PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 30, Fasc. 2 (2010), pp. 273–287

BOUNDARY BEHAVIOR OF A CONSTRAINED BROWNIAN MOTION BETWEEN REFLECTING-REPELLENT WALLS

BY

DOMINIQUE LÉPINGLE (ORLÉANS)

Abstract. Stochastic variational inequalities provide a unified treatment for stochastic differential equations living in a closed domain with normal reflection and/or singular repellent drift. When the domain is a convex polyhedron, we prove that the reflected-repelled Brownian motion does not hit the non-smooth part of the boundary. A sufficient condition for non-hitting a face of the polyhedron is derived from the one-dimensional situation. A full answer to the question of attainability of the walls of the Weyl chamber may be given for a radial Dunkl process.

2000 AMS Mathematics Subject Classification: Primary: 60G17; Secondary: 60H10.

Key words and phrases: Multivalued stochastic differential equation, reflected Brownian motion, particle collisions, Wishart process, radial Dunkl process, Weyl chamber.

1. INTRODUCTION

For a Brownian motion constrained to live in a convex polyhedral domain under action of a singular drift and/or normal reflection on the faces, a typical question is to ask whether it may hit the edges of the polyhedron. Our main result is that this hitting time is a.s. infinite. Then the possibility of hitting the single faces is discussed.

Stochastic differential equations with reflection on the faces of a convex polyhedron have been studied in several papers (e.g., [30], [31], [11], [12]). Their solution is a continuous process that may or may not hit the edges, depending on the drift and diffusion coefficients of the process and on the direction of reflection (normal or oblique). In particular, Williams [31] has proved that the Brownian motion with a skew symmetry condition on the direction of reflection does not touch the intersections of the faces of the polyhedron. Her result includes the case of normal reflection.

On the other hand, there is an extensive literature about non-colliding Brownian particles (e.g., [16]–[18], [25]). Most of these works originate in the study of the eigenvalues of Gaussian matrix processes. These eigenvalues are solutions to systems of stochastic differential equations with a singular drift that prevents the particles from colliding. Extensions of these systems are provided by Dunkl processes [26] that were recently developed in connection with harmonic analysis on symmetric spaces. The radial part of a Dunkl process may be considered as a Brownian motion perturbed by a singular drift which forces the process to live in a cone generated by the intersection of a finite set of half-spaces ([9], [10], [14]). Depending on the strength of the repulsion, the process may touch the walls of the cone or not.

Actually it is possible to unify both theories of (normal) reflection and strong repulsion within a common framework. This is carried out by stochastic variational inequalities, also called multivalued stochastic differential equations (MSDE) that were mainly developed by Cépa ([4], [5]). These equations are associated with a convex function in a domain of $\mathbf{R}^d$. Depending on the boundary behavior of this function the diffusion will (normally) reflect on the boundary, hit the boundary without local time, or live in the open domain. We shall here follow this way and concentrate on a Brownian motion living in a convex polyhedral domain, bounded or unbounded. With each face of the polyhedron there is associated a repelling force with normal reflection when the repulsion is not strong enough. In this setting we ask whether the process may hit the boundary of the domain. Our main task will be to rule out the possibility of hitting the intersection of two faces. Once this is achieved, the problem is now basically one-dimensional and we may use the ordinary scale function of real diffusions.

In some previous works ([20], [8]), this issue has been studied in the particular case of the limiting hyperplanes $H_{ij} := \{\mathbf{x} = (x_1, \ldots, x_d) \in \mathbf{R}^d \colon x_i = x_j\}, i \neq j$, and presented as the problem of collisions between Brownian particles. There is a *simple collision* if precisely two coordinates coincide and a *multiple collision* if at least three coordinates coincide at the same time. Because the d-dimensional Brownian motion does not hit the intersection of two hyperplanes, one can guess that an additional drift does not change anything. However, a rigorous proof is necessary because the singularity of the drift makes useless the usual Girsanov change of probability measure. And the counterexample of Bass and Pardoux [1] has shown that uniform nondegeneracy of the diffusion term does not preclude multiple collisions.

As in [8], where the particular case of electrostatic repulsion was considered, our proof only uses basic tools from stochastic calculus, mainly McKean's martingale method [23] which was already applied in [2] to prove non-collision for the eigenvalues of Wishart processes. Another way could be to use the theory of Dirichlet forms as done in [20] where a general condition of non-collision has been obtained.

The paper is organized as follows. In Section 2 we introduce basic definitions and notation. The main features about stochastic variational inequalities are also recalled. Section 3 is devoted to non-attainability of the edges of the polyhedron.

Probability and Mathematical Statistics 30/2010, z. 2

In Section 4 we give a sufficient condition of non-attainability of a single face. Section 5 presents some applications to Brownian particles with nearest neighbor interaction, Wishart processes and Dunkl processes.

2. MULTIVALUED STOCHASTIC DIFFERENTIAL EQUATION IN A POLYHEDRAL DOMAIN

Let $\big(\Omega, \mathcal{F}, (\mathcal{F}_t, t \geqslant 0), \mathbf{P}\big)$ be a filtered probability space endowed with the usual conditions and $B = (\mathbf{B}_t)$ be an $(\mathcal{F}_t)$-adapted d-dimensional Brownian motion starting from the origin. Let

$$\Phi : \mathbf{R}^d \to (-\infty, +\infty]$$

be a lower semi-continuous convex function such that

$$\mathrm{dom}(\Phi) := \{\mathbf{x} \colon \Phi(\mathbf{x}) < \infty\}$$

has nonempty interior. Let

$$D := \mathrm{Int}\big(\mathrm{dom}(\Phi)\big).$$

For simplicity of the notation, we will assume that Φ is C^1 on D. If $\mathbf{x} \in \partial D$, we say that the unit vector $\mathbf{n}(\mathbf{x})$ is a *unit inward normal* to D at $\mathbf{x}$ if

$$\mathbf{n}(\mathbf{x}) \cdot (\mathbf{x} - \mathbf{z}) \leqslant 0$$

for any $\mathbf{z} \in \overline{D}$. Based on the results in [4], the following theorem has been proved in [6] (see also Theorem 2.2 in [7]).

THEOREM 2.1. *For any $\mathcal{F}_0$-measurable random variable $\mathbf{X}_0$ with values in $\overline{D}$, there exist a unique continuous $(\mathcal{F}_t)$-adapted process $\mathbf{X} = \{\mathbf{X}_t, 0 \leqslant t < \infty\}$ with values in $\overline{D}$ and a unique continuous $(\mathcal{F}_t)$-adapted non-decreasing process $L = \{L_t, 0 \leqslant t < \infty\}$ such that*

$$\begin{aligned} \mathbf{X}_t &= \mathbf{X}_0 + \mathbf{B}_t - \int_0^t \nabla\Phi(\mathbf{X}_s)\, ds + \int_0^t \mathbf{n}_s\, dL_s, && 0 \leqslant t < \infty, \\ L_t &= \int_0^t \mathbf{1}_{\{\mathbf{X}_s \in \partial D\}} dL_s, && 0 \leqslant t < \infty, \end{aligned}$$

where $\mathbf{n}_s$ is dL_s-a.e. a unit inward normal to D at $\mathbf{X}_s$. For any $0 < T < \infty$,

$$\int_0^T \mathbf{1}_{\{\mathbf{X}_s \in \partial D\}} ds = 0 \tag{2.1}$$

and

$$\int_0^T |\nabla\Phi(\mathbf{X}_s)|\, ds < \infty. \tag{2.2}$$

From now on we concentrate on a particular polyhedral setting. Let $\mathbf{I} := \{1, \ldots, m\}$, where $m \geqslant 1$. We consider a convex function Φ of the form

$$\Phi(\mathbf{x}) := \sum_{i \in \mathbf{I}} \phi_i(\mathbf{x} \cdot \mathbf{n}_i - a_i), \tag{2.3}$$

where, for any $i \in \mathbf{I}$, ϕ_i is an l.s.c. convex function, $\phi_i = +\infty$ on $(-\infty, 0)$, ϕ_i is C^1 on $(0, +\infty)$, $\mathbf{n}_i$ is a unit vector, and a_i is a real number.

We may assume all $\mathbf{n}_i$ are different. Then

$$\nabla\Phi(\mathbf{x}) = \sum_{i \in \mathbf{I}} \mathbf{n}_i \, \phi_i'(\mathbf{x} \cdot \mathbf{n}_i - a_i),$$

$$D = \{\mathbf{x} \in \mathbf{R}^d \colon \ \mathbf{x} \cdot \mathbf{n}_i \ > \ a_i \ \ \forall i \in \mathbf{I}\},$$

$$\overline{D} = \{\mathbf{x} \in \mathbf{R}^d \colon \ \mathbf{x} \cdot \mathbf{n}_i \ \geqslant \ a_i \ \ \forall i \in \mathbf{I}\}.$$

Henceforth, we assume that D is not empty. There exists a ball with center $\mathbf{y} \in D$ and radius $b > 0$ included in D. Let $\mathbf{X}_t$ be the solution given by Theorem 2.1. For $i \in \mathbf{I}$ let

$$U_t^i := \mathbf{X}_t \cdot \mathbf{n}_i \ - \ a_i.$$

We will need a strengthening of inequality (2.2) ([7], Theorem 2.2).

LEMMA 2.1. *For any $i \in \mathbf{I}$, for any $0 < t < \infty$, we have*

$$\int_0^t |\phi_i'(U_s^i)| \, ds \ < \ \infty.$$

P r o o f. This is clear if $\phi_i'(0+) > -\infty$. Let

$$\mathbf{J} := \{j \in \mathbf{I} : \phi_j'(0+) = -\infty\}$$

and let $0 < \varepsilon < b$ be such that $\phi_j'(u) < 0$ for any $j \in \mathbf{J}$ and $u \in (0, \varepsilon)$. For $\mathbf{K} \subset \mathbf{J}$ let us define

$$A_{\mathbf{K}} := \{\mathbf{x} \in \overline{D} \colon \mathbf{x} \cdot \mathbf{n}_j < a_j + \varepsilon \ \ \forall j \in \mathbf{K}, \quad \mathbf{x} \cdot \mathbf{n}_j \geqslant a_j + \varepsilon \ \ \forall j \in \mathbf{J} \setminus \mathbf{K}\}.$$

Then for $t > 0$

$$\int_0^t \mathbf{1}_{A_{\mathbf{K}}}(\mathbf{X}_s) \Big| \sum_{j \notin \mathbf{K}} \mathbf{n}_j \, \phi_j'(U_s^j) \Big| ds \leqslant \sum_{j \notin \mathbf{K}} \int_0^t \mathbf{1}_{A_{\mathbf{K}}}(\mathbf{X}_s) |\phi_j'(U_s^j)| ds \ < \ \infty.$$

Using (2.2) we get

$$\int_0^t \mathbf{1}_{A_{\mathbf{K}}}(\mathbf{X}_s) \Big| \sum_{j \in \mathbf{K}} \mathbf{n}_j \, \phi_j'(U_s^j) \Big| ds \ < \ \infty,$$

and therefore

$$\begin{aligned}
-(b-\varepsilon)\sum_{j\in\mathbf{K}}\int_0^t \mathbf{1}_{A_{\mathbf{K}}}(\mathbf{X}_s)\phi_j'(U_s^j)\,ds &\leqslant \int_0^t \mathbf{1}_{A_{\mathbf{K}}}(\mathbf{X}_s)\sum_{j\in\mathbf{K}}(\mathbf{y}-\mathbf{X}_s)\cdot\mathbf{n}_j|\phi_j'(U_s^j)|ds\\
&\leqslant \int_0^t \mathbf{1}_{A_{\mathbf{K}}}(\mathbf{X}_s)|\mathbf{y}-\mathbf{X}_s|\big|\sum_{j\in\mathbf{K}}\mathbf{n}_j\phi_j'(U_s^j)\big|ds\\
&< \infty
\end{aligned}$$

by the continuity of $\mathbf{X}$ on $[0,t]$. Then for any $j\in\mathbf{J}$ we have

$$\begin{aligned}
\text{(2.4)}\quad \int_0^t |\phi_j'(U_s^j)|\,ds &= \int_0^t \mathbf{1}_{\{U_s^j<\varepsilon\}}|\phi_j'(U_s^j)|\,ds + \int_0^t \mathbf{1}_{\{U_s^j\geqslant\varepsilon\}}|\phi_j'(U_s^j)|\,ds\\
&= \sum_{j\in\mathbf{K}\subset\mathbf{J}}\int_0^t \mathbf{1}_{A_{\mathbf{K}}}(\mathbf{X}_s)\,|\phi_j'(U_s^j)|ds + \int_0^t \mathbf{1}_{\{U_s^j\geqslant\varepsilon\}}|\phi_j'(U_s^j)|\,ds<\infty. \quad\blacksquare
\end{aligned}$$

For any $\mathbf{J}\subset\mathbf{I}$, $\mathbf{J}\neq\emptyset$, we set

$$\begin{aligned}
H_{\mathbf{J}} &:= \{\mathbf{x}\in\mathbf{R}^d : \mathbf{x}\cdot\mathbf{n}_j=a_j\ \ \forall j\in\mathbf{J}\},\\
K_{\mathbf{J}} &:= \{\mathbf{x}\in\mathbf{R}^d : \mathbf{x}\cdot\mathbf{n}_j=a_j\ \ \forall j\in\mathbf{J},\ \ \mathbf{x}\cdot\mathbf{n}_j>a_j\ \ \forall j\notin\mathbf{J}\},\\
\sigma_{\mathbf{J}} &:= \inf\{t>0\ :\ \mathbf{X}_t\in H_{\mathbf{J}}\},\\
\tau_{\mathbf{J}} &:= \inf\{t>0\ :\ \mathbf{X}_t\in K_{\mathbf{J}}\},\\
V_{\mathbf{J}} &:= \operatorname{span}\{\mathbf{n}_j, j\in\mathbf{J}\},\\
q_{\mathbf{J}} &:= \dim V_{\mathbf{J}},\\
\pi_{\mathbf{J}} &:= \text{orthogonal projection onto } V_{\mathbf{J}}.
\end{aligned}$$

LEMMA 2.2. *If* $\mathbf{n}(\mathbf{x})$ *is a unit inward normal to* D *at* $\mathbf{x}\in K_{\mathbf{J}}$, *then we have* $\mathbf{n}(\mathbf{x})\in V_{\mathbf{J}}$.

Proof. Let $\mathbf{v}\perp V_{\mathbf{J}}$. For $\varepsilon>0$ small enough,

$$\mathbf{z}_1=\mathbf{x}+\varepsilon\mathbf{v},\qquad \mathbf{z}_2=\mathbf{x}-\varepsilon\mathbf{v}$$

satisfy

$$\begin{aligned}
\mathbf{z}_1\cdot\mathbf{n}_j=a_j\ \ \forall j\in\mathbf{J},&\qquad \mathbf{z}_1\cdot\mathbf{n}_i>a_i\ \ \forall i\notin\mathbf{J},\\
\mathbf{z}_2\cdot\mathbf{n}_j=a_j\ \ \forall j\in\mathbf{J},&\qquad \mathbf{z}_2\cdot\mathbf{n}_i>a_i\ \ \forall i\notin\mathbf{J}.
\end{aligned}$$

Then

$$\mathbf{n}(\mathbf{x})\cdot(\mathbf{x}-\mathbf{z}_1)\leqslant 0,\qquad \mathbf{n}(\mathbf{x})\cdot(\mathbf{x}-\mathbf{z}_2)\leqslant 0,$$

and therefore

$$\mathbf{n}(\mathbf{x})\cdot\mathbf{v}=0.\quad\blacksquare$$

Probability and Mathematical Statistics 30/2010, z. 2

3. NON-ATTAINABILITY OF THE EDGES

This section is devoted to the proof of the following theorem.

THEOREM 3.1. *For any* $\mathbf{J} \subset \mathbf{I}$ *with* $|\mathbf{J}| := \operatorname{card}(\mathbf{J}) \geqslant 2$,

$$\mathbf{P}(\sigma_{\mathbf{J}} = \infty) = 1.$$

P r o o f. (a) We first consider the initial condition $\mathbf{X}_0$. From (2.1) we deduce that for any $u > 0$ there exists $0 < v < u$ such that $\mathbf{X}_v \in D$ a.s. Using the continuity of paths and the Markov property we may and do assume that $\mathbf{X}_0 \in D$ in order to prove that $\sigma_{\mathbf{J}} = \infty$ a.s.

(b) We will also assume that

$$\max_{i \in \mathbf{I}} \phi_i'(0+) < 0. \tag{3.1}$$

If not, we introduce for any $0 < T < \infty$ the equivalent probability measure $\mathbf{Q}$ defined on $\mathcal{F}_T$ by

$$\frac{d\mathbf{Q}}{d\mathbf{P}} := \exp\Big\{ c\big(\mathbf{B}_T \cdot \sum_{i \in \mathbf{I}} \mathbf{n}_i\big) - \frac{1}{2} c^2 T \big| \sum_{i \in \mathbf{I}} \mathbf{n}_i \big|^2 \Big\},$$

where

$$c > \max_{i \in \mathbf{I}} \phi_i'(0+).$$

The continuous process

$$\mathbf{B}_t' := \mathbf{B}_t - ct \sum_{i \in \mathbf{I}} \mathbf{n}_i$$

is a $\mathbf{Q}$-Brownian motion on $[0, T]$ and now

$$d\mathbf{X}_t = d\mathbf{B}_t' - \sum_{i \in \mathbf{I}} \mathbf{n}_i \psi_i'(\mathbf{X}_t \cdot \mathbf{n}_i - a_i) dt + \mathbf{n}_t dL_t,$$

where

$$\psi_i(u) := \phi_i(u) - cu, \quad i \in \mathbf{I}.$$

If $\mathbf{Q}(\sigma_{\mathbf{J}} < T) = 0$, then $\mathbf{P}(\sigma_{\mathbf{J}} < T) = 0$, and if this is true for any T, we obtain $\mathbf{P}(\sigma_{\mathbf{J}} = \infty) = 1$.

(c) We are now going to prove that $\sigma_{\mathbf{I}} = \tau_{\mathbf{I}} = \infty$ a.s. (with $m \geqslant 2$). If $q_{\mathbf{I}} = 1$, then $m = 2$, $\mathbf{n}_1 + \mathbf{n}_2 = \mathbf{0}$ and $H_{\mathbf{I}} = K_{\mathbf{I}} = \emptyset$. Assume now $q_{\mathbf{I}} \geqslant 2$ and $H_{\mathbf{I}} \neq \emptyset$. Choose some $\mathbf{z} \in H_{\mathbf{I}}$ and set

$$\mathbf{Z}_t := \pi_{\mathbf{I}}(\mathbf{X}_t - \mathbf{z}).$$

Then

$$\mathbf{Z}_t = \mathbf{Z}_0 + \mathbf{C}_t - \sum_{i \in \mathbf{I}} \int_0^t \mathbf{n}_i \, \phi_i'(U_s^i) ds + \int_0^t \mathbf{n}_s \, dL_s,$$

where $\mathbf{C}$ is a $q_{\mathbf{I}}$-dimensional Brownian motion. Set

$$S_t := |\mathbf{Z}_t|^2.$$

Then

$$S_t = S_0 + 2\int_0^t \mathbf{Z}_s \cdot d\mathbf{C}_s - 2\sum_{i\in\mathbf{I}}\int_0^t U_s^i\, \phi_i'(U_s^i)ds + 2\int_0^t \mathbf{Z}_s \cdot \mathbf{n}_s\, dL_s + q_{\mathbf{I}}t.$$

From Lemma 2.2 we deduce that on $\partial D = \bigcup_{\mathbf{J}\subset\mathbf{I}} K_{\mathbf{J}}$

$$\mathbf{Z}_s \cdot \mathbf{n}_s \;=\; (\mathbf{X}_s - \mathbf{z}) \cdot \mathbf{n}_s \;=\; 0,$$

and thus

$$\int_0^t \mathbf{Z}_s \cdot \mathbf{n}_s\, dL_s = 0.$$

Let $0 < T < \infty$. For $t < \tau_{\mathbf{I}} \wedge T$,

$$\log S_t \;=\; \log S_0 \;+\; 2\int_0^t \frac{\mathbf{Z}_s \cdot d\mathbf{C}_s}{S_s} - 2\sum_{i\in\mathbf{I}}\int_0^t \frac{U_s^i\,\phi_i'(U_s^i)}{S_s}ds \;+\; (q_{\mathbf{I}}-2)\int_0^t \frac{ds}{S_s}. \tag{3.2}$$

By the assumption (3.1) there exists $0 < \beta \leqslant \infty$ such that $\phi_i' \leqslant 0$ on $(0,\beta]$ and

$$\begin{aligned}-\int_0^t \frac{U_s^i\,\phi_i'(U_s^i)}{S_s}ds &\geqslant -\int_0^t \frac{U_s^i\,\phi_i'(U_s^i)}{S_s}\,\mathbf{1}_{\{U_s^i\geqslant\beta\}}ds\\ &\geqslant -\frac{1}{\beta}\int_0^T |\phi_i'(U_s^i)|\, ds > -\infty.\end{aligned}$$

We now proceed as in [23], p. 47. As $t \to \tau_{\mathbf{I}} \wedge T$, the continuous local martingale part in the right-hand side of (3.2) either converges to a finite limit or oscillates between $+\infty$ and $-\infty$. Thus it does not converge to $-\infty$ and a.s. $S_{\tau_{\mathbf{I}}\wedge T} > 0$. Therefore

$$\mathbf{P}(\tau_{\mathbf{I}} \leqslant T) = 0$$

and the conclusion follows since T is arbitrary.

(d) Let now $\mathbf{J} \subset \mathbf{I}$ with $2 \leqslant |\mathbf{J}| \leqslant m-1$. We shall show by a backward induction on $|\mathbf{J}|$ that $\mathbf{P}(\tau_{\mathbf{J}} = \infty) = 1$. Remark that the backward induction assumption entails the equality $\sigma_{\mathbf{J}} = \tau_{\mathbf{J}}$ a.s. As previously done, we may assume $q_{\mathbf{J}} \geqslant 2$ and $K_{\mathbf{J}} \neq \emptyset$. Select now $\mathbf{z} \in K_{\mathbf{J}}$ and set

$$\begin{aligned}\mathbf{Z}_t &:= \pi_{\mathbf{J}}(\mathbf{X}_t - \mathbf{z})\\ &= \mathbf{Z}_0 \;+\; \mathbf{C}_t \;-\sum_{j\in\mathbf{J}}\int_0^t \mathbf{n}_j\,\phi_j'(U_s^j)ds - \sum_{i\notin\mathbf{J}}\int_0^t \pi_{\mathbf{J}}\mathbf{n}_i\,\phi_i'(U_s^i)ds + \int_0^t \pi_{\mathbf{J}}\mathbf{n}_s\, dL_s,\end{aligned}$$

where $\mathbf{C}$ is a $q_{\mathbf{J}}$-dimensional Brownian motion. Let again $S_t := |\mathbf{Z}_t|^2$. For $\varepsilon > 0$ and $r > 0$ we set

$$\begin{aligned}\tau_\varepsilon &:= \inf\{t > 0\colon\ S_t + \min_{i \notin \mathbf{J}} (U_t^i)^2 \leqslant 2\,\varepsilon^2\},\\ \rho_r &= \inf\{t > 0\colon\ |\mathbf{X}_t| \geqslant r\}.\end{aligned}$$

From the induction assumption we infer that $\tau_\varepsilon \to \infty$ as $\varepsilon \to 0$. Let $0 < T < \infty$. We introduce the equivalent probability measure $\mathbf{Q}$ defined on $\mathcal{F}_T$ by

$$\begin{aligned}\frac{d\mathbf{Q}}{d\mathbf{P}} = \exp\Big\{ \int\limits_0^{\tau_\varepsilon \wedge \rho_r \wedge T} \sum_{i \notin \mathbf{J}} \mathbf{1}_{\{U_s^i \geqslant \varepsilon\}} \phi_i'(U_s^i)\, \mathbf{n}_i \cdot d\mathbf{C}_s \\ - \frac{1}{2} \int\limits_0^{\tau_\varepsilon \wedge \rho_r \wedge T} \big| \sum_{i \notin \mathbf{J}} \mathbf{1}_{\{U_s^i \geqslant \varepsilon\}} \phi_i'(U_s^i)\, \pi_{\mathbf{J}} \mathbf{n}_i \big|^2\, ds \Big\}.\end{aligned}$$

Then

$$\mathbf{D}_t := \mathbf{C}_t - \int\limits_0^{\tau_\varepsilon \wedge \rho_r \wedge t} \sum_{i \notin \mathbf{J}} \mathbf{1}_{\{U_s^i \geqslant \varepsilon\}} \phi_i'(U_s^i) \pi_{\mathbf{J}} \mathbf{n}_i\, ds$$

is a $q_{\mathbf{J}}$-dimensional $\mathbf{Q}$-Brownian motion on $[0, T]$. For $t \leqslant \tau_\varepsilon \wedge \rho_r \wedge T$, we have

$$\begin{aligned}S_t = S_0 + 2 \int\limits_0^t \mathbf{Z}_s \cdot d\mathbf{D}_s - 2 \sum_{j \in \mathbf{J}} \int\limits_0^t U_s^j\, \phi_j'(U_s^j) ds \\ - 2 \sum_{i \notin \mathbf{J}} \int\limits_0^t \mathbf{1}_{\{U_s^i < \varepsilon\}}\, \mathbf{Z}_s \cdot \mathbf{n}_i\, \phi_i'(U_s^i)\, ds + 2 \sum_{\mathbf{L} \subset \mathbf{I}, \mathbf{L} \not\subset \mathbf{J}} \int\limits_0^t \mathbf{1}_{K_{\mathbf{L}}}(\mathbf{X}_s)\, \mathbf{Z}_s \cdot \mathbf{n}_s dL_s + q_{\mathbf{J}} t\end{aligned}$$

and for $t < \sigma_{\mathbf{J}} \wedge \tau_\varepsilon \wedge \rho_r \wedge T$, we get

$$\begin{aligned}\log S_t = \log S_0 + 2 \int\limits_0^t \frac{\mathbf{Z}_s \cdot d\mathbf{D}_s}{S_s} - 2 \sum_{j \in \mathbf{J}} \int\limits_0^t \frac{U_s^j\, \phi_j'(U_s^j)}{S_s} ds \\ - 2 \sum_{i \notin \mathbf{J}} \int\limits_0^t \mathbf{1}_{\{U_s^i < \varepsilon\}} \frac{\phi_i'(U_s^i)}{S_s} \mathbf{Z}_s \cdot \mathbf{n}_i\, ds \\ + 2 \sum_{\mathbf{L} \subset \mathbf{I}, \mathbf{L} \not\subset \mathbf{J}} \int\limits_0^t \mathbf{1}_{K_{\mathbf{L}}}(\mathbf{X}_s) \frac{\mathbf{Z}_s \cdot \mathbf{n}_s}{S_s} dL_s + (q_{\mathbf{J}} - 2) \int\limits_0^t \frac{ds}{S_s}.\end{aligned}$$

By the induction hypothesis and the continuity of paths, if $\sigma_{\mathbf{J}} < \infty$, then for any $\mathbf{L} \not\subset \mathbf{J}$ there exists an interval $(\sigma_{\mathbf{J}} - \delta, \sigma_{\mathbf{J}}]$ of positive length on which $\mathbf{X}_s \notin K_{\mathbf{L}}$. Therefore

$$\int\limits_0^{\sigma_{\mathbf{J}} \wedge \tau_\varepsilon \wedge \rho_r \wedge T} \mathbf{1}_{K_{\mathbf{L}}}(\mathbf{X}_s) \frac{\mathbf{Z}_s \cdot \mathbf{n}_s}{S_s} dL_s > -\infty.$$

For $s < \tau_\varepsilon$, if $U_s^i < \varepsilon$ for some $i \notin \mathbf{J}$, then $S_s \geqslant \varepsilon^2$ and we obtain as well

$$-\int_0^{\sigma_{\mathbf{J}} \wedge \tau_\varepsilon \wedge \rho_r \wedge T} \mathbf{1}_{\{U_s^i < \varepsilon\}} \frac{\phi_i'(U_s^i)}{S_s} \mathbf{Z}_s \cdot \mathbf{n}_i \, ds > -\infty.$$

The other terms behave as in (c), and thus

$$0 = \mathbf{Q}(\sigma_{\mathbf{J}} \leqslant \tau_\varepsilon \wedge \rho_r \wedge T) = \mathbf{P}(\sigma_{\mathbf{J}} \leqslant \tau_\varepsilon \wedge \rho_r \wedge T).$$

Letting $\varepsilon \to 0$ and $r, T \to \infty$ we get

$$\mathbf{P}(\sigma_{\mathbf{J}} = \infty) = 1,$$

and we are done. ■

4. KEEPING OFF FROM A WALL

We first recall some features from the one-dimensional setting [22]. Let $\phi : \mathbf{R} \to (-\infty, +\infty]$ be a convex lower semicontinuous function. Assume $\phi = +\infty$ on $(-\infty, 0)$ and C^1 on $(0, +\infty)$. Consider the one-dimensional MSDE

$$dY_t = dB_t - \phi'(Y_t)dt + \frac{1}{2} dL_t^0,$$
$$Y_t \geqslant 0,$$

where L^0 is the local time of Y at 0. There are three types of boundary behavior:

	repulsion
$\phi(0) < \infty$	weak: local time not zero
$\phi(0) = \infty, \int_{0+} \exp\{2\phi\} < \infty$	middle: local time zero
$\phi(0) = \infty, \int_{0+} \exp\{2\phi\} = \infty$	strong: boundary not hit

We shall check the behavior of the multidimensional process $\mathbf{X}$ according to this classification in the neighborhood of the faces of the polyhedron. For any $i \in \mathbf{I}$ we write $H_i, K_i, \sigma_i, \tau_i$ in place of $H_{\{i\}}, K_{\{i\}}, \sigma_{\{i\}}, \tau_{\{i\}}$, respectively.

PROPOSITION 4.1. *For any $i \in \mathbf{I}$ such that $\phi_i(0) = \infty$ and for any $t > 0$,*

$$\int_0^t \mathbf{1}_{H_i}(\mathbf{X}_s) \, dL_s = 0.$$

Proof. From the occupation times formula and Lemma 2.1 we obtain

$$\int_0^\infty L_t^a(U_i) \, |\phi_i'(a)| \, da = \int_0^t |\phi_i'(U_s^i)| \, ds < \infty$$

and from $\phi_i(0) = \infty$ and the continuity of $a \mapsto L_t^a(U_i)$ we deduce

$$L_t^0(U_i) = 0.$$

Thus

$$\begin{aligned}
0 &= U_t^i - (U_t^i)^+ \\
&= \int_0^t \mathbf{1}_{H_i}(\mathbf{X}_s)\mathbf{n}_i \cdot d\mathbf{B}_s - \int_0^t \mathbf{1}_{H_i}(\mathbf{X}_s) \sum_{j\in\mathbf{I}} \phi_j'(U_s^j)\,\mathbf{n}_i \cdot \mathbf{n}_j\, ds \\
&\quad + \int_0^t \mathbf{1}_{H_i}(\mathbf{X}_s)\,\mathbf{n}_i \cdot \mathbf{n}_s\, dL_s \\
&= \int_0^t \mathbf{1}_{K_i}(\mathbf{X}_s)\,\mathbf{n}_i \cdot \mathbf{n}_s\, dL_s = \int_0^t \mathbf{1}_{K_i}(\mathbf{X}_s)\, dL_s = \int_0^t \mathbf{1}_{H_i}(\mathbf{X}_s)\, dL_s. \quad \blacksquare
\end{aligned}$$

We now set for any $i \in \mathbf{I}$ and $x \geqslant 0$

$$p_i(x) := \int_1^x \exp\left\{2\big(\phi_i(u) - \phi_i(1)\big)\right\} du.$$

THEOREM 4.1. *For any* $i \in \mathbf{I}$ *such that* $p_i(0) = -\infty$ *or, equivalently,*

$$\int_{0+} \exp\{2\phi_i\} = \infty, \tag{4.1}$$

we have $\mathbf{P}(\sigma_i = \infty) = \mathbf{P}(\tau_i = \infty) = 1.$

P r o o f. From the Itô formula and Proposition 4.1 we obtain

$$\begin{aligned}
&p_i(U_t^i) = \\
&= p_i(U_0^i) + \int_0^t p_i'(U_s^i)\big[dC_s^i - \sum_{j\neq i} \mathbf{n}_i \cdot \mathbf{n}_j\, \phi_j'(U_s^j) ds + \sum_{j\neq i} \mathbf{1}_{K_j}(\mathbf{X}_s)\,\mathbf{n}_i \cdot \mathbf{n}_j\, dL_s\big],
\end{aligned}$$

where $C^i = \mathbf{B} \cdot \mathbf{n}_i$ is a one-dimensional Brownian motion. As in the proof of Theorem 3.1, let

$$\begin{aligned}
\tau_\varepsilon &:= \inf\{t > 0 \,:\, U_t^i + \min_{j\neq i}(U_t^j) \leqslant 2\,\varepsilon\}, \\
\rho_r &= \inf\{t > 0 \,:\, |\mathbf{X}_t| \geqslant r\}.
\end{aligned}$$

Let $0 < T < \infty$. We again introduce the equivalent probability measure $\mathbf{Q}$ defined on $\mathcal{F}_T$ by

$$\begin{aligned}
\frac{d\mathbf{Q}}{d\mathbf{P}} = \exp\Big\{ &\int_0^{\tau_\varepsilon \wedge \rho_r \wedge T} \sum_{j\neq i} \mathbf{1}_{\{U_s^j \geqslant \varepsilon\}} \phi_j'(U_s^j)\,\mathbf{n}_i \cdot \mathbf{n}_j\, dC_s^i \\
&- \frac{1}{2} \int_0^{\tau_\varepsilon \wedge \rho_r \wedge T} \big| \sum_{j\neq i} \mathbf{1}_{\{U_s^j \geqslant \varepsilon\}} \phi_j'(U_s^j)\,\mathbf{n}_i \cdot \mathbf{n}_j \big|^2 ds \Big\}.
\end{aligned}$$

Then

$$D_t^i := C_t^i - \int_0^{t\wedge\tau_\varepsilon\wedge\rho_r} \sum_{j\neq i} \mathbf{1}_{\{U_s^j \geqslant \varepsilon\}} \phi_j'(U_s^j)\,\mathbf{n}_i\cdot\mathbf{n}_j\,ds$$

is a $\mathbf{Q}$-Brownian motion on $[0,T]$ and for $t \leqslant \tau_\varepsilon \wedge \rho_r \wedge T$ we have

$$\begin{aligned} & p_i(U_t^i) = p_i(U_0^i) \\ & + \int_0^t p_i'(U_s^i)\big[dD_s^i - \sum_{j\neq i} \mathbf{1}_{\{U_s^j<\varepsilon\}}\mathbf{n}_i\cdot\mathbf{n}_j\,\phi_j'(U_s^j)ds + \sum_{j\neq i} \mathbf{1}_{K_j}(\mathbf{X}_s)\,\mathbf{n}_i\cdot\mathbf{n}_j\,dL_s\big]. \end{aligned}$$

For $j \neq i$,

$$-\int_0^{\sigma_i\wedge\tau_\varepsilon\wedge\rho_r\wedge T} \mathbf{1}_{\{U_s^j<\varepsilon\}}\,p_i'(U_s^i)\,\mathbf{n}_i\cdot\mathbf{n}_j\,\phi_j'(U_s^j)\,ds \;>-\infty$$

and

$$+\int_0^{\sigma_i\wedge\tau_\varepsilon\wedge\rho_r\wedge T} \mathbf{1}_{K_j}(\mathbf{X}_s)\,p_i'(U_s^i)\,\mathbf{n}_i\cdot\mathbf{n}_j\,dL_s \;>-\infty.$$

Then

$$0 \;=\; \mathbf{Q}(\sigma_i \leqslant \tau_\varepsilon\wedge\rho_r\wedge T) =\; \mathbf{P}(\sigma_i \leqslant \tau_\varepsilon\wedge\rho_r\wedge T),$$

meaning that $\mathbf{P}(\sigma_i=\infty)=1$. ∎

5. APPLICATIONS

5.1. Brownian particles with nearest neighbor repulsion. Rost and Vares [27] have considered the following system:

$$\begin{aligned} dX_t^1 &= dB_t^1 \;+\; \phi'(X_t^2 - X_t^1)\,dt, \\ dX_t^i &= dB_t^i \;+\; \big(\phi'(X_t^{i+1}-X_t^i) - \phi'(X_t^i - X_t^{i-1})\big)\,dt, \qquad i=2,\ldots,n-1, \\ dX_t^n &= dB_t^n \;-\; \phi'(X_t^n - X_t^{n-1})\,dt, \end{aligned}$$

where $X_t^1 < \ldots < X_t^n$ and ϕ is a positive convex function on $(0,\infty)$ satisfying

$$\phi(0)=\infty, \quad \phi(\infty)=0, \quad \int_0^1 \big(\phi'(x)\big)^2 e^{-2\phi(x)}\,dx \;<\; \infty. \tag{5.1}$$

This is an MSDE where the function Φ is given by (2.3) with $\phi_i(x)=\phi(\sqrt{2}\,x)$, $\mathbf{n}_i = \frac{1}{\sqrt{2}}(\mathbf{e}_{i+1}-\mathbf{e}_i)$, $a_i=0$ for $i=1,\ldots,n-1$, and $\mathbf{e}_j$ is the j-th basis vector. Condition (5.1) for non-collision is stronger than (4.1) as can be seen from the Schwarz inequality:

$$\infty \;=\; \big(\phi(0)-\phi(1)\big)^2 \;\leqslant\; \int_0^1 (\phi')^2\,e^{-2\phi} \int_0^1 e^{2\phi}.$$

5.2. Wishart and Laguerre processes. Wishart processes have been introduced in [2] and [3]. If B is an $n \times n$ Brownian matrix, a Wishart process with parameters n and $\delta \geqslant n+1$ may be obtained as a solution to the matrix-valued SDE

$$d\mathbf{S}_t \,=\, \sqrt{\mathbf{S}_t}\, d\mathbf{B}_t \,+\, d\mathbf{B}'_t \sqrt{\mathbf{S}_t} \,+\, \delta\, \mathbf{I}_n\, dt.$$

The eigenvalue process $(\lambda_t^1, \ldots, \lambda_t^n)$ of $\mathbf{S}_t$ satisfies

$$d\lambda_t^i \,=\, 2\sqrt{\lambda_t^i}\, dW_t^i \,+\, \Big(\delta \,+\, \sum_{j\neq i} \frac{\lambda_t^i + \lambda_t^j}{\lambda_t^i - \lambda_t^j}\Big)\, dt, \quad 1 \leqslant i \leqslant n, \tag{5.2}$$

and the square roots $r_t^i = \sqrt{\lambda_t^i}$,

$$dr_t^i \,=\, dW_t^i \,+\, \frac{1}{2}\frac{\delta - n}{r_t^i} dt \,+\, \frac{1}{2}\sum_{j\neq i}\Big(\frac{1}{r_t^i + r_t^j} \,+\, \frac{1}{r_t^i - r_t^j}\Big)\, dt, \tag{5.3}$$

where $(W^i, \ldots, W^n)$ is an n-dimensional Brownian motion. This system is an MSDE with

$$\begin{aligned} \Phi(r^1, \ldots, r^n) & \\ &= -\frac{1}{2}\big[(\delta - n)\sum_i \log\, r^i \,+\, \sum_{i>j}\log(r^i + r^j) \,+\, \sum_{i>j}\log(r^i - r^j)\big] \end{aligned} \tag{5.4}$$

on $\{0 < r^1 < \ldots < r^n\}$ and ∞ elsewhere. Systems (5.3) and (5.2) have strong solutions for $\delta > n$. If $\delta = n$, we must add to the right-hand side of (5.3) a local time at 0 that disappears in (5.2). It has been proved in [3] that the eigenvalues never collide, and if moreover $\delta \geqslant n+1$, the smallest one never vanishes. This is in accordance with Theorem 4.1.

Laguerre processes ([21], [13]) are Hermitian versions of Wishart processes. Constants are changed in (5.2), (5.3) and (5.4).

5.3. Reflection groups and Dunkl processes. We only give a short introduction to this topic and refer to [19] and [26] for more details. For $\alpha \in \mathbf{R}^N \setminus \{0\}$ we denote by s_α the orthogonal reflection with respect to the hyperplane H_α perpendicular to α:

$$s_\alpha(\mathbf{x}) \,=\, \mathbf{x} \,-\, 2\frac{\alpha \cdot \mathbf{x}}{|\alpha|^2}.$$

A finite subset $R \subset \mathbf{R}^N \setminus \{0\}$ is called a *root system* if for all $\alpha \in R$

$$R \cap \mathbf{R}\alpha \,=\, \{\alpha, -\alpha\}, \quad s_\alpha(R) \,=\, R.$$

The group $W \subset O(N)$ which is generated by the reflections $\{s_\alpha, \alpha \in R\}$ is called the *reflection group* associated with R. For $\beta \in \mathbf{R}^N \setminus \bigcup_{\alpha\in R} H_\alpha$, the hyperplane

$H_\beta := \{\mathbf{x} \in \mathbf{R}^N : \beta \cdot \mathbf{x} = 0\}$ separates the root system R into R_+ and R_-. Such a set R_+ is called a *positive subsystem* and defines the *positive Weyl chamber* C by the formula

$$C := \{\mathbf{x} \in \mathbf{R}^N : \alpha \cdot \mathbf{x} > 0 \ \ \forall \alpha \in R_+\}.$$

A subset S of R_+ is called *simple* if S is a vector basis for $\mathrm{span}(R)$. The elements of S are called *simple roots*. Such a subset exists, is unique and we actually get

$$C = \{\mathbf{x} \in \mathbf{R}^N : \alpha \cdot \mathbf{x} > 0 \ \ \forall \alpha \in S\}.$$

A function $k : R \to \mathbf{R}$ on the root system is called a *multiplicity function* if it is invariant under the natural action of W on R. If the multiplicity function k is positive on R_+, we define the radial Dunkl process $\mathbf{X}^W$ as the $\overline{C}$-valued continuous Markov process whose generator is given by

$$\mathcal{L}_k^W u(\mathbf{x}) = \frac{1}{2} \Delta u(\mathbf{x}) + \sum_{\alpha \in R_+} k(\alpha) \frac{\alpha \cdot \nabla u(\mathbf{x})}{\alpha \cdot \mathbf{x}}$$

for $u \in C^2(\overline{C})$ with the boundary condition $\alpha \cdot \nabla u(\mathbf{x}) = 0$ for $\mathbf{x} \in H_\alpha$. Demni ([14], [15]) has remarked that $\mathbf{X}^W$ may be viewed as the solution to the MSDE

$$d\mathbf{Y}_t = d\mathbf{B}_t - \nabla \Phi(\mathbf{Y}_t)\, dt,$$

where $\mathbf{B}$ is an N-dimensional Brownian motion and

$$\Phi(\mathbf{y}) = - \sum_{\alpha \in R_+} k(\alpha) \log(\alpha \cdot \mathbf{y})$$

on C and $\Phi = \infty$ elsewhere. From [9] or [10] we know that this equation has a unique strong solution, and if moreover $k(\alpha) \geqslant 1/2$ for any $\alpha \in R$, then the process never hits the walls H_α of the Weyl chamber. In [15], it is proved that if $k(\alpha) < 1/2$ for a simple root α, then the process hits H_α a.s. As a consequence of this result and of Theorem 4.1 (see also the statement at the bottom of p. 117 in [10]), we are in a position to classify the boundary behavior of the radial Dunkl process in the Weyl chamber.

PROPOSITION 5.1. *For any* $\alpha \in R_+$ *let* $\sigma_\alpha := \inf\{t > 0 : \mathbf{X}_t^W \in H_\alpha\}$.
- *If* $\alpha \in R_+ \setminus S$, *then* $\mathbf{P}(\sigma_\alpha = \infty) = 1$.
- *If* $\alpha \in S$ *and* $k(\alpha) \geqslant 1/2$, *then* $\mathbf{P}(\sigma_\alpha = \infty) = 1$.
- *If* $\alpha \in S$ *and* $k(\alpha) < 1/2$, *then* $\mathbf{P}(\sigma_\alpha < \infty) = 1$.

5.4. Trigonometric and hyperbolic interactions. Other interactions have been studied in [7].

The trigonometric system ([16], [18], [29]) reads

$$dX_t^j = dB_t^j + \frac{\gamma}{2} \sum_{k \neq j} \cot \frac{X_t^j - X_t^k}{2}, \quad 1 \leqslant j \leqslant n,$$
$$X_t^1 \leqslant X_t^2 \leqslant \ldots \leqslant X_t^n \leqslant X_t^1 + 2\pi.$$

This can be interpreted as the solution to the MSDE associated with

$$\Phi(\mathbf{x}) = \sum_{i>j} \phi\left(\mathbf{x} \cdot \frac{\mathbf{e}_i - \mathbf{e}_j}{\sqrt{2}}\right) + \sum_{i<j} \phi\left(\mathbf{x} \cdot \frac{\mathbf{e}_i - \mathbf{e}_j}{\sqrt{2}} + \pi\sqrt{2}\right),$$

where

$$\phi(u) = \begin{cases} \infty, & u \leqslant 0, \\ -\gamma \log\big(\sin(u/\sqrt{2})\big), & 0 < u < \pi/\sqrt{2}, \\ 0, & u \geqslant \pi/\sqrt{2}. \end{cases}$$

It has been proved in [7] that there exist a.s. collisions if $\gamma < 1/2$.

The hyperbolic system ([24], [28]) is

$$dX_t^j = dB_t^j + \gamma \sum_{k \neq j} \coth\,(X_t^j - X_t^k), \quad 1 \leqslant j \leqslant n,$$
$$X_t^1 \leqslant X_t^2 \leqslant \ldots \leqslant X_t^n.$$

In this case we have

$$\Phi(\mathbf{x}) = \sum_{1 \leqslant j < k \leqslant n} \phi\left(\mathbf{x} \cdot \frac{\mathbf{e}_k - \mathbf{e}_j}{\sqrt{2}}\right)$$

with

$$\phi(u) = \begin{cases} \infty, & u \leqslant 0, \\ -\gamma \log\big(\sinh(\sqrt{2}u)\big), & u > 0, \end{cases}$$

and collisions occur with positive probability if $\gamma < 1/2$.

REFERENCES

[1] R. F. Bass and E. Pardoux, *Uniqueness for diffusions with piecewise constant coefficients,* Probab. Theory Related Fields 76 (1987), pp. 557–572.

[2] M.-F. Bru, *Diffusions of perturbed principal component analysis,* J. Multivariate Anal. 29 (1989), pp. 127–136.

[3] M.-F. Bru, *Wishart processes*, J. Theoret. Probability 4 (1991), pp. 725–751.

[4] E. Cépa, *Equations différentielles stochastiques multivoques,* Sém. Probab. XXIX, Lecture Notes in Math. No 1613, Springer 1995, pp. 86–107.

[5] E. Cépa, *Problème de Skorohod multivoque,* Ann. Probab. 26 (1998), pp. 500–532.

[6] E. Cépa and D. Lépingle, *Diffusing particles with electrostatic repulsion,* Probab. Theory Related Fields 107 (1997), pp. 429–449.

[7] E. Cépa and D. Lépingle, *Brownian particles with electrostatic repulsion on the circle: Dyson's model for unitary random matrices revisited*, ESAIM: Probab. Stat. 5 (2001), pp. 203–224.

[8] E. Cépa and Lépingle, *No multiple collisions for mutually repelling Brownian particles*, Sém. Probab. XL, Lecture Notes in Math. No 1899, Springer 2007, pp. 241–246.

[9] O. Chybiryakov, *Processus de Dunkl et relation de Lamperti*, Ph.D. Thesis, Université de Paris VI (2006).

[10] O. Chybiryakov, L. Gallardo and M. Yor, *Dunkl processes and their radial parts relative to a root system,* Travaux en cours 71, Hermann 2008, pp. 113–197.

Probability and Mathematical Statistics 30/2010, z. 2

[11] J. G. Dai and R. J. Williams, *Existence and uniqueness of semimartingale reflecting Brownian motions in convex polyhedra*, Theory Probab. Appl. 40 (1996), pp. 1–40.
[12] F. Delarue, *Hitting time of a corner for a reflected diffusion in the square*, Ann. Inst. H. Poincaré Probab. Statist. 44 (2008), pp. 946–961.
[13] N. Demni, *The Laguerre process and generalized Hartman–Watson law,* Bernoulli 13 (2007), pp. 556–580.
[14] N. Demni, *A guided tour in the world of radial Dunkl processes*, Travaux en cours 71, Hermann 2008, pp. 199–226.
[15] N. Demni, *Radial Dunkl processes: existence, uniqueness and hitting time*, C. R. Acad. Sci. Paris, Sér. I 347 (2009), pp. 1125–1128.
[16] N. J. Dyson, *A Brownian-motion model for the eigenvalues of a random matrix*, J. Math. Phys. 3 (1962), pp. 1191–1198.
[17] D. J. Grabiner, *Brownian motion in a Weyl chamber, non-colliding particles, and random matrices*, Ann. Inst. H. Poincaré Probab. Statist. 35 (1999), pp. 177–204.
[18] D. Hobson and W. Werner, *Non-colliding Brownian motion on the circle*, Bull. London Math. Soc. 28 (1996), pp. 643–650.
[19] J. E. Humpreys, *Reflection groups and Coxeter groups*, Cambridge University Press, 1990.
[20] K. Inukai, *Collision or non-collision problem for interacting Brownian particles*, Proc. Japan Acad. Ser. A 82 (2006), pp. 66–70.
[21] W. König and N. O'Connell, *Eigenvalues of the Laguerre process as non-colliding squared Bessel processes*, Electron. Comm. Probab. 6 (2001), pp. 107–114.
[22] D. Lépingle and C. Marois, *Equations différentielles stochastiques multivoques unidimensionnelles*, Sém. Probab. XXI, Lecture Notes in Math. No 1247, Springer 1987, pp. 520–533.
[23] H. P. McKean *Stochastic Integrals*, Academic Press, New York 1969.
[24] J. R. Norris, L. C. G. Rogers and D. Williams, *Brownian motions of ellipsoids*, Trans. Amer. Math. Soc. 294 (1986), pp. 757–765.
[25] N. O'Connell, *Random matrices, non-colliding processes and queues*, Sém. Probab. XXXVI, Lecture Notes in Math. No 1801, Springer 2003, pp. 165–182.
[26] M. Rösler and M. Voit, *Markov processes related with Dunkl operators*, Adv. in Appl. Math. 21 (1998), pp. 575–643.
[27] H. Rost and M. E. Vares, *Hydrodynamics of a one-dimensional nearest neighbor model*, Contemp. Math. Amer. Math. Soc. 41 (1985), pp. 329–342.
[28] B. Schapira, *The Heckman–Opdam Markov process*, Probab. Theory Related Fields 138 (2007), pp. 495–519.
[29] H. Spohn, *Dyson's model of interacting Brownian motions at arbitrary coupling strength*, Markov Process. Related Fields 4 (1998), pp. 649–661.
[30] S. R. S. Varadhan and R. J. Williams, *Brownian motion in a wedge with oblique reflection*, Comm. Pure Appl. Math. 38 (1985), pp. 405–443.
[31] R. J. Williams, *Reflected Brownian motion with skew symmetric data in a polyhedral domain*, Probab. Theory Related Fields 75 (1987), pp. 459–485.

Université d'Orléans, MAPMO
CNRS, UMR 6628
Fédération Denis Poisson, FR 2924
B.P. 6759
45067 ORLEANS Cedex 2
E-mail: dominique.lepingle@univ-orleans.fr

Received on 5.11.2009;
revised version on 14.5.2010