac{1-\alpha}{2}\mathbb{E}\int_{T_1}^T \|Z_s^{n+m} - Z_s^n\|^2 ds \\
&\qquad + 2|\beta|\mathbb{E}\int_{T_1}^T |Y_s^{n+m} - Y_s^n|^2 dA_s \leqslant \Big(\frac{T-T_1}{\theta} + \frac{\theta+1}{M}\Big)\phi_{n-1}(T_1),
\end{aligned}$$

from which we deduce by Burkhölder–Davis–Gundy's inequality that

$$\begin{aligned}
\mathbb{E}(\sup_{T_1\leqslant t\leqslant T}|Y_t^{n+m} - Y_t^n|^2) + \mathbb{E}\int_{T_1}^T \|Z_s^{n+m} - Z_s^n\|^2 ds + \mathbb{E}\int_{T_1}^T |Y_s^{n+m} - Y_s^n|^2 dA_s& \\
\leqslant K\phi_{n-1}(T_1),&
\end{aligned}$$

where K is a positive constant dependent on C, T_1, T, α, $|\beta|$ and M. Since $\phi_n(t) \to 0$ for all $t \in [T_1, T]$ as $n \to \infty$, it follows that (Y^n, Z^n) is a Cauchy sequence in $\mathcal{E}^2(T_1, T)$. Let us set

$$Y = \lim_{n\to+\infty} Y^n, \qquad Z = \lim_{n\to+\infty} Z^n.$$

Then, since $\mathcal{E}^2(T_1,T)$ is a Banach space, $(Y,Z)\in\mathcal{E}^2(T_1,T)$. Passing to the limit in (3.1), we prove that (Y,Z) satisfies the BDSDE (2.1) on $[T_1,T]$.

If $T_1=0$, then we have proved the existence result. If $T_1\neq 0$, we consider the following equation:

$$(3.5)\quad Y_t = Y_{T_1} + \int_t^{T_1} f(s,Y_{s^-},Z_s)ds + \int_t^{T_1} h(s,Y_{s^-})dA_s + \int_t^{T_1} g(s,Y_{s^-},Z_s)\overleftarrow{dB_s} - \sum_{i=1}^{\infty}\int_t^{T_1} Z_s^{(i)}dH_s^{(i)}, \quad t\in[0,T_1].$$

We construct the Picard approximate sequence of the equation (3.5), as in (3.1). Using the same procedure as in the proof of Lemmas 3.1 and 3.2, for all $t\in[T_1,T]$, $n,m\geqslant 1$, we establish that

$$\mathbb{E}|Y_t^{n+m}-Y_t^n|^2 \leqslant \exp\Big(\frac{CT}{1-\alpha}\Big)\Big(\frac{1-\alpha}{C}+1\Big)\int_t^{T_1}\rho(s,\mathbb{E}|Y_s^{n+m-1}-Y_s^{n-1}|^2)ds,$$

and

$$\mathbb{E}\,|Y_t^n|^2 \leqslant \mu_t^2 + M\int_t^{T_1}\rho(s,\mathbb{E}|Y_s^{n-1}|^2)ds,$$

where

$$\mu_t^2 = \exp\Big(\frac{(2C+\alpha)T}{1-\alpha}\Big)\Big(\mathbb{E}\,|Y_{T_1}|^2 + \mathbb{E}\int_t^T\Big[\frac{2(1-\alpha)}{2C+\alpha}\,|f(s,0,0)|^2 + \Big(\frac{1+2C}{1-\alpha}\Big)|g(s,0,0)|^2\Big]ds + \frac{1}{|\beta|}\mathbb{E}\int_t^T|h(s,0)|^2\,dA_s\Big).$$

Let

$$M_2 = 2\mu_0^2 = 2\exp\Big(\frac{(2C+\alpha)T}{1-\alpha}\Big)\Big(\mathbb{E}\,|Y_{T_1}|^2 + \mathbb{E}\int_0^T\Big[\frac{2(1-\alpha)}{2C+\alpha}\,|f(s,0,0)|^2 + \Big(\frac{1+2C}{1-\alpha}\Big)|g(s,0,0)|^2\Big]ds + \frac{1}{|\beta|}\mathbb{E}\int_0^T|h(s,0)|^2\,dA_s\Big).$$

We can also find $T_2\in[0,T_1[$ such that

$$\mathbb{E}\,|Y_t^n|^2 \leqslant M_2, \qquad n\geqslant 1,\ t\in[T_2,T_1].$$

Here $T_2=0$ or $T_2\in]0,T_1[$ are such that $\int_{T_2}^{T_1}\rho(s,M_2)ds=\mu_0^2/M$. As above, we prove the existence of the solution to BDSDE (3.5) on $[T_2,T_1]$. If $T_2=0$, the proof of the existence is complete. Otherwise, we repeat the above procedures. Thus, we obtain a sequence $\{T_p,\ \mu_t^p,\ M_p,\ p \geqslant 1\}$ defined by

$$0 \leqslant T_p < T_{p-1} < \ldots < T_1 < T_0 = T,$$

$$\mu_t^p = \exp\left(\frac{(2C+\alpha)T}{1-\alpha}\right)\left[\mathbb{E}|Y_{T_{p-1}}|^2 + \mathbb{E}\int_t^T \left(\frac{2(1-\alpha)}{2C+\alpha}\,|f(s,0,0)|^2 + \left(\frac{1+2C}{1-\alpha}\right)|g(s,0,0)|^2\right)ds + \frac{1}{|\beta|}\mathbb{E}\int_t^T |h(s,0)|^2\,dA_s\right],$$

$$M_p = 2\mu_0^p = 2\exp\left(\frac{(2C+\alpha)T}{1-\alpha}\right)\left[\mathbb{E}|Y_{T_{p-1}}|^2 + \mathbb{E}\int_0^T \left(\frac{2(1-\alpha)}{2C+\alpha}\,|f(s,0,0)|^2 + \left(\frac{1+2C}{1-\alpha}\right)|g(s,0,0)|^2\right)ds + \frac{1}{|\beta|}\mathbb{E}\int_0^T |h(s,0)|^2\,dA_s\right],$$

and

$$\int_{T_p}^{T_{p-1}} \rho(s, M_p)ds = \frac{\mu_0^p}{M}.$$

Therefore, by iteration, we deduce the existence of a solution to BDSDE (2.1) on $[T_p, T]$.

Finally, setting

$$A = 2\exp\left(\frac{(2C+\alpha)T}{1-\alpha}\right)\left[\mathbb{E}\int_0^T \left(\frac{2(1-\alpha)}{2C+\alpha}\,|f(s,0,0)|^2 + \left(\frac{1+2C}{1-\alpha}\right)|g(s,0,0)|^2\right)ds + \frac{1}{|\beta|}\mathbb{E}\int_0^T |h(s,0)|^2\,dA_s\right]$$

and using the same argument as in [9], we prove the existence of a finite $p \geqslant 1$ such that $T_p = 0$. Thus, we obtain the existence of the solution on $[0, T]$.

Uniqueness. Let $(Y, Z), (Y', Z') \in \mathcal{S}^2([0,T]; \mathbb{R}^k) \times \mathcal{M}^2(0, T; \mathbb{R}^{k\times d})$ be two solutions of BDSDE (2.1). Let $\theta > 0$. By Itô's formula, we have

$$\begin{aligned}\mathbb{E}|Y_t - Y_t'|^2 e^{\theta t} + \theta\mathbb{E}\int_t^T |Y_s - Y_s'|^2 e^{\theta s}ds + \mathbb{E}\int_t^T \|Z_s - Z_s'\|^2 e^{\theta s}ds \\ = 2\mathbb{E}\int_t^T \langle Y_s - Y_s', f(s, Y_s, Z_s) - f(s, Y_s', Z_s')\rangle e^{\theta s}ds \\ + 2\mathbb{E}\int_t^T \langle Y_s - Y_s', h(s, Y_s) - h(s, Y_s')\rangle e^{\theta s}dA_s \\ + \mathbb{E}\int_t^T |g(s, Y_s, Z_s) - g(s, Y_s', Z_s')|^2 e^{\theta s}ds.\end{aligned}$$

Using the assumptions (H3) and (H4) and Young's inequality $2ab \leqslant \theta^{-1}a^2 + \theta b^2$, we derive

$$\mathbb{E}|Y_t - Y_t'|^2 e^{\theta t} + \left(1 - \alpha - \frac{1}{\theta}C\right)\mathbb{E}\int_t^T \|Z_s - Z_s'\|^2 e^{\theta s} ds$$
$$+ 2|\beta|\mathbb{E}\int_t^T |Y_s - Y_s'|^2 e^{\theta s} dA_s \leqslant \left(\frac{1}{\theta} + 1\right)\mathbb{E}\int_t^T \rho(s, |Y_s - Y_s'|^2) e^{\theta s} ds.$$

Choosing $\theta > C/(1-\alpha)$ and noting that $1 \leqslant e^{\theta t} \leqslant e^{\theta T}$ for all $t \in [0,T]$, we get

$$\text{(3.6)} \quad \mathbb{E}|Y_t - Y_t'|^2 + \left(1-\alpha-\frac{1}{\theta}C\right)\mathbb{E}\int_t^T \|Z_s - Z_s'\|^2 ds + 2|\beta|\mathbb{E}\int_t^T |Y_s - Y_s'|^2 dA_s$$
$$\leqslant \left(\frac{1}{\theta} + 1\right) e^{\theta T}\mathbb{E}\int_t^T \rho(s, |Y_s - Y_s'|^2) ds.$$

Therefore

$$\mathbb{E}|Y_t - Y_t'|^2 \leqslant \left(\frac{1}{\theta} + 1\right) e^{\theta T}\int_t^T \rho(s, \mathbb{E}|Y_s - Y_s'|^2) ds.$$

Using the comparison theorem for ODE, we have

$$\mathbb{E}|Y_t - Y_t'|^2 \leqslant r(t) \quad \text{for all } t \in [0,T],$$

where $r(t)$ is the maximum left shift solution of the following equation:

$$u' = -(\theta^{-1} + 1)e^{\theta T}\rho(t,u),$$
$$u(T) = 0.$$

By the assumption (H3), $r(t) = 0$, $t \in [0,T]$. Thus $\mathbb{E}|Y_t - Y_t'|^2 = 0$, $t \in [0,T]$, which implies $Y_t = Y_t'$ a.s. It then follows from (3.6) that $Z_t = Z_t'$ a.s. for any $t \in [0,T]$. ■

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