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# EXTREMES OF ORDER STATISTICS OF STATIONARY GAUSSIAN PROCESSES*

BY

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*Abstract.* Let $\{X_i(t), t \geqslant 0\}$, $1 \leqslant i \leqslant n$, be mutually independent and identically distributed centered stationary Gaussian processes. Under some mild assumptions on the covariance function, we derive an asymptotic expansion of

$$\mathbb{P}\big(\sup_{t \in [0, x m_r(u)]} X_{(r)}(t) \leqslant u\big) \quad \text{as } u \to \infty,$$

where

$$m_r(u) = \big(\mathbb{P}\big(\sup_{t \in [0,1]} X_{(r)}(t) > u\big)\big)^{-1}\big(1 + o(1)\big),$$

and $\{X_{(r)}(t), t \geqslant 0\}$ is the $r$th order statistic process of $\{X_i(t), t \geqslant 0\}$, $1 \leqslant i, r \leqslant n$. As an application of the derived result, we analyze the asymptotics of supremum of the order statistic process of stationary Gaussian processes over random intervals.

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## 1. INTRODUCTION

Let $\{X(t) : t \geqslant 0\}$ be a centered stationary Gaussian process with continuous sample paths. One of the classical results in extreme value theory states that, under some mild conditions on the covariance function of $X$,

$$\lim_{u \to \infty} \mathbb{P}\Big(\sup_{t \in [0, x m(u)]} X(t) \leqslant u\Big) = e^{-x} \tag{1.1}$$

for $x > 0$ and $m(u) = \mathbb{P}\big(\sup_{t \in [0,1]} X(t) > u\big)^{-1}$; see, e.g., Leadbetter et al. [11], Theorem 12.3.4; Arendarczyk and Dębicki [4], Lemma 4.3; Tan and Hashorva [13], Lemma 3.3.

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Consider a vector-valued Gaussian stochastic process $\{\mathbf{X}(t) : t \geqslant 0\}$, where $\mathbf{X}(t) = \big(X_1(t), \ldots, X_n(t)\big)$ with $\{X_i(t) : t \geqslant 0\}$, $i = 1, \ldots, n$, being mutually independent copies of $\{X(t) : t \geqslant 0\}$. Denote by $\{X_{(r)}(t), t \geqslant 0\}$, $r = 1, 2, \ldots, n$, the $r$th smallest order statistic process, i.e., for each $t \geqslant 0$,

$$X_{(1)}(t) = \min_{1 \leqslant i \leqslant n} X_i(t) \leqslant X_{(2)}(t) \leqslant \ldots \leqslant \max_{1 \leqslant i \leqslant n} X_i(t) = X_{(n)}(t). \tag{1.2}$$

In this contribution we derive a counterpart of (1.1) for $\{X_{(r)}(t), t \geqslant 0\}$.

One of important motivations to analyze asymptotic properties of extremes of order statistic processes is their relation with the *conjunction problem*. Following [14], the set of conjunctions $C_{T,u}$ is defined as

$$C_{T,u} := \{t \in [0, T] : \min_{1 \leqslant i \leqslant n} X_i(t) > u\},$$

so

$$\mathbb{P}\left(C_{T,u} = \emptyset\right) = \mathbb{P}\big(\sup_{t \in [0,T]} \min_{1 \leqslant i \leqslant n} X_i(t) \leqslant u\big).$$

We refer to [2], [3], [6], [9], [14] for recent results on asymptotic properties of $\mathbb{P}\left(C_{T,u} \neq \emptyset\right)$.

As an application of the obtained result we provide the exact asymptotics of

$$\mathbb{P}\big(\sup_{t \in [0,\mathcal{T}]} X_{(r)}(t) > u\big) \quad \text{as } u \to \infty$$

for $\mathcal{T}$ being a nonnegative random variable independent of $\mathbf{X}(t)$. The obtained asymptotics extends the recent results of Arendarczyk and Dębicki [4].

## 2. PRELIMINARIES

Suppose that $\mathbf{X}(t) = \big(X_1(t), \ldots, X_n(t)\big)$ and $\{X_i(t) : t \geqslant 0\}$, $i = 1, \ldots, n$, are mutually independent centered stationary Gaussian processes with covariance function $r(t)$ satisfying the following conditions:

(A1) $r(t) = 1 - t^\alpha + o(t^\alpha)$ as $t \to 0$;

(A2) $r(t) < 1$ if $t > 0$;

(A3) $r(t) \log t \to 0$ as $t \to \infty$.

Following Dębicki et al. [9], let us introduce the *generalized Pickands constant* as

$$\mathcal{H}_{\alpha,k} = \lim_{S \to \infty} S^{-1} \mathcal{H}_{\alpha,k}(S) \in (0, \infty),$$

where

$$\mathcal{H}_{\alpha,k}(S)$$
$$= \int_{R^n} \exp\big(\sum_{i=1}^{k} w_j\big) \mathbb{P}\Big(\sup_{t \in [0,S]} \min_{1 \leqslant i \leqslant k} \big(\sqrt{2} B_\alpha^{(i)}(t) - t^\alpha - w_i\big) > 0\Big) d\mathbf{w} \in (0, \infty),$$

and $B_\alpha^{(i)}$, $i = 1, \ldots, n$, are mutually independent standard fractional Brownian motions with Hurst index $\alpha/2 \in (0,1]$, i.e., centered Gaussian processes with stationary increments and variance function $t^\alpha$.

Let

$$m_r(u) := \frac{(2\pi)^{(n+1-r)/2}}{c_{n,r-1}\mathcal{H}_{\alpha,n+1-r}} u^{n+1-r-2/\alpha} \exp\left(\frac{n+1-r}{2}u^2\right), \tag{2.1}$$

where

$$c_{n,r-1} = \frac{n!}{(r-1)!(n+1-r)!}.$$

It follows from Theorem 2.2 in [8] that, for each $T > 0$ and $1 \leqslant r \leqslant n$,

$$\begin{aligned} \mathbb{P}\big(\sup_{t\in[0,T]} X_{(r)}(t) > u\big) &= c_{n,r-1}\mathcal{H}_{\alpha,n+1-r} T u^{2/\alpha} \big(\Psi(u)\big)^{n+1-r}\big(1+o(1)\big) \\ &= \frac{T}{m_r(u)}\big(1+o(1)\big) \quad \text{as } u \to \infty, \end{aligned} \tag{2.2}$$

where $\Psi(u) = \frac{1}{\sqrt{2\pi}} \int_u^\infty \exp(-x^2/2)dx$.

## 3. MAIN RESULTS

The following theorem constitutes the main result of this contribution.

THEOREM 3.1. *Let $\{X_j(t), t \geqslant 0\}$ be independent and identically distributed centered stationary Gaussian processes with convariance function $r(t)$ satisfying the conditions* (A1)–(A3) *and assume that $0 < A < B < \infty$ and $x > 0$. Then*

$$\mathbb{P}\big(\sup_{t\in[0,x m_r(u)]} X_{(r)}(t) \leqslant u\big) \to e^{-x} \quad \text{as } u \to \infty, \tag{3.1}$$

*uniformly for $x \in [A, B]$.*

Let $\mathcal{T}$ be a nonnegative random variable which is independent of $\mathbf{X}$. In the following theorem we discuss the asymptotic behavior of $\mathbb{P}\big(\sup_{t\in[0,\mathcal{T}]} X_{(r)}(t) > u\big)$ as $u \to \infty$. It appears that the qualitative form of the asymptotics strongly depends on *heaviness* of the tail of $\mathcal{T}$.

THEOREM 3.2. *Let $\{X_j(t), t \geqslant 0\}$ be independent and identically distributed centered stationary Gaussian processes with convariance function $r(t)$ satisfying the conditions* (A1)–(A3), *and let $\mathcal{T}$ be a nonnegative random variable independent of $X$.*

(i) *If $\mathbb{E}\mathcal{T} < \infty$, then, as $u \to \infty$,*

$$\mathbb{P}\big(\sup_{t\in[0,\mathcal{T}]} X_{(r)}(t) > u\big) = \mathbb{E}\mathcal{T} c_{n,r-1}\mathcal{H}_{\alpha,n+1-r} u^{2/\alpha}\big(\Psi(u)\big)^{n+1-r}\big(1+o(1)\big). \tag{3.2}$$

(ii) *If* $\mathcal{T}$ *has a regularly varying tail distribution at infinity with index* $\lambda \in (0,1)$, *then*, *as* $u \to \infty$,

$$\mathbb{P}\big(\sup_{t\in[0,\mathcal{T}]} X_{(r)}(t) > u\big) = \Gamma(1-\lambda)\mathbb{P}\big(\mathcal{T} > m_r(u)\big)\big(1+o(1)\big). \tag{3.3}$$

(iii) *If* $\mathcal{T}$ *has a slowly varying tail distribution at infinity*, *then*, *as* $u \to \infty$,

$$\mathbb{P}\big(\sup_{t\in[0,\mathcal{T}]} X_{(r)}(t) > u\big) = \mathbb{P}\big(\mathcal{T} > m_r(u)\big)\big(1+o(1)\big). \tag{3.4}$$

The proofs of Theorems 3.1 and 3.2 are given in Section 4.

## 4. PROOFS

Before proceeding to the proofs of Theorems 3.1 and 3.2, we give some preliminary lemmas. Let us put $\mathcal{T}_r = xm_r(u)$ and $n_r = \lfloor \mathcal{T}_r \rfloor$. For any $\varepsilon \in (0,1)$ and $1 \leqslant l \leqslant n_r$, we write $I_l = [l-1+\varepsilon, l]$ and $I_l^* = [l-1, l-1+\varepsilon]$.

LEMMA 4.1. *For each* $B > A > 0$,

$$\lim_{u\to\infty} \Big|\mathbb{P}\big(\sup_{t\in[0,n_r]} X_{(r)}(t) \leqslant u\big) - \mathbb{P}\big(\sup_{t\in\bigcup_{l=1}^{n_r} I_l} X_{(r)}(t) \leqslant u\big)\Big| \leqslant \rho_1(\varepsilon), \tag{4.1}$$

*uniformly for* $x \in [A, B]$, *where* $\rho_1(\varepsilon) \to 0$ *as* $\varepsilon \to 0$.

P r o o f. Suppose that $x \in [A, B]$. By stationarity, Bonferroni's inequality (see, e.g., [10]) and (2.2), we have

$$\begin{aligned} 0 &\leqslant \mathbb{P}\big(\sup_{t\in\bigcup_{l=1}^{n_r} I_l} X_{(r)}(t) \leqslant u\big) - \mathbb{P}\big(\sup_{t\in[0,n_r]} X_{(r)}(t) \leqslant u\big) \\ &= \mathbb{P}\big(\sup_{t\in[0,n_r]} X_{(r)}(t) > u\big) - \mathbb{P}\big(\sup_{t\in\bigcup_{l=1}^{n_r} I_l} X_{(r)}(t) > u\big) \\ &\leqslant \mathbb{P}\big(\sup_{t\in\bigcup_{l=1}^{n_r} I_l^*} X_{(r)}(t) > u\big) \leqslant n_r \mathbb{P}\big(\sup_{t\in[0,\varepsilon]} X_{(r)}(t) > u\big) \\ &= xm_r(u)\frac{\varepsilon}{m_r(u)}\big(1+o(1)\big) \leqslant B\varepsilon =: \rho_1(\varepsilon) \quad \text{as } u \to \infty. \end{aligned}$$

This completes the proof. ∎

LEMMA 4.2. *Let* $q = q(u) = au^{-2/\alpha}$ *for some* $a > 0$. *Then*

$$\limsup_{u\to\infty} \Big|\mathbb{P}\big(\sup_{t\in\bigcup_{l=1}^{n_r} I_l} X_{(r)}(t) \leqslant u\big) - \mathbb{P}\big(\max_{iq\in\bigcup_{l=1}^{n_r} I_l} X_{(r)}(iq) \leqslant u\big)\Big| \leqslant \rho_2(a),$$

*uniformly for* $x \in [A, B]$, *where* $\rho_2(a) \to 0$ *as* $a \to 0$.

Proof. Since $X_i(t)$ are independent and identically distributed, we obtain

$$\begin{aligned}
&\mathbb{P}\big(\max_{iq\in I_1} X_{(r)}(iq) > u\big)\\
&= \mathbb{P}\big(\bigcup_{iq\in I_1}\bigcup_{j=n-r+1}^{n}\{\exists k_1,\ldots,k_j, X_{k_1}(iq) > u,\ldots,X_{k_j}(iq) > u\}\big)\\
&= \mathbb{P}\big(\bigcup_{iq\in I_1}\bigcup_{j=n-r+1}^{n}\{\exists k_1,\ldots,k_j, X_{k_1}(iq) > u,\ldots,X_{k_j}(iq) > u,\\
&\qquad\qquad X_k(iq) \leqslant u, k\neq k_1,\ldots,k_j\}\big)\\
&= \sum_{j=n-r+1}^{n} c_{n,j}\mathbb{P}\big(\exists_{iq\in I_1}, X_1(iq) > u,\ldots,X_j(iq) > u, X_k(iq)\leqslant u, k>j\big)\\
&= \sum_{j=n-r+1}^{n} c_{n,j}\mathbb{P}\big(\max_{iq\in I_1}\min_{1\leqslant i\leqslant j} X_i(iq) > u\big)\big(1+o(1)\big).
\end{aligned}$$

Following Dębicki et al. [8] we define

$$\mathcal{H}'_{\alpha,j}(a) = \frac{1}{a}P\Big(\max_{k\geqslant 1}\min_{1\leqslant m\leqslant j}\big(\sqrt{2}B_\alpha^{(m)}(ak) - (ak)^\alpha + \eta_m\big) \leqslant 0\Big), \tag{4.2}$$

where $j = 1,2,\ldots,n$, and $\{B_\alpha^{(m)}, t\geqslant 0\}$, $m\geqslant 1$, are independent and identically distributed standard fractional Brownian motions which are further independent of independent unit exponential random variables $\eta_m$. Using analogous arguments to those in the proof of Theorem 1.1 in Dębicki et al. [8] or Lemma 1 in Albin and Choi [1], we have

$$\begin{aligned}
\mathbb{P}\big(\max_{iq\in I_1} X_{(r)}(iq) > u\big) &= \sum_{j=n-r+1}^{n}\frac{\mathcal{H}'_{\alpha,j}(a)}{\mathcal{H}_{\alpha,j}}\frac{1-\varepsilon}{m_{n+1-j}(u)}\\
&= \frac{\mathcal{H}'_{\alpha,n+1-r}(a)}{\mathcal{H}_{\alpha,n+1-r}}\frac{1-\varepsilon}{m_r(u)}\big(1+o(1)\big) \quad \text{as } u\to\infty,
\end{aligned}$$

where $\mathcal{H}'_{\alpha,k}(a)\to\mathcal{H}_{\alpha,k}$ as $a\to 0$. Therefore, by stationarity, we obtain

$$\begin{aligned}
0 &\leqslant \mathbb{P}\big(\max_{iq\in\bigcup_{l=1}^{n_r} I_l} X_{(r)}(iq)\leqslant u\big) - \mathbb{P}\big(\sup_{t\in\bigcup_{l=1}^{n_r} I_l} X_{(r)}(t)\leqslant u\big)\\
&\leqslant n_r\max_{1\leqslant l\leqslant n_r}\Big(\mathbb{P}\big(\max_{iq\in I_l} X_{(r)}(iq)\leqslant u\big) - \mathbb{P}\big(\sup_{t\in I_l} X_{(r)}(t)\leqslant u\big)\Big)\\
&\leqslant n_r\mathbb{P}\big(X_{(r)}(0) > u\big) + n_r\mathbb{P}\big(\sup_{t\in[0,1-\varepsilon]} X_{(r)}(t) > u\big)\\
&\quad - n_r\mathbb{P}\big(\max_{iq\in[0,1-\varepsilon]} X_{(r)}(iq) > u\big)\\
&= xm_r(u)\left(o\left(\frac{1}{m_r(u)}\right) + \frac{1-\varepsilon}{m_r(u)} - \frac{\mathcal{H}'_{\alpha,n+1-r}(a)}{\mathcal{H}_{\alpha,n+1-r}}\frac{1-\varepsilon}{m_r(u)}\right)\big(1+o(1)\big)\\
&\leqslant B\left(1 - \frac{\mathcal{H}'_{\alpha,n+1-r}(a)}{\mathcal{H}_{\alpha,n+1-r}}\right) =: \rho_2(a),
\end{aligned}$$

where the penultimate expression is due to (2.2). Since $\rho_2(a) \to 0$ as $a \to 0$, the proof is completed. ∎

For each $1 \leqslant j \leqslant n$, let $\{X_j^{(k)}(t), t \geqslant 0\}_{k=1}^{\infty}$ be a sequence of independent and identically distributed centered stationary Gaussian processes that satisfy the conditions (A1)–(A3). Define

$$Y_j(t) = X_j^{(k)}(t) \quad \text{if } t \in [k-1, k),$$

and, for $t \geqslant 0$,

$$Y_{(1)}(t) = \min_{1 \leqslant j \leqslant n} Y_j(t) \leqslant Y_{(2)}(t) \leqslant \ldots \leqslant \max_{1 \leqslant j \leqslant n} Y_j(t) = Y_{(n)}(t).$$

LEMMA 4.3. *We have*

$$\lim_{u \to \infty} \left| \mathbb{P}\big( \sup_{iq \in \bigcup_{l=1}^{n_r} I_l} X_{(r)}(iq) \leqslant u \big) - \mathbb{P}\big( \sup_{iq \in \bigcup_{l=1}^{n_r} I_l} Y_{(r)}(iq) \leqslant u \big) \right| = 0.$$

P r o o f. Define $A = \mathbb{N} \cap \bigcup_{l=1}^{n_r} I_l q^{-1} = \{i_1, i_2, \ldots, i_d\}$, where $1 \leqslant i_1 < i_2 < \ldots < i_d < \infty$, and observe that

$$\begin{aligned} \Delta_{(r)} &= \left| \mathbb{P}\big( \sup_{iq \in \bigcup_{l=1}^{n_r} I_l} X_{(r)}(iq) \leqslant u \big) - \mathbb{P}\big( \sup_{iq \in \bigcup_{l=1}^{n_r} I_l} Y_{(r)}(iq) \leqslant u \big) \right| \\ &= \left| \mathbb{P}\big( \sup_{i \in A} X_{(r)}(iq) \leqslant u \big) - \mathbb{P}\big( \sup_{i \in A} Y_{(r)}(iq) \leqslant u \big) \right|. \end{aligned}$$

For $i \in A$ and $1 \leqslant j \leqslant n$, we put $X_{ij} = X_j(iq)$ and $Y_{ij} = Y_j(iq) = X_j^{(\lfloor iq \rfloor + 1)}(iq)$. Note that

$$\begin{aligned} \sigma_{ij,lk}^X &= \mathbb{E} X_{ij} X_{lk} = \mathbb{E} X_j(iq) X_k(lq) = r\big((i-l)q\big) \mathbb{I}\{j = k\} := \sigma_{il}^X \mathbb{I}\{j = k\}, \\ \sigma_{ij,lk}^Y &= \mathbb{E} Y_{ij} Y_{lk} = \mathbb{E} X_j^{(\lfloor iq \rfloor + 1)}(iq) X_k^{(\lfloor lq \rfloor + 1)}(lq) \\ &= r\big((i-l)q\big) \mathbb{I}\{\lfloor iq \rfloor = \lfloor lq \rfloor\} \mathbb{I}\{j = k\} := \sigma_{il}^Y \mathbb{I}\{j = k\}. \end{aligned}$$

It follows from Theorem 2.4 in [7] that

$$\Delta_{(r)} \leqslant \frac{n(c_{n-1,r-1})^2}{(2\pi)^{n+1-r}} u^{-2(n-r)} \sum_{i,l \in A, i \neq l} |A_{il}^{(r)}| \exp\left( -\frac{(n+1-r)u^2}{1+\rho_{il}} \right),$$

where

$$\rho_{il} = \max\{|\sigma_{il}^X|, |\sigma_{il}^Y|\} = \left| r\big((i-l)q\big) \right|,$$

$$\begin{aligned} A_{il}^{(r)} &= \int_{\sigma_{il}^Y}^{\sigma_{il}^X} \frac{(1+|h|)^{2(n-r)}}{(1-h^2)^{(n+1-r)/2}} dh \\ &= \int_0^{r((i-l)q)} \frac{(1+|h|)^{2(n-r)}}{(1-h^2)^{(n+1-r)/2}} dh \mathbb{I}\{\lfloor iq \rfloor \neq \lfloor lq \rfloor\}. \end{aligned}$$

Since $\delta := \sup\{|r(t)|, t \geqslant \varepsilon\} < 1$, for $i, l \in A$ satisfying $\lfloor iq \rfloor \neq \lfloor lq \rfloor$, one has $|(i-l)q| \geqslant \varepsilon$, and $\left|r\big((i-l)q\big)\right| \leqslant \delta < 1$. Notice that the integrand in the definition of $A_{il}^{(r)}$ is continuous and bounded on $[0, \delta]$, so there exists a constant $K_1$ such that

$$|A_{il}^{(r)}| \leqslant K_1 \left|r\big((i-l)q\big)\right| \mathbb{I}\{\lfloor iq \rfloor \neq \lfloor lq \rfloor\}.$$

Hence,

$$\begin{aligned}
\Delta_{(r)} &\leqslant \frac{n(c_{n-1,r-1})^2 K_1}{(2\pi)^{n+1-r}} u^{-2(n-r)} \frac{\mathcal{T}_r}{q} \sum_{\varepsilon \leqslant kq \leqslant \mathcal{T}_r} |r(kq)| \exp\left(-\frac{(n+1-r)u^2}{1+|r(kq)|}\right) \\
&= \frac{n(c_{n-1,r-1})^2 K_1}{(2\pi)^{n+1-r}} u^{-2(n-r)} \frac{\mathcal{T}_r}{q} \sum_{\varepsilon \leqslant kq \leqslant \mathcal{T}_r^\beta} |r(kq)| \exp\left(-\frac{(n+1-r)u^2}{1+|r(kq)|}\right) \\
&\quad + \frac{n(c_{n-1,r-1})^2 K_1}{(2\pi)^{n+1-r}} u^{-2(n-r)} \frac{\mathcal{T}_r}{q} \sum_{\mathcal{T}_r^\beta < kq \leqslant \mathcal{T}_r} |r(kq)| \exp\left(-\frac{(n+1-r)u^2}{1+|r(kq)|}\right) \\
&=: \mathbb{P}_1 + \mathbb{P}_2,
\end{aligned}$$

where $0 < \beta < (1-\delta)/(1+\delta)$.

First, we prove that $\mathbb{P}_1 \to 0$ as $u \to \infty$. Indeed,

$$\begin{aligned}
\mathbb{P}_1 &\leqslant \frac{n(c_{n-1,r-1})^2 K_1}{(2\pi)^{n+1-r}} u^{-2(n-r)} \frac{\mathcal{T}_r^{\beta+1}}{q^2} \exp\left(-\frac{(n+1-r)u^2}{1+\delta}\right) \\
&= \frac{n(c_{n-1,r-1})^2 K_1}{(2\pi)^{n+1-r} a^2} u^{4/\alpha - 2(n-r)} \mathcal{T}_r^{\beta+1} \exp\left(-\frac{(n+1-r)u^2}{2}\right)^{2/(1+\delta)} \\
&\leqslant K_2 u^{4/\alpha - 2(n-r) + (\beta+1)(n+1-r-2/\alpha)} \exp\left(\frac{(n+1-r)u^2}{2}\right)^{\beta - (1-\delta)/(1+\delta)} \\
&\to 0 \quad \text{ as } u \to \infty.
\end{aligned}$$

In order to show that $\mathbb{P}_2 \to 0$, we put $\delta(t) = \sup\{|r(s)\log s|, s \geqslant t\}$. By (A3), we have $|r(t)| \leqslant \delta(t)/\log t$ and $\delta(t) \downarrow 0$ as $t \to \infty$. Moreover,

$$\log \mathcal{T}_r = \frac{n+1-r}{2} u^2 \big(1 + o(1)\big) \quad \text{ for } kq > \mathcal{T}_r^\beta.$$

Thus,

$$\begin{aligned}
\exp\left(-\frac{(n+1-r)u^2}{1+|r(kq)|}\right) &\leqslant \exp\left(-(n+1-r)u^2\left(1 - \frac{\delta(\mathcal{T}_r^\beta)}{\log \mathcal{T}_r^\beta}\right)\right) \\
&\leqslant K_3 \exp\big(-(n+1-r)u^2\big).
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathbb{P}_2 &\leqslant \left\{ K_4 u^{-2(n-r)} \frac{\mathcal{T}_r^2}{q^2} \exp\left(-(n+1-r)u^2\right) \frac{1}{\log \mathcal{T}_r^\beta} \right\} \\
&\quad \times \frac{q}{\mathcal{T}_r} \sum_{\mathcal{T}_r^\beta < kq \leqslant \mathcal{T}_r} |r(kq)| \log(kq) \\
&\leqslant K_5 u^{-2(n-r)} \frac{u^{2(n+1-r-2/\alpha)} \exp\left((n+1-r)u^2\right)}{u^{-4/\alpha}} \exp\left(-(n+1-r)u^2\right) \frac{1}{u^2} \\
&\quad \times \frac{q}{\mathcal{T}_r} \sum_{\mathcal{T}_r^\beta < kq \leqslant \mathcal{T}_r} |r(kq)| \log(kq) \\
&\leqslant K_5 \frac{q}{\mathcal{T}_r} \sum_{\mathcal{T}_r^\beta < kq \leqslant \mathcal{T}_r} |r(kq)| \log(kq) \to 0 \quad \text{as } u \to \infty.
\end{aligned}$$

This completes the proof. ■

LEMMA 4.4. *We have*

$$\limsup_{u\to\infty} \Big|\mathbb{P}\big(\sup_{iq \in \bigcup_{l=1}^{n_r} I_l} Y_{(r)}(iq) \leqslant u\big) - \mathbb{P}\big(\sup_{t\in[0,n_r]} Y_{(r)}(t) \leqslant u\big)\Big| \leqslant x\big(\rho_3(a) + \varepsilon\big),$$

*where* $\rho_3(a) \to 0$ *as* $a \to 0$.

Proof. Since $I_l$, $l = 1, 2, \ldots, n_r$, are disjoint, $\{Y_{(r)}(t), t \in I_l\}$ are independent, and, by stationarity,

$$\begin{aligned}
0 &\leqslant \mathbb{P}\big(\sup_{iq\in\bigcup_{l=1}^{n_r} I_l} Y_{(r)}(iq) \leqslant u\big) - \mathbb{P}\big(\sup_{t\in\bigcup_{l=1}^{n_r} I_l} Y_{(r)}(t) \leqslant u\big) \\
&= \mathbb{P}\big(\sup_{iq\in[0,1-\varepsilon]} Y_{(r)}(iq) \leqslant u\big)^{n_r} - \mathbb{P}\big(\sup_{t\in[0,1-\varepsilon]} Y_{(r)}(t) \leqslant u\big)^{n_r} \\
&\leqslant n_r \Big(\mathbb{P}\big(\sup_{iq\in I_1} Y_{(r)}(iq) \leqslant u\big) - \mathbb{P}\big(\sup_{t\in I_1} Y_{(r)}(t) \leqslant u\big)\Big) \\
&\leqslant n_r \Big(\mathbb{P}\big(Y_{(r)}(0) > u\big) + \mathbb{P}\big(\sup_{iq\in[0,1-\varepsilon]} Y_{(r)}(iq) \leqslant u\big) \\
&\qquad\qquad - \mathbb{P}\big(\sup_{t\in[0,1-\varepsilon]} Y_{(r)}(t) \leqslant u\big)\Big) \\
&= x m_r(u) \left( o\left(\frac{1}{m_r(u)}\right) + \left(1 - \frac{\mathcal{H}'_{\alpha,n+1-r}(a)}{\mathcal{H}_{\alpha,n+1-r}}\right) \frac{1-\varepsilon}{m_r(u)} \right) \big(1 + o(1)\big) \\
&\leqslant x \left(1 - \frac{\mathcal{H}'_{\alpha,n+1-r}(a)}{\mathcal{H}_{\alpha,n+1-r}}\right) =: x\rho_3(a),
\end{aligned}$$

where $\rho_3(a) \to 0$ as $a \to 0$. Moreover,

$$\begin{aligned}0 &\leqslant \mathbb{P}\big(\sup_{t\in\bigcup_{l=1}^{n_r} I_l} Y_{(r)}(t) \leqslant u\big) - \mathbb{P}\big(\sup_{t\in[0,n_r]} Y_{(r)}(t) \leqslant u\big)\\
&\leqslant \mathbb{P}\big(\sup_{t\in[0,1-\varepsilon]} Y_{(r)}(t) \leqslant u\big)^{n_r} - \mathbb{P}\big(\sup_{t\in[0,1]} Y_{(r)}(t) \leqslant u\big)^{n_r}\\
&\leqslant n_r P\big(\sup_{t\in[0,\varepsilon]} Y_{(r)}(t) > u\big)\\
&= x m_r(u)\frac{\varepsilon}{m_r(u)}\big(1+o(1)\big) = x\varepsilon\big(1+o(1)\big).\end{aligned}$$

The combination of the above displays completes the proof. ■

LEMMA 4.5. *We have*

$$\lim_{u\to\infty} \mathbb{P}\big(\sup_{t\in[0,n_r]} Y_{(r)}(t) \leqslant u\big) = e^{-x}.$$

Proof. Since

$$\begin{aligned}\mathbb{P}\big(\sup_{t\in[0,n_r]} Y_{(r)}(t) \leqslant u\big) &= \mathbb{P}\big(\sup_{t\in[0,1]} X_{(r)}(t) \leqslant u\big)^{n_r}\\
&= \Big(1 - \mathbb{P}\big(\sup_{t\in[0,1]} X_{(r)}(t) > u\big)\Big)^{n_r}\\
&= \big(1 - m_r(u)^{-1}\big)^{x m_r(u)}\big(1+o(1)\big) \to e^{-x},\end{aligned}$$

the proof is completed. ■

Proof of Theorem 3.1. The proof of the theorem follows directly from Lemmas 4.1–4.5. ■

LEMMA 4.6. *For any $S > 0$, we have*

$$\mathbb{P}\big(\sup_{t\in[0,Su^{-2/\alpha}]} X_{(r)}(t) > u\big) = c_{n,r-1}\mathcal{H}_{\alpha,n+1-r}(S)\big(\Psi(u)\big)^{n+1-r}\big(1+o(1)\big) \tag{4.3}$$

*as $u \to \infty$.*

The proof of Lemma 4.6 follows line-by-line the same reasoning as the proof of Theorem 2.2 in [8], and thus we omit it.

Proof of Theorem 3.2. (i) For any $t, u, S > 0$, let us put

$$N_t = \Big\lfloor \frac{t}{Su^{-2/\alpha}} \Big\rfloor \quad \text{and} \quad \Delta_k = [kSu^{-2/\alpha}, (k+1)Su^{-2/\alpha}] \text{ with } k = 0, 1, \ldots, N_t.$$

Upper bound. By stationarity of the process $\{X_{(r)}(t), t \geqslant 0\}$ and Lemma 4.6, we obtain

$$\begin{aligned}\mathbb{P}\big(\sup_{t\in[0,\mathcal{T}]} X_{(r)}(t) > u\big) &= \int_0^\infty \mathbb{P}\big(\sup_{s\in[0,t]} X_{(r)}(s) > u\big) d\mathbb{P}(\mathcal{T} \leqslant t)\\ &\leqslant \mathbb{P}\big(\sup_{s\in\Delta_0} X_{(r)}(s) > u\big)\Big(\frac{u^{2/\alpha}}{S}\int_0^\infty t d\mathbb{P}(\mathcal{T} \leqslant t) + 1\Big)\\ &= \frac{\mathcal{H}_{\alpha,n+1-r}(S)}{S} c_{n,r-1}\mathbb{E}\mathcal{T} u^{2/\alpha}\big(\Psi(u)\big)^{n+1-r}\big(1+o(1)\big)\end{aligned}$$

as $u \to \infty$. Thus, letting $S \to \infty$, we get

$$\mathbb{P}\big(\sup_{t\in[0,\mathcal{T}]} X_{(r)}(t) > u\big) = c_{n,r-1} s\mathcal{H}_{\alpha,n+1-r} u^{2/\alpha}\mathbb{E}\mathcal{T}\big(\Psi(u)\big)^{n+1-r}\big(1+o(1)\big).$$

Lower bound. By Bonferroni's inequality, we have

$$\begin{aligned}\mathbb{P}\big(\sup_{t\in[0,\mathcal{T}]} X_{(r)}(t) > u\big) &= \int_0^\infty \mathbb{P}\big(\sup_{s\in[0,t]} X_{(r)}(s) > u\big) d\mathbb{P}(\mathcal{T} \leqslant t)\\ &\geqslant \int_0^u \mathbb{P}\big(\sup_{s\in[0,t]} X_{(r)}(s) > u\big) d\mathbb{P}(\mathcal{T} \leqslant t)\\ &\geqslant \mathbb{P}\big(\sup_{s\in\Delta_0} X_{(r)}(s) > u\big)\Big(\frac{u^{2/\alpha}}{S}\int_0^u t d\mathbb{P}(\mathcal{T} \leqslant t) - 1\Big)\\ &\quad - \int_0^u \sum_{0\leqslant i<j\leqslant N_t} \mathbb{P}\big(\sup_{s\in\Delta_i} X_{(r)}(s) > u, \sup_{s\in\Delta_j} X_{(r)}(s) > u\big) d\mathbb{P}(\mathcal{T} \leqslant t)\\ &=: I_1 - I_2.\end{aligned} \tag{4.4}$$

Note that

$$I_1 = \frac{\mathcal{H}_{\alpha,n+1-r}(S)}{S} c_{n,r-1}\mathbb{E}\mathcal{T} u^{2/\alpha}\big(\Psi(u)\big)^{n+1-r}\big(1+o(1)\big)$$

as $u \to \infty$. Thus, letting $S \to \infty$, we obtain

$$I_1 \geqslant c_{n,r-1}\mathcal{H}_{\alpha,n+1-r} u^{2/\alpha}\mathbb{E}\mathcal{T}\big(\Psi(u)\big)^{n+1-r}. \tag{4.5}$$

Hence, in order to complete the proof it suffices to show that $I_2 = o(I_1)$ as $u \to \infty$.

Indeed, we have

$$
\begin{aligned}
I_2 &= \int_0^u \sum_{k=1}^{N_t} (N_t - k)\mathbb{P}\big( \sup_{s\in\Delta_0} X_{(r)}(s) > u, \sup_{s\in\Delta_k} X_{(r)}(s) > u\big) d\mathbb{P}(\mathcal{T} \leqslant t) \\
&\leqslant \frac{u^{2/\alpha}}{S} \int_0^u t d\mathbb{P}(\mathcal{T} \leqslant t) \sum_{k=1}^{N_u} \mathbb{P}\big( \sup_{s\in\Delta_0} X_{(r)}(s) > u, \sup_{s\in\Delta_k} X_{(r)}(s) > u\big) \\
&\leqslant \frac{u^{2/\alpha}}{S} \mathbb{E}\mathcal{T} \sum_{k=1}^{N_u} \mathbb{P}\big( \sup_{s\in\Delta_0} X_{(r)}(s) > u, \sup_{s\in\Delta_k} X_{(r)}(s) > u\big) \\
&\leqslant c_{n,r-1} \frac{u^{2/\alpha}}{S} \mathbb{E}\mathcal{T} \sum_{k=1}^{N_u} \mathbb{P}\big( \sup_{s\in\Delta_0} \min_{1\leqslant i\leqslant n+1-r} X_i(s) > u, \sup_{s\in\Delta_k} X_{(r)}(s) > u\big) \\
&\leqslant c_{n,r-1} \frac{u^{2/\alpha}}{S} \mathbb{E}\mathcal{T} \sum_{k=1}^{N_u} \mathbb{P}\big( \sup_{s\in\Delta_0} \min_{1\leqslant i\leqslant n+1-r} X_i(s) > u, \sup_{s\in\Delta_k} \min_{1\leqslant i\leqslant n+1-r} X_i(s) > u\big) \\
&\quad + c_{n,r-1} \frac{u^{2/\alpha}}{S} \mathbb{E}\mathcal{T} \sum_{k=1}^{N_u} \mathbb{P}\big( \sup_{s\in\Delta_0} \min_{1\leqslant i\leqslant n+1-r} X_i(s) > u, \sup_{s\in\Delta_k} X_{(r)}(s) > u, \\
&\qquad\qquad\qquad \sup_{s\in\Delta_k} \min_{1\leqslant i\leqslant n+1-r} X_i(s) \leqslant u\big) \\
&=: I_{21} + I_{22}.
\end{aligned}
$$

Since

$$
\begin{aligned}
\sum_{k=1}^{N_u} \mathbb{P}\big( \sup_{s\in\Delta_0} \min_{1\leqslant i\leqslant n+1-r} X_i(s) > u, \sup_{s\in\Delta_k} \min_{1\leqslant i\leqslant n+1-r} X_i(s) \leqslant u, \sup_{s\in\Delta_k} X_{(r)}(s) > u\big) \\
\leqslant N_u \mathbb{P}\big( \sup_{s\in\Delta_0} X_1(s) > u\big)^{n+2-r},
\end{aligned}
$$

we get $I_{22} = o(I_1)$ as $u \to \infty$. Moreover, using the relations

$$
\begin{aligned}
I_{21} &\leqslant c_{n,r-1} \frac{u^{2/\alpha}}{S} \mathbb{E}\mathcal{T} \sum_{k=1}^{N_u} \mathbb{P}\big( \sup_{s\in\Delta_0} X_1(s) > u, \sup_{s\in\Delta_k} X_1(s) > u\big)^{n+r-1} \\
&\leqslant c_{n,r-1} u^{2/\alpha} \mathbb{E}\mathcal{T} \bigg( \frac{1}{S^{1/(n+r-1)}} \sum_{k=1}^{N_u} \mathbb{P}\big( \sup_{s\in\Delta_0} X_1(s) > u, \sup_{s\in\Delta_k} X_1(s) > u\big)\bigg)^{n+r-1},
\end{aligned}
$$

we are left with finding a tight asymptotic bound for

$$
\frac{1}{S^{1/(n+r-1)}} \sum_{k=1}^{N_u} \mathbb{P}\big( \sup_{s\in\Delta_0} X_1(s) > u, \sup_{s\in\Delta_k} X_1(s) > u\big),
$$

which follows by the same argument as that given in the proof of Theorem D.2 in [12] (see also the proof of Theorem 3.1 in [4]), with the minor exception that the first term in the above summand is bounded by

$$\begin{aligned}
&\mathbb{P}\big(\sup_{s\in\Delta_0} X_1(s)>u, \sup_{s\in\Delta_1} X_1(s)>u\big)\\
&\qquad\leqslant \mathbb{P}\big(\sup_{s\in[0,Su^{-2/\alpha}]} X_1(s)>u, \sup_{\substack{[(S+S^{1/(2(n+r-1))})u^{-2/\alpha},\\ (2S+S^{1/(2(n+r-1))})u^{-2/\alpha}]}} X_1(s)>u\big)\\
&\qquad\quad + \mathbb{P}\big(\sup_{s\in[0,S^{1/(2(n+r-1))}u^{-2/\alpha}]} X_1(s)>u\big).
\end{aligned}$$

This completes the proof of Theorem 3.1(i).

(ii) For any $0<A<B<\infty$ and sufficiently large $u$, we make the following decomposition:

$$\begin{aligned}
&\mathbb{P}\big(\sup_{t\in[0,\mathcal{T}]} X_{(r)}(t)>u\big)\\
&\qquad= \Big(\int_0^{Am_r(u)} + \int_{Am_r(u)}^{Bm_r(u)} + \int_{Bm_r(u)}^{\infty}\Big)\mathbb{P}\big(\sup_{s\in[0,t]} X_{(r)}(s)>u\big)d\mathbb{P}(\mathcal{T}\leqslant t)\\
&\qquad=: I_1+I_2+I_3.
\end{aligned}$$

We analyze $I_1, I_2, I_3$ separately.

Integral $I_1$. Since the process $\{X_{(r)}(t), t\geqslant 0\}$ is stationary, by Bonferroni's inequality, we have

$$\begin{aligned}
I_1 &\leqslant \mathbb{P}\big(\sup_{s\in[0,1]} X_{(r)}(s)>u\big)\Big(\int_0^{Am_r(u)} t d\mathbb{P}(\mathcal{T}\leqslant t)+1\Big)\\
&= \mathbb{P}\big(\sup_{s\in[0,1]} X_{(r)}(s)>u\big)\\
&\quad\times \Big(\int_0^{Am_r(u)} \mathbb{P}(\mathcal{T}>t)dt - Am_r(u)\mathbb{P}\big(\mathcal{T}>Am_r(u)\big)+1\Big).
\end{aligned}\tag{4.6}$$

Using Karamata's theorem, we get

$$\int_0^{Am_r(u)} \mathbb{P}(\mathcal{T}>t)dt = \frac{1}{\lambda}Am_r(u)\mathbb{P}\big(\mathcal{T}>Am_r(u)\big)\big(1+o(1)\big) \quad \text{as } u\to\infty,$$

which, combined with (4.6) and Theorem 2.2 in [8], implies that

$$\begin{aligned}
I_1 &\leqslant \frac{\lambda}{1-\lambda}A\mathbb{P}\big(\mathcal{T}>Am_r(u)\big)\big(1+o(1)\big)\\
&= \frac{\lambda}{1-\lambda}A^{1-\lambda}\mathbb{P}\big(\mathcal{T}>m_r(u)\big)\big(1+o(1)\big) \quad \text{as } u\to\infty.
\end{aligned}$$

Integral $I_3$. It is straightforward that

$$I_3 \leqslant \mathbb{P}\big(\mathcal{T} > Bm_r(u)\big)\big(1+o(1)\big) = B^{-\lambda}\mathbb{P}\big(\mathcal{T} > m_r(u)\big)\big(1+o(1)\big) \quad \text{as } u \to \infty.$$

Integral $I_2$. For any $\varepsilon > 0$ and sufficiently large $u$, applying Theorem 3.1, we get the upper bound

$$\begin{aligned} I_2 &= \int_A^B \mathbb{P}\big(\sup_{s\in[0,xm_r(u)]} X_{(r)}(s) > u\big) d\mathbb{P}\big(\mathcal{T} \leqslant xm_r(u)\big) \\ &\leqslant (1+\varepsilon)\int_A^B (1-e^{-x}) d\mathbb{P}\big(\mathcal{T} \leqslant xm_r(u)\big) \\ &= (1+\varepsilon)\int_A^B e^{-x}\mathbb{P}\big(\mathcal{T} > xm_r(u)\big)dx - (1+\varepsilon)(1-e^{-B})\mathbb{P}\big(\mathcal{T} > Bm_r(u)\big) \\ &\quad + (1+\varepsilon)(1-e^{-A})\mathbb{P}\big(\mathcal{T} > Am_r(u)\big), \end{aligned}$$

and similarly we obtain the lower bound

$$\begin{aligned} I_2 \geqslant (1-\varepsilon)&\int_A^B e^{-x}\mathbb{P}\big(\mathcal{T} > xm_r(u)\big)dx - (1-\varepsilon)(1-e^{-B})\mathbb{P}\big(\mathcal{T} > Bm_r(u)\big) \\ &+ (1-\varepsilon)(1-e^{-A})\mathbb{P}\big(\mathcal{T} > Am_r(u)\big). \end{aligned}$$

Since $\mathcal{T}$ has a regularly varying tail distribution at infinity, by Theorem 1.5.2 in [5], we get

$$\int_A^B e^{-x}\mathbb{P}\big(\mathcal{T} > xm_r(u)\big)dx = \mathbb{P}\big(\mathcal{T} > m_r(u)\big)\int_A^B e^{-x}x^{-\lambda}dx\big(1+o(1)\big) \quad \text{as } u \to \infty.$$

Thus, for any $\varepsilon > 0$ and $0 < A < B < \infty$, we obtain

$$\begin{aligned} &\limsup_{u\to\infty} \frac{I_2}{\mathbb{P}\big(\mathcal{T} > m_r(u)\big)} \\ &\qquad \leqslant (1+\varepsilon)\Big(\int_0^B x^{-\lambda}e^{-x}dx - (1-e^{-B})B^{-\lambda} + (1-e^{-A})A^{-\lambda}\Big) \end{aligned}$$

and

$$\begin{aligned} &\liminf_{u\to\infty} \frac{I_2}{\mathbb{P}\big(\mathcal{T} > m_r(u)\big)} \\ &\qquad \leqslant (1-\varepsilon)\Big(\int_0^B x^{-\lambda}e^{-x}dx - (1-e^{-B})B^{-\lambda} + (1-e^{-A})A^{-\lambda}\Big). \end{aligned}$$

Therefore, letting $A \to 0$, $B \to \infty$, and $\varepsilon \to 0$, we find that $I_1$ and $I_3$ are negligible, and

$$I_2 = \Gamma(1-\lambda)\mathbb{P}\big(\mathcal{T} > m_r(u)\big)\big(1+o(1)\big) \quad \text{as } u \to \infty,$$

which completes the proof of Theorem 3.2(ii).

(iii) Lower bound. From Theorem 3.1, for any given $B > 0$, it follows that

$$\begin{aligned}\mathbb{P}\big(\sup_{t\in[0,\mathcal{T}]} X_{(r)}(t) > u\big) &\geqslant \mathbb{P}\big(\sup_{s\in[0,Bm_r(u)]} X_{(r)}(s) > u\big)\mathbb{P}\big(\mathcal{T} > Bm_r(u)\big)\\ &= (1-e^{-B})\mathbb{P}\big(\mathcal{T} > m_r(u)\big)\big(1+o(1)\big)\end{aligned}$$

as $u \to \infty$. Thus, letting $B \to \infty$, we obtain the asymptotic lower bound

$$\mathbb{P}\big(\sup_{t\in[0,\mathcal{T}]} X_{(r)}(t) > u\big) \geqslant \mathbb{P}\big(\mathcal{T} > m_r(u)\big)\big(1+o(1)\big) \quad \text{as } u \to \infty.$$

Upper bound. For given $A > 0$, we get

$$\begin{aligned}&\mathbb{P}\big(\sup_{t\in[0,\mathcal{T}]} X_{(r)}(t) > u\big)\\ &\leqslant \int_0^{Am_r(u)} \mathbb{P}\big(\sup_{s\in[0,t]} X_{(r)}(s) > u\big)d\mathbb{P}(\mathcal{T} \leqslant t) + \mathbb{P}\big(\mathcal{T} > Am_r(u)\big)\\ &= \int_0^{Am_r(u)} \mathbb{P}\big(\sup_{s\in[0,t]} X_{(r)}(s) > u\big)d\mathbb{P}(\mathcal{T} \leqslant t) + \mathbb{P}\big(\mathcal{T} > m_r(u)\big)\big(1+o(1)\big)\end{aligned}$$

as $u \to \infty$. Due to the stationarity of the process $\{X_{(r)}(t), t \geqslant 0\}$ and Bonferroni's inequality, we have

$$\begin{aligned}(4.7)\quad &\int_0^{Am_r(u)} \mathbb{P}\big(\sup_{s\in[0,t]} X_{(r)}(s) > u\big)d\mathbb{P}(\mathcal{T} \leqslant t)\\ &\leqslant \mathbb{P}\big(\sup_{s\in[0,1]} X_{(r)}(s) > u\big)\Big(\int_0^{Am_r(u)} t\,d\mathbb{P}(\mathcal{T} \leqslant t) + 1\Big)\\ &\leqslant \mathbb{P}\big(\sup_{s\in[0,1]} X_{(r)}(s) > u\big)\Big(\int_0^{Am_r(u)} \mathbb{P}(\mathcal{T} > t)dt + 1\Big).\end{aligned}$$

From Karamata's theorem (see, e.g., Proposition 1.5.8 in [5]), we get

$$\int_0^{Am_r(u)} \mathbb{P}(\mathcal{T} > t)dt = Am_r(u)\mathbb{P}\big(\mathcal{T} > Am_r(u)\big)\big(1+o(1)\big)$$

as $u \to \infty$, which, combined with (4.7) and Theorem 2.2 in [8], implies that

$$\mathbb{P}\big(\sup_{t\in[0,\mathcal{T}]} X_{(r)}(t) > u\big) \leqslant (1+A)\mathbb{P}\big(\mathcal{T} > m_r(u)\big)\big(1+o(1)\big)$$

as $u \to \infty$. Letting $A \to 0$, we obtain (3.4). This completes the proof of Theorem 3.2. ■

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