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ON EXTREMAL INDEX OF MAX-STABLE STATIONARY PROCESSES

BY

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To Tomasz Rolski,
thankful for all the support and ideas you shared with us!

Abstract. In this contribution we discuss the relation between Pickands-type constants defined for certain Brown–Resnick stationary process $W(t), t \in \mathbb{R}$, as

$$\mathcal{H}_W^\delta = \lim_{T\to\infty} T^{-1}\mathbb{E}\{\sup_{t\in\delta\mathbb{Z}\cap[0,T]} e^{W(t)}\}, \quad \delta \geqslant 0$$

(set $0\mathbb{Z} = \mathbb{R}$ if $\delta = 0$) and the extremal index of the associated max-stable stationary process ξ_W. We derive several new formulas and obtain lower bounds for $\mathcal{H}_W^\delta$ if W is a Gaussian or a Lévy process. As a by-product we show an interesting relation between Pickands constants and lower tail probabilities for fractional Brownian motions.

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1. INTRODUCTION

The motivation for this contribution comes from the importance and the intriguing properties of the classical Pickands constants $\mathcal{H}_W^\delta$ which are defined for any $\delta \geqslant 0$ by (interpret $0\mathbb{Z}$ as $\mathbb{R}$)

$$\mathcal{H}_W^\delta = \lim_{T\to\infty} \frac{1}{T}\mathbb{E}\{\sup_{t\in\delta\mathbb{Z}\cap[0,T]} e^{W(t)}\}, \tag{1.1}$$

where

$$W(t) = \sqrt{2}B_\alpha(t) - |t|^\alpha, \quad t \in \mathbb{R},$$

with B_α a standard fractional Brownian motion with Hurst index $\alpha \in (0, 2]$, that is a centered Gaussian process with stationary increments and variance function $\mathrm{Var}\big(B_\alpha(t)\big) = |t|^\alpha, t \in \mathbb{R}$.

It is well known (but not trivial to prove) that $\mathcal{H}_W^\delta$ is finite and positive for any $\delta \geqslant 0$. The only values known for $\mathcal{H}_W^\delta$ are for $\delta = 0$ and $\alpha \in \{1, 2\}$, see, e.g., [41], [42]. Surprisingly, Pickands and related constants appear in numerous unrelated asymptotic problems, see, e.g., the recent papers [17], [25], [26], [15].

The contribution [19] derived a new formula for Pickands constants, which in fact indicates a direct connection between those constants and max-stable stationary processes, see [12]. The definition of $\mathcal{H}_W^\delta$ in (1.1) is extended in [12] for some general process W provided that it defines a max-stable and stationary process. More precisely, assume throughout in the sequel that

$$W(t) = B(t) - \ln \mathbb{E}\{e^{B(t)}\}, \quad t \in \mathbb{R}, \tag{1.2}$$

where $B(t), t \in \mathbb{R}$, is a random process on the space D of càdlàg functions $f : \mathbb{R} \to \mathbb{R}$ with

$$B(0) = 0, \quad \mathbb{E}\{e^{B(t)}\} < \infty, \quad t \in \mathbb{R}. \tag{1.3}$$

Hence $X(t) = e^{W(t)}$ satisfies $X(0) = 1$ almost surely, and $\mathbb{E}\{X(t)\} = 1, t \in \mathbb{R}$. If $\Pi = \sum_{i=1}^\infty \varepsilon_{P_i}$ is a Poisson point process (PPP) with intensity $x^{-2}dx$ on $(0, \infty)$, and $X_i = e^{W_i}, i \geqslant 1$, are independent copies of the random process $X = e^W$ being independent of Π, then the random process ξ_W defined by

$$\xi_W(t) = \max_{i \geqslant 1} P_i X_i(t) = \max_{i \geqslant 1} P_i e^{W_i(t)}, \quad t \in \mathbb{R}, \tag{1.4}$$

has unit Fréchet marginals and is max-stable. Here ε_x denotes the unit Dirac measure at $x \in \mathbb{R}$. Adopting the definition from [30], we shall refer to W as the *Brown–Resnick stationary process* whenever the associated max-stable process ξ_W is stationary. Note that stationarity of ξ_W means that $\{\xi_W(t), t \in \mathbb{R}\}$ and $\{\xi_W(t + h), t \in \mathbb{R}\}$ have the same distribution for any $h \in \mathbb{R}$.

In the sequel, for the case $\delta = 0$ we shall assume that

$$\mathbb{E}\{\sup_{t \in K} e^{W(t)}\} < \infty$$

for any compact $K \subset \mathbb{R}$. A direct consequence of stationarity of ξ_W and the fact that for any $t_1, \ldots, t_n \in \mathbb{R}$ and $x_1, \ldots, x_n > 0$ (see, e.g., [18], [39])

$$\mathbb{P}\{\xi_W(t_i) \leqslant x_i, \forall i \leqslant n\} = \exp\big(-\mathbb{E}\{\max_{1 \leqslant i \leqslant n} (e^{W(t_i)}/x_i)\}\big) \tag{1.5}$$

is that, for any $b \geqslant 0, \delta \geqslant 0, T > 0$, we have

$$H_W^\delta([0, T]) := \mathbb{E}\{\sup_{t \in \delta\mathbb{Z} \cap [0, T]} e^{W(t)}\} = \mathbb{E}\{\sup_{t \in \delta\mathbb{Z} \cap [b, b+T]} e^{W(t)}\}.$$

Consequently, $\mathcal{H}_W^\delta$ defined in (1.1) exists and is given by (see [12])

$$\mathcal{H}_W^\delta = \inf_{T>0} \frac{1}{T} H_W^\delta([0,T]) \in [0,\infty). \tag{1.6}$$

Note that if $\delta > 0$, then (1.6) implies that

$$\mathcal{H}_W^\delta \leqslant \frac{H_W^\delta([0,\delta-\varepsilon])}{\delta-\varepsilon} = \frac{1}{\delta-\varepsilon}$$

for any $\varepsilon \in (0,\delta)$, hence letting ε tend to zero yields $\mathcal{H}_W^\delta \in [0, 1/\delta]$.

Interestingly, $\mathcal{H}_W^\delta$ is related to the extremal index of the stationary process

$$\xi_W^\delta(t) = \xi_W(\delta t), \quad t \in \mathbb{Z}, \delta > 0,$$

where we set $\xi_W^\delta(t) = \xi_W(t)$ if $\delta = 0$. Indeed, by (1.5),

$$\begin{aligned}\lim_{T\to\infty} \mathbb{P}\{\max_{i\in\delta\mathbb{Z}\cap[0,T]} \xi_W(t) \leqslant Tx\} &= \exp\big(-\lim_{T\to\infty} \mathbb{E}\{\max_{i\in\delta\mathbb{Z}\cap[0,T]} (e^{W(i)}/T)\}x^{-1}\big) \\ &= (e^{-1/x})^{\mathcal{H}_W^\delta}, \quad x > 0.\end{aligned} \tag{1.7}$$

Thus the Fréchet limit result in (1.7), which is already shown in [50] (see also Proposition 3.1 in [10] and [18]), states that the extremal index of the stationary process $\xi_W^\delta(t), t \in \mathbb{Z}$, is given for any $\delta > 0$ by

$$\theta_W^\delta = \delta \mathcal{H}_W^\delta \in [0,1]. \tag{1.8}$$

Clearly, the constant $\mathcal{H}_W^\delta$ is positive if and only if the extremal index θ_W^δ of the stationary process ξ_W^δ is positive.

Numerous papers in the literature have discussed the calculation and estimation of extremal index of stationary processes, see, e.g., the recent articles [46], [10], [38], [35], [33], [21] and the references therein. The primary goal of this contribution is to study Pickands-type constants $\mathcal{H}_W^\delta$ by exploring the properties of the extremal index θ_W^δ. In particular, we are interested in establishing tractable conditions that guarantee the positivity of $\mathcal{H}_W^\delta$.

By our assumptions it is clear that ξ_W^δ is stationary and *jointly regularly varying*, hence in view of Theorem 2.1 in [5] (see also [29]), there exists a so-called *tail process*

$$Y^\delta(i), \quad i \in \mathbb{Z},$$

of the stationary process X, which was introduced in [5]. It turns out that for any $m \leqslant n, m, n \in \mathbb{Z}$, we have the stochastic representation

$$\big(Y^\delta(m), \ldots, Y^\delta(n)\big) \stackrel{d}{=} \big(\mathcal{P}X^\delta(m), \ldots, \mathcal{P}X^\delta(n)\big), \tag{1.9}$$

where $X^\delta(i) := e^{W(\delta i)}, i \in \mathbb{Z}$, with $\mathcal{P}$ a unit Pareto random variable with survival function $1/x, x > 1$, being independent of the process X.

Under the *finite mean cluster size condition* (see Condition 2.1 below) and condition $\mathcal{A}(a_n)$ (see [5], [4], [32]) it follows that θ_W^δ is positive, see the seminal contribution [5].

We shall show the positivity of the extremal index under a weaker condition, namely supposing that

$$\lim_{|z|\to\infty, z\in\mathbb{Z}} W(z\delta) = -\infty \tag{1.10}$$

holds almost surely for $\delta \in (0, \infty)$. In our derivations the following simple result is crucial:

LEMMA 1.1. *If* $r_n, n \geqslant 1$, *are positive integers such that*

$$\lim_{n\to\infty} r_n = \lim_{n\to\infty} n/r_n = \infty,$$

then for any $\delta \in (0, \infty)$ *we have*

$$\widetilde{\theta_W^\delta} := \lim_{n\to\infty} \frac{n}{r_n} \mathbb{P}\{\max_{i\in\{0,\delta,\ldots,\delta r_n\}} \xi_W(t) > n\} = \theta_W^\delta \in [0, \infty). \tag{1.11}$$

In the next section we shall show that the new expression for the extremal index in (1.11) is positive under (1.10). Using the explicit form of the *tail process*, we shall derive several new interesting formulas for $\mathcal{H}_W^\delta$.

Brief outline of the rest of the paper is the following. In Section 2 we give our main results which establish the positivity of the Pickands-type constants and some new formulas. In Section 3 we shall discuss the connection with mixed moving maxima (M3) representation of Brown–Resnick processes. Then we derive some explicit lower bounds for $\mathcal{H}_W^\delta$ in case that B in (1.2) is a Gaussian or a Lévy process, and discuss the relation between $\mathcal{H}_W^0$ and the *mean cluster index*. Further, we shall show that the classical Pickands constants are related to a small ball problem. All the proofs are relegated to Section 4.

2. MAIN RESULTS

We keep the same setup as in the Introduction, and denote additionally by $\mathcal{E}$ a unit exponential random variable which is independent of everything else. According to [5] a candidate for the extremal index is given by the formula

$$\widehat{\theta_W^\delta} = \lim_{m\to\infty} \mathbb{P}\{\max_{1\leqslant i\leqslant m} Y^\delta(i) \leqslant 1\}, \tag{2.1}$$

where $Y^\delta(i), i \in \mathbb{Z}$, is the *tail process* of ξ_W^δ, see [5]. As in the aforementioned paper we shall impose the *finite mean cluster size condition* of [5], Condition 4.1:

CONDITION 2.1. *Given* $\delta > 0$, *there exists a sequence of positive integers* $r_n, n \in \mathbb{N}$, *satisfying* $\lim_{n\to\infty} r_n/n = 1/\lim_{n\to\infty} r_n = 0$ *such that*

$$\lim_{m\to\infty} \limsup_{n\to\infty} \mathbb{P}\Big\{ \max_{m\leqslant |k|\leqslant r_n} \xi_W(k\delta) > nx \Big| \xi_W(0) > nx \Big\} = 0 \tag{2.2}$$

holds for any $x > 0$.

By Proposition 4.2 in [5] we infer that $\widehat{\theta_W^\delta} > 0$ follows from Condition 2.1. Our main result below establishes new formulas for $\mathcal{H}_W^\delta$. Moreover, from the above-mentioned reference, Condition 2.1 together with well-known $\mathcal{A}(a_n)$ conditions of Hsing and Davis implies that the candidate for the extremal index is equal to the extremal index, i.e., $\widehat{\theta_W^\delta} = \theta_W^\delta > 0$. It is well known that $\mathcal{A}(a_n)$ is implied by the strong mixing of ξ_W^δ. However, our results derived below do not require strong mixing but just mixing of ξ_W^δ.

THEOREM 2.1. *Let* $X(t) = e^{W(t)}, t \in \mathbb{R}$, *with* W *as in* (1.2) *be such that* (1.3) *holds and* $\xi_W(t), t \in \mathbb{R}$, *is max-stable and stationary. Then* (1.10) *holds for* $\delta > 0$ *if and only if Condition* 2.1 *holds. Moreover*, *if* (1.10) *holds for* $\delta > 0$, *then we have consecutively*

$$\mathcal{H}_W^\delta = \frac{1}{\delta}\mathbb{P}\{\sup_{i<0} W^\delta(i) < 0 = \sup_{i\in\mathbb{Z}} W^\delta(i)\}, \tag{2.3}$$

$$\mathcal{H}_W^\delta = \frac{1}{\delta}\mathbb{P}\big\{ \sup_{i\geqslant 1}\big(\mathcal{E} + W^\delta(i)\big) \leqslant 0\big\}, \tag{2.4}$$

$$\mathcal{H}_W^\delta = \frac{1}{\delta}[\mathbb{E}\{\sup_{i\geqslant 0} e^{W^\delta(i)}\} - \mathbb{E}\{\sup_{i\geqslant 1} e^{W^\delta(i)}\}] \in (0, 1/\delta), \tag{2.5}$$

where $W^\delta(t) = W(t\delta), t \in \mathbb{Z}$, *and* $\mathcal{E}$ *is a unit exponential random variable independent of* W.

REMARK 2.1. (a) *If* $\mathbb{P}\left\{W^\delta(i) = 0\right\} = 0$ *for any negative integer* i, *then*

$$\mathbb{P}\{\sup_{-m\leqslant i<0} W^\delta(i) < 0 = \sup_{-m\leqslant j\leqslant m} W^\delta(j)\} = \mathbb{P}\{\sup_{-m\leqslant j\leqslant m} W^\delta(j) = 0\}$$

holds for any integer $m > 1$. *Consequently*, *by* (2.3) *we have*

$$\begin{aligned} \mathcal{H}_W^\delta &= \frac{1}{\delta}\lim_{m\to\infty} \mathbb{P}\{\sup_{-m\leqslant i<0} W^\delta(i) < 0 = \sup_{-m\leqslant j\leqslant m} W^\delta(j)\} \\ &= \frac{1}{\delta}\mathbb{P}\{\sup_{i\in\mathbb{Z}} W^\delta(i) = 0\} > 0, \end{aligned} \tag{2.6}$$

which has been shown in [19] *for the case where* B *is a standard fractional Brownian motion. The assumption* $W(0) = 0$ *can be removed* (see [27]).

(b) *We assumed above that* ξ_W *has càdlàg sample paths in order to define* H_W^0. *For the results of Theorem* 2.1, *this assumption is not needed.*

(c) *In* [12] *it is shown that under the assumptions of Theorem* 2.1 *we have*

$$\mathcal{H}_W^\delta = \mathbb{E}\left\{ \frac{\sup_{t\in\delta\mathbb{Z}} e^{W(t)}}{\delta \sum\limits_{t\in\delta\mathbb{Z}} e^{W(t)}} \right\}. \tag{2.7}$$

According to (2.5), *for calculation of* $\mathcal{H}_W^\delta$ *it suffices to know* $W(t), t \in \delta\mathbb{Z}, t > 0$, *i.e., only the values of* W *for positive* t *matter. This is not the case for formula* (2.7). *Both* (2.7) *and* (2.5) *are given in terms of expectations and not as limits, which is a great advantage for simulations. To this end, we mention that simulation of Pickands constants has been the topic of many works* (see, e.g., [9], [36], [19]).

(d) *If* $X(t) = e^{W(t)}, t \in \mathbb{R}$, *is Brown–Resnick stationary, i.e., the associated max-stable process with* ζ_W *is max-stable and stationary, then the time reversed process* $V(t) = W(-t), t \in \mathbb{R}$, *also defines Brown–Resnick stationary processes. Moreover, for any* $\delta \geqslant 0$

$$\mathcal{H}_W^\delta = \mathcal{H}_V^\delta.$$

Consequently, the formulas in Theorem 2.1 *can be stated with* V *instead of* W, *for instance we have*

$$\begin{aligned} \mathcal{H}_W^\delta &= \frac{1}{\delta}\mathbb{P}\big\{ \sup_{i\leqslant -1} \big(\mathcal{E} + W^\delta(i)\big) \leqslant 0\big\} \\ &= \frac{1}{\delta}\mathbb{P}\{W^\delta(i) < 0, i \in \mathbb{N}; W^\delta(i) \leqslant 0, i \in \mathbb{Z}\}. \end{aligned} \tag{2.8}$$

(e) *If* $W(t) = \sqrt{2}tL - t^2$ *with* L *an* $N(0,1)$ *random variable with distribution* Φ *and probability density function* φ, *by* (2.4) *we have*

$$\begin{aligned} \mathcal{H}_W^\delta &= \frac{1}{\delta}\int_0^\infty \mathbb{P}\big\{\mathcal{E} + \sup_{i\geqslant 1}\big(\sqrt{2}\delta i b - (\delta i)^2\big) \leqslant 0\big\}\varphi(b)db \\ &= \frac{1}{\delta}\int_{-\delta/\sqrt{2}}^{\delta/\sqrt{2}} \varphi(b)db = \frac{1}{\delta}\big[\Phi(\delta/\sqrt{2}) - \Phi(-\delta/\sqrt{2})\big] \end{aligned} \tag{2.9}$$

for any $\delta > 0$. *Consequently, letting* $\delta \to 0$ *we obtain the well-known result*

$$\mathcal{H}_W^0 = \sqrt{2}\varphi(0) = \frac{1}{\sqrt{\pi}}.$$

A canonical example for W with representation (1.2) is the case when B is a centered Gaussian process with stationary increments, continuous sample paths, and variance function σ^2. Then the max-stable process ξ_W is stationary, see [40]. Using a direct argument, we establish in the next theorem the positivity of $\mathcal{H}_W^0$.

THEOREM 2.2. *If*

$$\liminf_{t\to\infty} \frac{\sigma^2(t)}{\ln t} > 8,$$

then $\mathcal{H}_W^0 > 0$.

Since (2.2) implies (1.10) (see Corollary 2.4 in [34] or [30]), using $\mathcal{H}_W^0 \geqslant \mathcal{H}_W^\delta$ for any $\delta > 0$ we immediately establish the positivity of $\mathcal{H}_W^0$.

Indeed, the positivity of $\mathcal{H}_W^0$ is crucial for the study of extremes of Gaussian processes. Condition (2.2) can be easily checked, e.g., if $W(t) = \sqrt{2}B_\alpha(t) - |t|^\alpha$. Consequently, the classical Pickands constants $\mathcal{H}_W^\delta$ are positive for any $\delta \geqslant 0$. This fact is highly non-trivial; after announced in Pickands' pioneering work [41], correct proofs were obtained later by Pickands himself, and in [7], [43], see, e.g., Theorem B3 in [8]. We note in passing that under general conditions on σ^2 the positivity of $\mathcal{H}_W^0$ is established in [11].

Apart from the alternative proof for the positiveness of the original Pickands constants, Theorem 2.1 extends to non-Gaussian processes W. For the above Gaussian setup, direct calculations show the positivity of $\mathcal{H}_W^\delta$ under a slightly weaker condition than (2.2).

3. DISCUSSIONS AND EXTENSIONS

3.1. Relation with lower tail probabilities. For the classical case of Piterbarg constants $\mathcal{H}_{B_\alpha}$, i.e., for $W(t) = \sqrt{2}B_\alpha(t) - |t|^\alpha, t \in \mathbb{R}, \alpha \in (0,2]$, we show below that (2.6) implies a nice relation with a small ball problem.

PROPOSITION 3.1. *For any* $\alpha \in (0,2]$ *we have*

$$\lim_{\eta\to 0} \eta^{-2/\alpha}\mathbb{P}\{\forall_{k\in\mathbb{Z}\setminus\{0\}} B_\alpha(1/k) \leqslant \eta\} = 2^{1/\alpha}\mathcal{H}_{B_\alpha}.$$

The above result strongly relates to the self-similarity property of fractional Brownian motion. In case of a general Gaussian W, ξ_W is still stationary if W has stationary increments. However, fBm is the only centered Gaussian process with stationary increments being further self-similar. Hence, no obvious extensions of the above relation with lower tails can be derived for general W.

3.2. Non-Gaussian W. The classical Pickands constants are defined for $W(t) = \sqrt{2}B_\alpha(t) - |t|^\alpha$ with B_α a standard fBm with Hurst index $\alpha/2 \in (0,1]$. The more general case where B_α is substituted by a centered Gaussian process with stationary increments is discussed in detail in [11].

Our setup clearly allows for any random process W, not necessarily Gaussian, which is Brown–Resnick stationary. Along with the Gaussian case of W, the Lévy one has also been dealt already in the literature. In view of [23], [49], if $B(t), t \geqslant 0$, is a Lévy process such that

$$\Phi(1) < \infty, \quad \Phi(\theta) := \ln \mathbb{E}\{e^{\theta B(1)}\}, \tag{3.1}$$

then $W(t) = B(t) - \Phi(1)t, t \geqslant 0$, is Brown–Resnick stationary, i.e., $\xi_W(t), t \geqslant 0$, is max-stable stationary with unit Gumbel marginals.

In [31] an important constant appears in the asymptotic analysis of the maximum of standardised increments of random walks, which in fact is the Pickands constant $\mathcal{H}_W^\delta, \delta > 0$, introduced here for W as above. In [31], Lemma 5.16, a new formula for $\mathcal{H}_W^\delta$ is derived, which is identical with our formula in (2.8). Another instance of the Pickands constant given by formula (2.3) is displayed in [44], Theorem 5.3. With the notation of that theorem, for $\delta = 1$ we have

$$W(i) = \sum_{j=1}^{i} A_i,$$

where A_i's are i.i.d. with the same distribution as $Z\mathbb{I}(U \leqslant e^{-\eta Z})$ for some $\eta > 0$ with U uniformly distributed on $(0, 1)$ being independent of Z which has some pdf symmetric around zero.

Pickands constants appear also in the context of semi-min-stable processes, see [51]. In view of the aforementioned paper, several results derived here for max-stable processes are extendable to semi-min-stable processes.

3.3. Finite mean cluster size condition. As noted in [45], Condition 2.1 is implied by the following so-called *short-lasting exceedance condition*:

CONDITION 3.1. *Given* $\delta > 0$, *there exists a sequence of integers* $r_n, n \in \mathbb{N}$, *satisfying* $\lim_{n\to\infty} r_n/n = 1/\lim_{n\to\infty} r_n = 0$ *such that*

$$\lim_{m\to\infty} \limsup_{n\to\infty} \sum_{k=m}^{r_n} \mathbb{P}\left\{\xi_W(k\delta) > nx \middle| \xi_W(0) > nx\right\} = 0 \tag{3.2}$$

is valid for any $x > 0$.

This latter condition is a rephrasing of the so-called B condition (see, e.g., [1], [13], [2]) which was formulated by discretising the original Berman's condition, see [6]. Condition 3.1 is weaker than the $D'(xn)$ condition of Leadbetter as discussed in [22], Section 5.3.2.

Commonly, Condition 2.1 assumed for $x = 1$ is referred to as the *anti-clustering condition*, see, e.g., [46], [47]. Clearly, the finite mean cluster size condition is stronger than the anti-clustering condition. The latter appears in various contexts related to extremes of stationary processes, see, e.g., [3], [37], [46], [5], [47] and the references therein.

3.4. M3 representation. Since we assume that ξ_W is max-stable stationary with càdlàg sample paths, and W with representation (1.2) is such that B satisfies (1.3), it follows that assuming the almost sure convergence

$$W(t) \to -\infty$$

as $|t| \to \infty$ is equivalent to the fact that ξ_W has a *mixed moving maxima representation* (for short, M3), see [20], Theorem 3, and [52]. More specifically, under (3.4) we have the equality of finite-dimensional distributions

$$\xi_W(t) \stackrel{d}{=} \max_{i \geqslant 1} P_i e^{F_i(t-T_i)}, \quad t \in \mathbb{R}, \tag{3.3}$$

between the right-hand side and the left-hand side in (3.3), where the F_i's are independent copies of a measurable càdlàg process $F_W(t), t \in \mathbb{R}$, satisfying almost surely

$$\sup_{t \in \mathbb{R}} F_W(t) = F_W(0) = 0, \tag{3.4}$$

and $\sum_{i=1}^{\infty} \varepsilon_{(P_i,T_i)}$ is a PPP in $(0, \infty) \times \mathbb{R}$ with intensity $C_W \cdot p^{-2} dp \cdot dt$ with

$$C_W = \big(\mathbb{E}\{ \int_{\mathbb{R}} e^{F_W(t)}\, dt\}\big)^{-1} \in (0, \infty). \tag{3.5}$$

Moreover, ξ_W^δ, the restriction of ξ_W on $\delta\mathbb{Z}$, has an M3 representation for any $\delta > 0$, see [12] for more details. Denote the corresponding constant in the intensity of this PPP by $C_W^\delta > 0$ (and thus C_W^0 is just C_W given in (3.5)).

In view of [12], Proposition 1, if $\xi_W^\delta, \delta > 0$, admits an M3 representation as mentioned above, then

$$\mathcal{H}_W^\delta = C_W^\delta, \tag{3.6}$$

provided that (1.10) holds. Hence Theorem 2.1 presents new formulas for C_W^δ. Note in passing that (3.6) has been shown in [40]. Therein it is proved that C_W^δ is given by the right-hand side of (2.6) assuming further that $W(t) = B(t) - \mathbb{E}\{e^{\ln B(t)}\}, t \in \mathbb{R}$, with B a centered Gaussian process with stationary increments satisfying $W(0) = 0$ almost surely.

In view of [12], Theorem 1, if (1.10) holds, then we have

$$\mathcal{H}_W^\delta = \mathbb{E}\left\{ \frac{M^\delta}{S^\delta} \right\} = C_W^\delta, \tag{3.7}$$

with $M^\delta := \max_{i \in \mathbb{Z}} e^{W(i\delta)}$ and $S^\delta := \delta \sum_{t \in \delta\mathbb{Z}} e^{W(t)}$. Thus $\mathcal{H}_W^\delta > 0$.

The representation of $\mathcal{H}_W^\delta$ as an expectation of the ratio M^δ / S^δ is crucial for its simulation. Such a representation has initially been shown in [19] for classical Pickands constants.

3.5. Lower bounds. In Theorem 2.1 we present new formulas for $\mathcal{H}_W^\delta$, which in turn establish the positivity of $\mathcal{H}_W^\delta$ and thus the positivity for the extremal index of ξ_W^δ. If only the positivity of $\mathcal{H}_W^\delta$ is of primary interest, then the conditions of Theorem 2.1 can be relaxed. Next, we consider two important classes of processes for B, that is, centered Gaussian processes with stationary increments and Lévy processes. Results for the Lévy case have already been given in [12].

For particular values of δ, we show that it is possible to derive a positive lower bound for $\mathcal{H}_W^\delta$ and thus establishing the positivity of $\mathcal{H}_W^\delta$. Let $x_+ := \max(x, 0)$.

THEOREM 3.1. (i) *Let* $W(t) = B(t) - \sigma^2(t)/2$, $t \geqslant 0$, *where* $B(t)$ *is a centered Gaussian process with stationary increments and variance function* σ^2 *such that* $\sigma(0) = 0$. *Then for any* $\delta > 0$

$$\mathcal{H}_W^\delta \geqslant \frac{1}{\delta} \max\left(0, 1 - \sum_{k=1}^{\infty} \exp\left(-\frac{\sigma^2(\delta k)}{8}\right)\right). \tag{3.8}$$

(ii) *Let* $W(t) = B(t) - \Phi(1)t$, $t \geqslant 0$, *where* $B(t)$ *is a Lévy process satisfying the condition* (3.1). *Then for any* $\delta > 0$

$$\mathcal{H}_W^\delta \geqslant \frac{1}{\delta} \frac{\max\big(0, 1 - 2\exp\big(\big(\Phi(1/2) - \frac{1}{2}\Phi(1)\big)\delta\big)\big)}{1 - \exp\big(\big(\Phi(1/2) - \frac{1}{2}\Phi(1)\big)\delta\big)}. \tag{3.9}$$

REMARK 3.1. (a) *It follows from* (i) *of Theorem* 3.1 *that if* $\sigma(\delta k) \geqslant C(\delta k)^{\kappa/2}$ *for all* $k \in \mathbb{N}$ *and some* $\kappa > 0$, *then*

$$\mathcal{H}_W^\delta \geqslant \frac{1}{\delta}\left(1 - \frac{1}{\delta} \frac{\Gamma(1/\kappa)}{\kappa\,(C^2/8)^{1/\kappa}}\right). \tag{3.10}$$

Since $\mathcal{H}_W^0 \geqslant \mathcal{H}_W^\delta$ *for any* $\delta > 0$, *the above implies* $\mathcal{H}_W^0 > 0$.

(b) *If* B *is a Lévy process as in* (ii) *of Theorem* 3.1, *then* (see the proof in Section 4)

$$\mathcal{H}_W^0 \geqslant \frac{1}{8}[\Phi(1) - 2\Phi(1/2)] > 0. \tag{3.11}$$

3.6. Case $\delta = 0$. Since (1.7) holds also for $\delta = 0$ and $\mathcal{H}_W^0 \geqslant \mathcal{H}_W^\delta$, it follows that the extremal index of the continuous process ξ_W is

$$\widetilde{\theta}_W = \mathcal{H}_W^0 \geqslant 0,$$

which is positive provided that (1.10) holds. In the special case where $W(t) = \sqrt{2}B_\alpha(t) - |t|^\alpha$ we have

$$\lim_{\delta \downarrow 0} \mathcal{H}_W^\delta = \mathcal{H}_W^0 =: \mathcal{H}_W, \tag{3.12}$$

hence for such W and for any $\alpha \in (0,2]$

$$\widetilde{\theta}_W = \lim_{\delta \downarrow 0} \frac{\theta_W^\delta}{\delta}. \tag{3.13}$$

Recall that we denote by $\theta_W^\delta, \delta > 0$, the extremal index of ξ_W^δ. Using the terminology of [28] we refer to $\overline{H}_W$ defined by (assuming that the limit exists)

$$\lim_{\delta \downarrow 0} \frac{\theta_W^\delta}{\delta} = \lim_{\delta \downarrow 0} \mathcal{H}_W^\delta = \overline{H}_W$$

as the *mean cluster index of the process* W. Since for any $T > 0$ and $\delta > 0$

$$0 \leqslant \mathbb{E}\{ \sup_{t \in \delta\mathbb{Z} \cap [0,T]} e^{W(t)} \} =: \mathcal{H}_W^0([0,T]),$$

we have clearly $\overline{H}_W \in [0, \mathcal{H}_W]$.

We show next that if ξ_W has an M3 representation, then $\overline{H}_W$ is positive.

PROPOSITION 3.2. *Suppose ξ_W is max-stable and stationary with $W(0) = 0$. If ξ_W has an* M3 *representation and $\overline{H}_W$ exists, then*

$$\overline{H}_W \geqslant \mathbb{E}\left\{ \frac{\sup_{t \in \mathbb{R}} e^{W(t)}}{\eta \sum\limits_{t \in \eta\mathbb{Z}} e^{W(t)}} \right\} > 0 \tag{3.14}$$

holds for any $\eta > 0$.

REMARK 3.2. (a) *In view of Theorems* 2 *and* 3 *in* [12] *we have for some general W as in* (1.2), *with B being a Gaussian or Lévy process*,

$$\mathcal{H}_W^0 = \mathbb{E}\left\{ \frac{\sup_{t \in \mathbb{R}} e^{W(t)}}{\eta \sum\limits_{t \in \eta\mathbb{Z}} e^{W(t)}} \right\} = \mathbb{E}\left\{ \frac{\sup_{t \in \mathbb{R}} e^{W(t)}}{\int_{t \in \mathbb{R}} e^{W(t)}\, dt} \right\} \tag{3.15}$$

for any $\eta > 0$. Consequently, under these conditions and the setup of Proposition 3.2,

$$\mathcal{H}_W^0 = \overline{H}_W. \tag{3.16}$$

(b) *If $W(t) = \sqrt{2}B_\alpha(t) - |t|^\alpha$, $t \in \mathbb{R}$, by* (3.12) *and* (2.4) *for any $\alpha \in (0,2]$*

$$\mathcal{H}_W^0 = \overline{H}_W = \lim_{\delta \downarrow 0} \frac{1}{\delta} \mathbb{P}\big\{ \sup_{i \geqslant 1} \big(\mathcal{E} + W^\delta(i)\big) \leqslant 0 \big\}, \tag{3.17}$$

with $\mathcal{E}$ a unit exponential random variable independent of W. Expression (3.17) *of the classical Pickands constant was initially derived in* [1] *for some general W*, (see also a recent contribution [2]). *In* [28], *Proposition* 3, *or the formula in* [24], *p.* 44, *the classical Pickands constant is the limit of a cluster index.*

4. PROOFS

Proof of Lemma 1.1. Since $\lim_{n\to\infty} r_n = \infty$, by (1.7) and (1.8) we obtain

$$\lim_{n\to\infty} r_n^{-1}\mathbb{E}\{\max_{i\in\{0,\delta,\ldots,\delta r_n\}} e^{W(i)}\} = \delta\mathcal{H}_W^\delta = \theta_W^\delta.$$

For any $n \in \mathbb{N}$ we have

$$\begin{aligned}\frac{\mathbb{P}\{\max_{i\in\{0,\delta,\ldots,\delta r_n\}} \xi_W(i) > n\}}{r_n\mathbb{P}\{\xi_W(0) > n\}} &= \frac{\mathbb{P}\{\max_{i\in\{0,\delta,\ldots,\delta r_n\}} \xi_W(i) > n\}}{r_n[1-e^{-1/n}]} \\ &\sim n r_n^{-1}[1-\mathbb{P}\{\max_{i\in\{0,\delta,\ldots,\delta r_n\}} \xi_W(i) \leqslant n\}] \\ &= n r_n^{-1}[1-e^{-c_n/n}], \quad c_n := \mathbb{E}\{\max_{i\in\{0,\delta,\ldots,\delta r_n\}} e^{W(i)}\},\end{aligned}$$

where the last equality follows from (1.5). The assumption $\lim_{n\to\infty} n/r_n = \infty$ and $\mathbb{E}\{e^{W(i)}\} = 1, i \in \delta\mathbb{Z}$, imply

$$\frac{c_n}{n} \leqslant \frac{1}{n}\mathbb{E}\{\sum_{i\in\{0,\delta,\ldots,\delta r_n\}} e^{W(i)}\} = \frac{r_n+1}{n} \to 0, \quad n\to\infty. \tag{4.1}$$

Consequently,

$$\frac{\mathbb{P}\{\max_{i\in\{0,\delta,\ldots,\delta r_n\}} \xi_W(i) > n\}}{r_n\mathbb{P}\{\xi_W(0) > n\}} \sim r_n^{-1}\mathbb{E}\{\max_{i\in\{0,\delta,\ldots,\delta r_n\}} e^{W(i)}\} \sim \theta_W^\delta, \quad n\to\infty,$$

hence the claim follows. ■

Proof of Theorem 2.1. We show first the stochastic representation (1.9). Recall that $X(t) = e^{W(t)}$ and for $\delta > 0$ we set

$$W^\delta(t) = W(\delta t), \quad X^\delta(t) = e^{W^\delta(t)}, \quad t\in\mathbb{Z}.$$

By (1.5), the fact that $\mathbb{P}\{\xi_W(0) \leqslant x\} = e^{-1/x}, x > 0$, and the assumption that $X(0) = 1$ almost surely, for any $y_1,\ldots,y_n$ positive and $y_0 > 1$ we have

$$\begin{aligned}&\mathbb{P}\{\xi_W^\delta(i) \leqslant Ty_i, i=0,\ldots,n \big| \xi_W^\delta(0) > T\} \\ &\quad= \frac{1-\mathbb{P}\{\xi_W^\delta(0) \leqslant T, \xi_W^\delta(i) \leqslant Ty_i, i\in\{0,\ldots,n\}\}}{\mathbb{P}\{\xi_W^\delta(0) > T\}} \\ &\quad\quad - \frac{1-\mathbb{P}\{\xi_W^\delta(i) \leqslant Ty_i, i\in\{0,\ldots,n\}\}}{\mathbb{P}\{\xi_W^\delta(0) > T\}} \\ &\quad= \frac{1-\exp\big(-\mathbb{E}\{\max\big(X^\delta(0), \max_{i\in\{1,\ldots,n\}} X^\delta(i)/y_i\big)\}T^{-1}\big)}{1-e^{-1/T}}\end{aligned}$$

$$
\begin{aligned}
&- \frac{1-\exp\left(-\mathbb{E}\{\max_{i\in\{1,\dots,n\}} X^\delta(i)/y_i\}T^{-1}\right)}{1-e^{-1/T}}\\
&\sim T\Bigg[1-\Big[1-\frac{1}{T}\mathbb{E}\big\{\max\big(1, \max_{i\in\{0,\dots,n\}} X^\delta(i)/y_i\big)\big\}\Big]\\
&\quad -\Big(1-\Big[1-\frac{1}{T}\mathbb{E}\big\{\max_{i\in\{0,\dots,n\}} X^\delta(i)/y_i\big\}\Big]\Big)\Bigg]\\
&\to \mathbb{E}\big\{\big(1-\max_{i\in\{0,\dots,n\}} X^\delta(i)/y_i\big)_+\big\}, \quad T\to\infty,\\
&= \mathbb{P}\big\{\mathcal{P}\leqslant y_0, \mathcal{P}X^\delta(i)\leqslant y_i, \forall i\in\{1,\dots,n\}\big\},
\end{aligned}
$$

where $\mathcal{P}$ is a unit Pareto random variable with survival function $1/s, s>1$, independent of the process X. Hence the claim in (1.9) follows by Theorem 2.1 (ii) in [5]. Next, by the above derivations for any sequence of integers $r_n > m\in\mathbb{N}$ for any $x>0$ (recall $X^\delta(0)=1$ almost surely), we have

$$
\begin{aligned}
&1-\mathbb{P}\big\{\max_{m\leqslant|i|\leqslant r_n} \xi_W^\delta(i) > nx \big| \xi_W^\delta(0) > nx\big\}\\
&\quad = \frac{\mathbb{P}\{\max_{m\leqslant|i|\leqslant r_n} \xi_W^\delta(i)\leqslant nx, \xi_W^\delta(0)>nx\}}{\mathbb{P}\{\xi_W^\delta(0)>nx\}}\\
&\quad = \frac{1-\exp\left(-\mathbb{E}\left\{\max\left(X^\delta(0), \max_{|i|\in\{m,\dots,r_n\}} X^\delta(i)\right)\right\}(nx)^{-1}\right)}{1-e^{-1/(nx)}}\\
&\qquad - \frac{1-\exp\left(-\mathbb{E}\{\max_{|i|\in\{m,\dots,r_n\}} X^\delta(i)\}(nx)^{-1}\right)}{1-e^{-1/(nx)}}\\
&\quad \sim nx\Bigg[1-\Big[1-\frac{1}{nx}\mathbb{E}\big\{\max\big(1, \max_{|i|\in\{m,\dots,r_n\}} X^\delta(i)\big)\big\}\Big]\\
&\qquad -\Big(1-\Big[1-\frac{1}{nx}\mathbb{E}\big\{\max_{|i|\in\{m,\dots,r_n\}} X^\delta(i)\big\}\Big]\Big)\Bigg]\\
&\quad \sim \mathbb{E}\big\{\big(1-\max_{|i|\in\{m,\dots,r_n\}} X^\delta(i)\big)_+\big\},
\end{aligned}
$$

where we used the fact that, as in (4.1), the condition $\lim_{n\to\infty} r_n = \lim_{n\to\infty}\frac{n}{r_n} = \infty$ implies

$$
\lim_{n\to\infty}\frac{1}{n}\mathbb{E}\big\{\max\big(X^\delta(0), \max_{|i|\in\{m,\dots,r_n\}} X^\delta(i)\big)\big\}=0,
$$

and

$$
\lim_{n\to\infty}\frac{1}{n}\mathbb{E}\{\max_{|i|\in\{m,\dots,r_n\}} X^\delta(i)\}=0.
$$

Consequently,

$$\begin{aligned}\lim_{m\to\infty}\limsup_{n\to\infty}\mathbb{P}\Big\{\max_{m\leqslant|i|\leqslant r_n}\xi_W^\delta(i)>nx\Big|\xi_W^\delta(0)>nx\Big\}\\ =\lim_{m\to\infty}\limsup_{n\to\infty}\Big[1-\mathbb{E}\Big\{\big(1-\max_{|i|\in\{m,\dots,r_n\}}X^\delta(i)\big)_+\Big\}\Big]\\ =1-\lim_{m\to\infty}\mathbb{E}\Big\{\big(1-\max_{|i|\in\mathbb{Z},i\geqslant m}X^\delta(i)\big)_+\Big\}=0,\end{aligned}$$

where we used the assumption (1.10). Hence Condition 2.1 holds.

By Proposition 4.2 in [5], we see that Condition 2.1 implies (1.10). Moreover, since

$$\mathbb{P}\{\xi_W(0)>n\}=1-e^{-1/n}\sim\frac{1}{n},\quad n\to\infty,$$

Proposition 4.2 in [5] and Lemma 1.1 imply

$$\theta_W^\delta=\widetilde{\theta_W^\delta}=\widehat{\theta_W^\delta}>0.$$

Consequently,

$$\widehat{\theta_W^\delta}=\mathbb{P}\{\sup_{i\geqslant 1}Y^\delta(i)\leqslant 1\}=\lim_{n\to\infty}\mathbb{P}\{\mathcal{P}\sup_{n\geqslant i\geqslant 1}X^\delta(i)\leqslant 1\},\tag{4.2}$$

and so

$$\begin{aligned}\widehat{\theta_W^\delta}&=\lim_{n\to\infty}\mathbb{E}\Big\{\big(1-\sup_{n\geqslant i\geqslant 1}X^\delta(i)\big)_+\Big\}=\mathbb{E}\Big\{\big(1-\sup_{i\geqslant 1}X^\delta(i)\big)_+\Big\}\\ &=\mathbb{E}\{\sup_{i\geqslant 0}X^\delta(i)-\sup_{i\geqslant 1}X^\delta(i)\}\in(0,1],\end{aligned}$$

where the second last equality follows from the monotone convergence theorem. In fact, the above claim readily follows also from Remark 4.7 in [5]. Further from (4.2) we obtain

$$\begin{aligned}\lim_{n\to\infty}\mathbb{P}\{\mathcal{P}\sup_{n\geqslant i\geqslant 1}X^\delta(i)\leqslant 1\}=\lim_{n\to\infty}\mathbb{P}\Big\{\sup_{n\geqslant i\geqslant 1}\big(\ln\mathcal{P}+\ln X^\delta(i)\big)\leqslant 0\Big\}\\ =\lim_{n\to\infty}\mathbb{P}\Big\{\sup_{n\geqslant i\geqslant 1}\big(\mathcal{E}+W^\delta(i)\big)\leqslant 0\Big\}=\mathbb{P}\Big\{\sup_{i\geqslant 1}\big(\mathcal{E}+W^\delta(i)\big)\leqslant 0\Big\},\end{aligned}$$

with $\mathcal{E}=\ln\mathcal{P}$ a unit exponential random variable independent of X.

Next, (2.3) follows from eq. (16) in [45]. Since further we assume (1.2), we infer that (2.3) implies

$$\mathcal{H}_W^\delta\in(0,1/\delta)\tag{4.3}$$

for any $\delta>0$, which completes the proof. ∎

P r o o f o f T h e o r e m 2.2. By our assumption, for all large k we get

$$\frac{\sigma^2(\delta k)}{8} > \ln(\delta k)^a.$$

Consequently, by (3.8) we have for all δ large and some $a > 1$

$$\mathcal{H}_W^0 \geqslant \mathcal{H}_W^\delta \geqslant \frac{1}{\delta}\left(1 - \sum_{k=1}^{\infty} \exp\left(-\frac{\sigma^2(\delta k)}{8}\right)\right) \geqslant \frac{1}{\delta}\left(1 - \frac{1}{\delta^a}\sum_{k=1}^{\infty}\frac{1}{k^a}\right) > 0.$$

Hence the proof is complete. ■

P r o o f o f P r o p o s i t i o n 3.1. Since $B_\alpha(0) = 0$ almost surely, in view of (2.6) (see also [19], Proposition 4) we obtain

$$\lim_{\delta \to 0} \delta^{-1}\mathbb{P}\{\forall_{k\in\mathbb{Z}\setminus\{0\}} B_\alpha(\delta k) \leqslant |\delta k|^\alpha/\sqrt{2}\} = \mathcal{H}_{B_\alpha}.$$

Moreover, by the self-similarity of B_α, we have

$$\mathbb{P}\{\forall_{k\in\mathbb{Z}\setminus\{0\}} B_\alpha(\delta k) \leqslant |\delta k|^\alpha/\sqrt{2}\} = \mathbb{P}\left\{\forall_{k\in\mathbb{Z}\setminus\{0\}} |\delta k|^\alpha B_\alpha\left(\frac{1}{\delta k}\right) \leqslant |\delta k|^\alpha/\sqrt{2}\right\}$$
$$= \mathbb{P}\left\{\forall_{k\in\mathbb{Z}\setminus\{0\}} B_\alpha\left(\frac{1}{\delta k}\right) \leqslant 1/\sqrt{2}\right\} = \mathbb{P}\left\{\forall_{k\in\mathbb{Z}\setminus\{0\}} B_\alpha\left(\frac{1}{k}\right) \leqslant \delta^{\alpha/2}/\sqrt{2}\right\},$$

hence the proof follows easily. ■

P r o o f o f T h e o r e m 3.1. (i) The proof is based on a technique developed in Lemma 16 and Corollary 17 in [16] and in Lemma 7 in [48], therefore we omit some details. For any $\delta > 0$ and T a positive integer, using Bonferroni's inequality, we have for any process W such that $\mathbb{E}\left\{e^{W(k\delta)}\right\} = 1, k \geqslant 1$,

$$\begin{aligned}
&\mathbb{E}\{\sup_{t\in\delta\mathbb{Z}\cap[0,\delta T]} e^W(t)\} \\
&= \int_{\mathbb{R}} e^s \mathbb{P}\{\sup_{t\in\delta\mathbb{Z}\cap[0,\delta T]} W(t) > s\} ds \geqslant \int_{\mathbb{R}} e^s \mathbb{P}\left\{\exists_{1\leqslant k\leqslant T} W(k\delta) > s\right\} ds \\
&\geqslant \sum_{k=1}^{T} \int_{\mathbb{R}} e^s \mathbb{P}\left\{W(k\delta) > s\right\} ds - \sum_{k=1}^{T-1}\sum_{l=k+1}^{T} \int_{\mathbb{R}} e^s \mathbb{P}\left\{W(k\delta) > s, W(l\delta) > s\right\} ds \\
&\geqslant \sum_{k=1}^{T} \mathbb{E}\{e^{W(k\delta)}\} \sum_{k=1}^{T-1}\sum_{l=k+1}^{T} \int_{\mathbb{R}} e^s \mathbb{P}\left\{W(k\delta) + W(l\delta) > 2s\right\} ds,
\end{aligned}$$

and so

$$\mathbb{E}\{\sup_{t\in\delta\mathbb{Z}\cap[0,\delta T]} e^W(t)\} = T - \sum_{k=1}^{T-1}\sum_{l=k+1}^{T} \int_{\mathbb{R}} e^s \mathbb{P}\left\{W(k\delta) + W(l\delta) > 2s\right\} ds \tag{4.4}$$

$$= T - \sum_{k=1}^{T-1} \sum_{l=k+1}^{T} \mathbb{E}\Big\{ \exp\Big(\frac{W(k\delta)+W(l\delta)}{2} \Big) \Big\}$$
$$= T - \sum_{k=1}^{T-1} \sum_{l=k+1}^{T} \exp\Big(-\frac{\sigma^2(\delta\,|k-l|)}{8} \Big)$$
$$\geqslant T - T \sum_{k=1}^{T} \exp\Big(-\frac{\sigma^2(\delta k)}{8} \Big),$$

where the last equality follows by the stationarity of increments of the random process B. Along the lines of the proof in [14] we obtain

$$\mathcal{H}_W^\delta = \lim_{T\to\infty} \frac{1}{T} \mathbb{E}\{ \sup_{t\in\delta\mathbb{Z}\cap[0,T]} e^W(t)\} \geqslant \lim_{T\to\infty} \frac{1}{T} \lfloor T/\delta \rfloor \Big(1 - \sum_{k=1}^{\infty} \exp\Big(-\frac{\sigma^2(\delta k)}{8} \Big)\Big)_+$$
$$= \frac{1}{\delta} \Big(1 - \sum_{k=1}^{\infty} \exp\Big(-\frac{\sigma^2(\delta k)}{8} \Big)\Big)_+ .$$

(ii) In view of (4.4), in order to establish the proof we need to calculate

$$a_{kl} = \int_{\mathbb{R}} e^s \mathbb{P}\{ W(\delta k) + W(\delta l) > 2s \}\, ds.$$

By independence of the increments and the fact that $W(\delta l) - W(\delta k) \stackrel{d}{=} W\big(\delta(l-k)\big)$, we have

$$a_{kl} = \mathbb{E}\Big\{ \exp\Big(\frac{W(\delta k)+W(\delta l)}{2} \Big) \Big\} = \mathbb{E}\{e^{W(\delta k)}\}\mathbb{E}\Big\{ \exp\Big(\frac{W(\delta l)-W(\delta k)}{2} \Big) \Big\}$$
$$= \mathbb{E}\{e^{W(\delta k)}\}\mathbb{E}\Big\{ \exp\Big(\frac{W\big(\delta(l-k)\big)}{2} \Big) \Big\}$$
$$= \mathbb{E}\Big\{ \exp\Big(\frac{B\big(\delta(l-k)\big) - \Phi(1)\delta(l-k)}{2} \Big) \Big\}$$
$$= \exp\big(-\delta(l-k)\lambda \big),$$

where $\lambda := \frac{1}{2}\Phi(1) - \Phi(1/2) > 0$ by Jensen's inequality and the independence and stationarity of increments of the Lévy process B. Consequently, for $N \in \mathbb{N}$ we obtain

$$\text{(4.5)} \qquad \int_{\mathbb{R}} e^s \mathbb{P}\{ \sup_{t\in\delta\mathbb{Z}\cap[0,N]} W(t) > s\} ds \geqslant \frac{N}{\delta}\big(1 - \sum_{k=1}^{\infty} e^{-\delta k\lambda} \big)$$
$$= \frac{N}{\delta} \frac{1-2\exp(-\delta\lambda)}{1-\exp(-\delta\lambda)},$$

which leads to

$$\mathcal{H}_W^\delta \geqslant \frac{1}{\delta} \frac{1-2\exp(-\delta\lambda)}{1-\exp(-\delta\lambda)},$$

and thus the proof is complete. ■

P r o o f o f (3.11). By (4.5) and letting $\lambda = \frac{1}{2}\Phi(1) - \Phi(1/2) > 0$, we have

$$\mathcal{H}_W^0 \geqslant \lim_{N\to\infty} \frac{1}{N} \int_{\mathbb{R}} e^s \mathbb{P}\{\sup_{t\in\delta\mathbb{Z}\cap[0,N]} W(t) > s\} ds$$
$$\geqslant \frac{1}{\delta}\big(1 - \sum_{k=1}^{\infty} e^{-\delta k\lambda}\big) \geqslant \frac{1}{\delta}\big(1 - \int_0^{\infty} e^{-\delta x\lambda} dx\big) = \frac{1}{\delta}\left(1 - \frac{1}{\delta\lambda}\right) \geqslant \frac{\lambda}{4} > 0,$$

establishing the proof. ■

P r o o f o f P r o p o s i t i o n 3.2. By [12], for any $\delta > 0$ and any integer $k \in \mathbb{N}$ we have

$$\mathcal{H}_W^\delta \geqslant \mathbb{E}\left\{\frac{\sup_{t\in\delta\mathbb{Z}} e^{W(t)}}{k\delta \sum_{t\in k\delta\mathbb{Z}} e^{W(t)}}\right\},$$

hence choosing $\delta_n = \eta l^{-n}$ with $\eta > 0$ and $l > 1$ some integer and for $k = l^n$ which is clearly integer for any $n \geqslant 1$, we have

$$\mathcal{H}_W^{\delta_n} \geqslant \mathbb{E}\left\{\frac{\sup_{t\in\delta_n\mathbb{Z}} e^{W(t)}}{k\delta_n \sum_{t\in k\delta_n\mathbb{Z}} e^{W(t)}}\right\}$$
$$= \mathbb{E}\left\{\frac{\sup_{t\in\delta_n\mathbb{Z}} e^{W(t)}}{\eta \sum_{t\in\eta\mathbb{Z}} e^{W(t)}}\right\} \to \mathbb{E}\left\{\frac{\sup_{t\in\mathbb{R}} e^{W(t)}}{\eta \sum_{t\in\eta\mathbb{Z}} e^{W(t)}}\right\}, \quad n\to\infty,$$

where the last limit follows by the monotone convergence theorem and the fact that W has continuous sample paths. Since, by construction, $\mathcal{H}_W^{\delta_n}$ is non-decreasing in n, and we assume that $\lim_{\delta\downarrow 0} \mathcal{H}_W^\delta = \overline{H}_W$, the claim follows. ■

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