EXTREMES OF MULTIDIMENSIONAL STATIONARY
GAUSSIAN RANDOM FIELDS

BY

NATALIA SOJA-KUKIELA (TORUS)

Abstract. Let \( \{ X(t) : t = (t_1, t_2, \ldots, t_d) \in [0, \infty)^d \} \) be a centered stationary Gaussian field with almost surely continuous sample paths, unit variance and correlation function \( r \) satisfying \( r(t) < 1 \) for every \( t \neq 0 \) and \( r(t) = 1 - \sum_{i=1}^d |t_i|^\alpha_i + o(\sum_{i=1}^d |t_i|^\alpha_i) \), as \( t \to 0 \), with some \( \alpha_1, \alpha_2, \ldots, \alpha_d \in (0, 2] \). The main result of this contribution is the description of the asymptotic behaviour of \( P(\sup_{t \in J} X(t) > u) \), as \( u \to \infty \), for some Jordan-measurable sets \( J \) of volume proportional to \( P(\sup_{t \in [0, 1]^d} X(t) > u)^{-1}(1 + o(1)) \).

2010 AMS Mathematics Subject Classification: Primary: 60G15; Secondary: 60G70, 60G60.

Key words and phrases: Gaussian random field, supremum, limit theorem, asymptotics, Berman condition, strong dependence.

1. INTRODUCTION

In extreme value theory of Gaussian processes, we have the following seminal result (see Leadbetter et al. [4], Theorem 12.3.4; Arendarczyk and Dębicki [1], Lemma 4.3; Tan and Hashorva [7], Lemma 3.3) concerning the asymptotics of the distribution of supremum of a centered stationary Gaussian process \( \{ X(t) : t \geq 0 \} \) with correlation function satisfying

\[
r(t) = \text{Cov} \left( X(t), X(0) \right) = 1 - |t|^\alpha + o(|t|^\alpha), \quad \text{as } t \to 0,
\]

for some \( \alpha \in (0, 2] \), over intervals with length proportional to

\[
\mu(u) = P \left( \sup_{t \in [0,1]} X(t) > u \right)^{-1}(1 + o(1)), \quad \text{as } u \to \infty.
\]

Theorem 1.1. Let \( \{ X(t) : t \geq 0 \} \) be a zero-mean, unit-variance stationary Gaussian process with a.s. continuous sample paths and correlation function \( r \) satisfying (1.1) and \( r(t) \log t \to R \in [0, \infty) \) as \( t \to \infty \). Let \( 0 < A < B < \infty \).
Then
\[ P \left( \sup_{t \in [0, x\mu(u)]} X(t) \leq u \right) - E \exp \left( -x \exp \left( -R + \sqrt{2RW} \right) \right), \]
as \( u \to \infty, \) uniformly for \( x \in [A, B], \) with \( W \) an \( N(0, 1) \) random variable.

It is natural to study a similar problem in the \( d \)-dimensional setting for arbitrary \( d \in \mathbb{N}. \) In this case one considers a centered stationary Gaussian process \( \{ X(t_1, t_2, \ldots, t_d) : t_1, t_2, \ldots, t_d \geq 0 \} \) with unit variance and correlation function \( r(t_1, t_2, \ldots, t_d) = \text{Cov} \left( X(t_1, t_2, \ldots, t_d), X(0, 0, \ldots, 0) \right) \) satisfying
\[ r(t_1, t_2, \ldots, t_d) = 1 - \sum_{i=1}^{d} |t_i|^{\alpha_i} + o\left( \sum_{i=1}^{d} |t_i|^{\alpha_i} \right), \]
as \( t_1, t_2, \ldots, t_d \to 0, \) with \( \alpha_1, \alpha_2, \ldots, \alpha_d \in (0, 2]. \) The subject of interest is then the distribution of supremum of the field \( \{ X(t_1, t_2, \ldots, t_d) \} \) over sets of volume proportional to
\[ m(u) = P \left( \sup_{(t_1, t_2, \ldots, t_d) \in [0,1]^d} X(t_1, t_2, \ldots, t_d) > u \right)^{-1} (1 + o(1)). \]

In this paper we investigate suprema over sets of the form
\[ \mathcal{J}_m^x := \left\{ (t_1, t_2, \ldots, t_d) \in \mathbb{R}^d : \frac{t_1}{x_1m_1(u)}, \frac{t_2}{x_2m_2(u)}, \ldots, \frac{t_d}{x_dm_d(u)} \right\} \in \mathcal{J} \right\}, \]
where \( \mathcal{J} \subset \mathbb{R}^d \) is a Jordan-measurable set with Lebesgue measure \( \lambda(\mathcal{J}) > 0, \)
\( x = (x_1, x_2, \ldots, x_d) \in (0, \infty)^d \) and \( m = (m_1, m_2, \ldots, m_d) \) with \( m_1, m_2, \ldots, m_d \)
some positive functions satisfying \( m_1(u)m_2(u) \ldots m_d(u) = m(u). \) Let us put
\[ \mathcal{J}_m := \mathcal{J}_m^{[1, \ldots, 1]}. \] One interesting case is \( \mathcal{J} = [0, 1]^d \) with \( \mathcal{J}_m^x = \prod_{i=1}^{d} [0, x_im_i(u)]. \)

In a recent paper Dębicki et al. \[ [3] \] consider the case \( d = 2. \) They assume that the functions \( m_1 \) and \( m_2 \) tend to infinity and satisfy
\[ \log m_1(u) \to 1 \quad \text{as} \quad u \to \infty. \]

The authors establish the following two-dimensional counterpart ([5], Theorem 2) of Theorem [1].

**Theorem 1.2.** Let \( \{ X(t_1, t_2) : t_1, t_2 \geq 0 \} \) be a zero-mean, unit-variance stationary Gaussian field with a.s. continuous sample paths and correlation function \( r \) satisfying (1.2) and \( r(t_1, t_2) \log \sqrt{t_1^2 + t_2^2} \to R \in [0, \infty) \) as \( t_1^2 + t_2^2 \to \infty. \)

Let \( m_1 \) and \( m_2 \) be positive functions such that \( m_1(u)m_2(u) = m(u) \) and (1.3) hold. Then:
(i) for each $0 < A < B < \infty$, we have
\[
P \left( \sup_{(t_1, t_2) \in [0, x_m]} X(t_1, t_2) \leq u \right) \to e^{-x_1x_2 \exp(-2R+2\sqrt{\|W\|})},
\]
as $u \to \infty$, uniformly for $(x_1, x_2) \in [A, B]^2$, with $W$ an $N(0, 1)$ random variable.

(ii) for every Jordan-measurable set $J \subset \mathbb{R}^2$ with Lebesgue measure $\lambda(J) > 0$, we have
\[
P \left( \sup_{(t_1, t_2) \in J_m} X(t_1, t_2) \leq u \right) \to e^{-\lambda(J) \exp(-2R+2\sqrt{\|W\|})},
\]
as $u \to \infty$, with $W$ an $N(0, 1)$ random variable.

Our goal is to derive a general limit theorem for the distribution of supremum of the field \( \{X(t_1, t_2, \ldots, t_d)\} \) over sets $J_m^d$, for arbitrary $d \in \mathbb{N}$ and for a wide class of families \( \{m_1, m_2, \ldots, m_d\} \) of functions, uniform for $x \in [A, B]^d$, for all $0 < A < B < \infty$. The main result is Theorem 3.1. In the paper we do not assume that every $m_i$ tends to infinity like Dębicki et al. \([3]\) do. We fully explain the case when all $m_i$s are separated from zero (see Theorem 3.1 and Remark 3.1) and give some partial results in the case when some of $m_i$s tend to zero (see Corollaries 3.4 and 3.5).

2. PRELIMINARIES

We consider $\mathbb{R}^d$ with coordinatewise order $\leq$, write $t = (t_1, t_2, \ldots, t_d)$ for an element $t \in \mathbb{R}^d$, put $0 := (0, 0, \ldots, 0)$ and $1 := (1, 1, \ldots, 1)$, and denote by $\| \cdot \|_\infty$ the sup-norm in $\mathbb{R}^d$, i.e., $\|t\|_\infty = \max \{|t_1|, |t_2|, \ldots, |t_d|\}$ for any $t \in \mathbb{R}^d$.

Let $\{X(t) : t \in [0, \infty)^d\}$ be a centered stationary Gaussian field with a.s. continuous sample paths, unit variance and correlation function
\[
r(t) = \text{Cov}(X(t), X(0)).
\]

We will often assume that the correlation function satisfies:

A1: $r(t) = 1 - \sum_{i=1}^d |t_i|^{\alpha_i} + o(\sum_{i=1}^d |t_i|^{\alpha_i})$ as $t_1, t_2, \ldots, t_d \to 0$;

A2: $r(t) < 1$ for $t \neq 0$;

A3: $r(t) \log \sqrt{t_1^2 + t_2^2 + \ldots + t_d^2} \to R$ as $t_1^2 + t_2^2 + \ldots + t_d^2 \to \infty$,

with some constants $\alpha_1, \alpha_2, \ldots, \alpha_d \in (0, 2]$ and $R \in [0, \infty)$.

The above conditions are analogous to the ones given in \([4]\), \([1]\), \([7]\), \([3]\).

Condition A1 implies that the correlation function $r$ is continuous. A1 and A2 give $|r(t)| < 1$ for $t \neq 0$. Moreover, condition A2 follows from A1 and A3. Notice that we study both weakly dependent fields, satisfying A3 with $R = 0$, and strongly dependent fields, satisfying A3 with $R \in (0, \infty)$.

For every $\alpha \in (0, 2]$, we denote by $\mathcal{H}_\alpha$ the Pickands constant (see \([5]\)), i.e.,
\[
\mathcal{H}_\alpha := \lim_{T \to \infty} E \exp \left( \max_{0 \leq t \leq T} B_{\alpha/2}(t) - |t|^{\alpha/2} \right),
\]
where $\{B_{\alpha/2}(t) : t \geq 0\}$ is a fractional Brownian motion with Hurst index $\alpha/2$. 

Probability and Mathematical Statistics 38, z. 1, 2018
© for this edition by CNS
Let $W$ be a standard normal random variable and let $\Phi(u) := P(W \leq u)$, $\Psi(u) := P(W > u)$. We recall that

$$\Psi(u) = \frac{1}{\sqrt{2\pi}} \exp\left( -\frac{u^2}{2} \right) (1 + o(1)) \quad \text{as } u \to \infty.$$  

If the considered field $\{X(t)\}$ satisfies A1 and A2, then, for arbitrary Jordan-measurable set $J \subset \mathbb{R}^d$ with Lebesgue measure $\lambda(J) > 0$, we have

$$(2.1) \quad P\left( \max_{t \in J} X(t) > u \right) = \lambda(J) \prod_{i=1}^{d} (\mathcal{H}_{\alpha_i} u^{2/\alpha_i}) \Psi(u) (1 + o(1)),$$  

as $u \to \infty$, due to Piterbarg [6], Theorem 7.1. Thus

$$m(u) := \left( \prod_{i=1}^{d} (\mathcal{H}_{\alpha_i} u^{2/\alpha_i}) \Psi(u) \right)^{-1} = P\left( \max_{t \in [0,1]^d} X(t) > u \right)^{-1} (1 + o(1)).$$  

Let $m_1, m_2, \ldots, m_d$ be positive functions such that

$$m_1(u)m_2(u) \cdots m_d(u) = m(u)$$  

and for some $k \in \{0, 1, \ldots, d-1\}$:

1. for every $i \in \{1, 2, \ldots, k\}$ there exists an $M_i \in (0, \infty)$ such that
   $$m_i(u) \to M_i \quad \text{as } u \to \infty;$$

2. for every $i \in \{k+1, k+2, \ldots, d\}$ we have
   $$m_i(u) \to \infty \quad \text{as } u \to \infty \quad \text{and} \quad m_i(u) = \exp(\gamma_i u^2) c_i(u),$$

for some constant $\gamma_i \in [0,1/2]$ and positive function $c_i$ with $\log c_i(u) = o(u^2)$. Then $\gamma_{k+1} + \gamma_{k+2} + \cdots + \gamma_d = 1/2$. We put $\gamma := \max_i \gamma_i$.

For arbitrary $x \in (0, \infty)^d$, we define $R^x := [0, x_1] \times [0, x_2] \times \cdots \times [0, x_d]$ and $R^x_m := [0, x_1 m_1(u)] \times [0, x_2 m_2(u)] \times \cdots \times [0, x_d m_d(u)]$ for each $u \in \mathbb{R}$. Note that $R^x_m = J^x_m$ for $J = [0, 1]^d$.

3. RESULTS

Below, in Section 3.1, we present Theorem 3.1, which is the main result. Its proof is given in Sections 3.3 and 3.4. Some consequences of Theorem 3.1 can be found in Sections 3.1 and 3.2.
3.1. Main theorem. The following theorem describes the asymptotic behaviour of \( P \left( \sup \{X(t) : t \in J_m^x \} \leq u \} \), as \( u \to \infty \), for Jordan-measurable sets \( J_m^x \) of volume proportional to \( m(u) \).

**Theorem 3.1.** Let \( \{X(t) : t \in [0, \infty)^d \} \) be a centered stationary Gaussian field with a.s. continuous sample paths, unit variance and correlation function \( r \) that satisfies A1 and A3 with some \( R \in [0, \infty) \). Then, for every Jordan-measurable set \( J \subset \mathbb{R}^d \) with \( \lambda(J) > 0 \), for each \( 0 < A < B < \infty \),

\[
P \left( \sup_{t \in J_m^x} X(t) \leq u \right) \to \exp \left( -x_1 x_2 \ldots x_d \lambda(J) \exp \left( -\frac{R}{2\gamma} + \sqrt{\frac{R}{\gamma}} W \right) \right),
\]

as \( u \to \infty \), uniformly for \( x \in [A, B]^d \).

Applying the above theorem for \( J = [0, 1]^d \), we obtain the following result.

**Corollary 3.1.** Let \( \{X(t)\} \) satisfy the assumptions of Theorem 3.1. Then, for each \( 0 < A < B < \infty \),

\[
P \left( \sup_{t \in J_m^x} X(t) \leq u \right) \to \exp \left( -x_1 x_2 \ldots x_d \exp \left( -\frac{R}{2\gamma} + \sqrt{\frac{R}{\gamma}} W \right) \right),
\]

as \( u \to \infty \), uniformly for \( x \in [A, B]^d \).

In the special case, when \( k = 0 \) and the functions \( m_1, m_2, \ldots, m_d \) are chosen so that \( \gamma_1 = \gamma_2 = \ldots = \gamma_d \) (and thus a \( d \)-dimensional analog of (1.3) holds), we have the following corollary. Note that for \( d = 2 \) it coincides with Theorem 1.2.

**Corollary 3.2.** Let the assumptions of Theorem 3.1 be satisfied and let

\[
(3.1) \quad \frac{\log m_i(u)}{\log m_j(u)} \to 1 \quad \text{as} \quad u \to \infty, \quad \text{for} \ i, j \in \{1, 2, \ldots, d\}.
\]

Then, for every Jordan-measurable set \( J \subset \mathbb{R}^d \) with \( \lambda(J) > 0 \),

\[
P \left( \sup_{t \in J_m^x} X(t) \leq u \right) \to \exp \left( -x_1 x_2 \ldots x_d \lambda(J) \exp \left( -dR + \sqrt{2dRW} \right) \right),
\]

as \( u \to \infty \), uniformly for \( x \in [A, B]^d \), for each \( 0 < A < B < \infty \).

3.2. Some consequences of the main theorem. Let the field \( \{X(t)\} \) satisfy the assumptions of Theorem 3.1. In this section we ask for the asymptotic behaviour of the supremum of \( \{X(t)\} \) over sets \( J_m^x \), for \( J \subset \mathbb{R}^d \) a Jordan-measurable set with \( \lambda(J) > 0 \), \( x \in (0, \infty)^d \), \( \mathbf{m} = (\tilde{m}_1, \tilde{m}_2, \ldots, \tilde{m}_d) \) and \( \tilde{m}_1, \tilde{m}_2, \ldots, \tilde{m}_d \) some positive functions with \( \tilde{m}_1(u) \tilde{m}_2(u) \ldots \tilde{m}_d(u) = m(u) \). Note, we do not assume that \( \tilde{m}_1, \tilde{m}_2, \ldots, \tilde{m}_d \) fulfill all the conditions, which have to be satisfied by the functions \( m_1, m_2, \ldots, m_d \) introduced in Section 2.
First, we consider the case when the functions \( \bar{m}_1, \bar{m}_2, \ldots, \bar{m}_d \) are separated from zero, i.e., \( \bar{m}_1(u), \bar{m}_2(u), \ldots, \bar{m}_d(u) > \varepsilon \) for some \( \varepsilon > 0 \). Then, it is easy to show that every sequence \( \{u_n\}_{n \in \mathbb{N}} \) tending to infinity contains a subsequence \( \{u_{n_j}\}_{j \in \mathbb{N}} \) such that for each \( i \in \{1, 2, \ldots, d\} \) we have \( \bar{m}_i(u_{n_j}) \to M_i \in [\varepsilon, \infty) \), as \( j \to \infty \), or, alternatively, \( \bar{m}_i(u_{n_j}) = \exp(\gamma_i u_{n_j}^2) \tilde{c}_i(u_{n_j}) \to \infty \), as \( j \to \infty \), for some constant \( \gamma_i \in [0, 1/2] \) and some function \( \tilde{c}_i \) with \( \log \tilde{c}_i(u_{n_j}) = o(u_{n_j}^2) \). We can apply Theorem 3.1 for such subsequences. This justifies the following remark.

**Remark 3.1.** Theorem 3.1 fully explains the case when \( \bar{m}_1, \bar{m}_2, \ldots, \bar{m}_d \) are positive functions separated from zero, such that \( \bar{m}_1(u)\bar{m}_2(u) \cdots \bar{m}_d(u) = m(u) \). It gives the asymptotics for convergent subsequences.

Since for weakly dependent Gaussian fields the limit in Theorem 3.1 does not depend on \( \gamma \), the above considerations entail a concise corollary.

**Corollary 3.3.** Let \( \{X(t)\} \) satisfy the assumptions of Theorem 3.1 with \( R = 0 \) and let \( \bar{m}_1, \bar{m}_2, \ldots, \bar{m}_d \) be positive functions separated from zero, such that \( \bar{m}_1(u)\bar{m}_2(u) \cdots \bar{m}_d(u) = m(u) \). Then, for each \( 0 < A < B < \infty \),

\[
\mathbb{P} \left( \sup_{t \in J_m^x} X(t) \leq u \right) \to \exp \left( -x_1 x_2 \cdots x_d \lambda(J) \right),
\]

as \( u \to \infty \), uniformly for \( x \in [A, B]^d \).

Next, we focus on the case when \( \bar{m}_i \)'s are allowed to tend to zero. In general, such weakening of the assumptions enforces a different approach. However, basing on Theorem 3.1, we can give the limit theorems in two special opposite cases: when \( \bar{m}_i \to 0 \) sufficiently fast and when \( \bar{m}_i \to 0 \) sufficiently slow.

Suppose that for some \( 0 \leq j \leq k < d \):

1. for every \( i \in \{j + 1, j + 2, \ldots, k\} \) there exists an \( \bar{M}_i \in (0, \infty) \) such that

   \[
   \bar{m}_i(u) \to \bar{M}_i \quad \text{as} \quad u \to \infty;
   \]

2. for every \( i \in \{k + 1, k + 2, \ldots, d\} \)

   \[
   \bar{m}_i(u) \to \infty \quad \text{(as} \quad u \to \infty) \quad \text{and} \quad \bar{m}_i(u) = \exp(\tilde{\gamma}_i u^2) \tilde{c}_i(u)
   \]

hold for some constant \( \tilde{\gamma}_i \geq 0 \) and function \( \tilde{c}_i \) such that \( \log \tilde{c}_i(u) = o(u^2) \). Then \( \tilde{\gamma}_{k+1} + \tilde{\gamma}_{k+2} + \cdots + \tilde{\gamma}_d \geq 1/2 \). We put \( \tilde{\gamma} := \max_i \tilde{\gamma}_i \).

Note that the above conditions are very similar to the conditions given in Section 3 for the functions \( m_1, m_2, \ldots, m_d \). Under these assumptions (and some extra ones) we can prove the following results.
Corollary 3.4. Assume that \( \bar{m}_1, \bar{m}_2, \ldots, \bar{m}_d \) satisfy the above conditions and, moreover, 
\[
\bar{m}_1(u) = \exp(-\kappa u^2) c(u)
\]
for some constant \( \kappa > 0 \) and function \( c \) satisfying \( \log c(u) = o(u^2) \). Then,
\[
P \left( \sup_{t \in J_m} X(t) \leq u \right) \to 0, \quad \text{as} \; u \to \infty,
\]
uniformly for \( x \in [A, \infty)^d \), for each \( A > 0 \).

Proof. Let \( x \in (0, \infty)^d \). Since the set \( J \subset \mathbb{R}^d \) is Jordan-measurable and \( \lambda(J) > 0 \), there exist \( y \in \mathbb{R}^d \) and \( z \in (0, \infty)^d \) such that \( y + R^x \subset J \). Thus
\[
P \left( \sup_{t \in J_m} X(t) \leq u \right) \leq P \left( \sup_{t \in (y+R^x)_{J_m}} X(t) \leq u \right) = P \left( \sup_{t \in R^x_{J_m}} X(t) \leq u \right),
\]
with \( zx := (z_1 x_1, z_2 x_2, \ldots, z_d x_d) \), where the last equality is a consequence of stationarity. Furthermore,
\[
P \left( \sup_{t \in R^x_{J_m}} X(t) \leq u \right) \leq P \left( \sup_{0 \leq t_i \leq z_i x_i} X(0, \ldots, 0, t_{k+1}, t_{k+2}, \ldots, t_d) \leq u \right).
\]
We will show that the right-hand side of the above inequality tends to zero, using Theorem 3.1 for the field \( \tilde{X}(t_{k+1}, t_{k+2}, \ldots, t_d) := X(0, \ldots, 0, t_{k+1}, t_{k+2}, \ldots, t_d) \).

Since \( \kappa > 0 \), we have \( \sigma := \gamma_{k+1} + \gamma_{k+2} + \ldots + \gamma_d > 1/2 \). Hence
\[
\frac{\bar{m}_{k+1}(u) \bar{m}_{k+2}(u) \ldots \bar{m}_d(u)}{\bar{m}(u)} \to \infty, \quad \text{as} \; u \to \infty,
\]
where
\[
\bar{m}(u) := \left( \prod_{i=k+1}^d (H_{a_i} u^{2/a_i}) \Psi(u) \right)^{-1}.
\]
For every \( i \in \{k + 1, k + 2, \ldots, d\} \), we put
\[
\bar{m}_i(u) := \exp(\gamma_i u^2) \hat{c}_i(u),
\]
with \( \gamma_i := (2\sigma)^{-1} \gamma_i \) and \( \hat{c}_i(u) := \left( \bar{m}(u) \exp(-u^2/2) \right)^{1/(d-k)} \). Then \( \gamma_i \in [0, 1/2] \), \( \log \hat{c}_i(u) = o(u^2) \) and \( \gamma_{k+1} + \gamma_{k+2} + \ldots + \gamma_d = 1/2 \). Moreover, the functions \( \bar{m}_i \) satisfy \( \bar{m}_{k+1}(u) \bar{m}_{k+2}(u) \ldots \bar{m}_d(u) = \bar{m}(u) \) and we have
\[
\frac{\bar{m}_i(u)}{\bar{m}_i(u)} \to \infty \quad \text{as} \; u \to \infty.
\]
Let $C > 0$ be arbitrary. Since $\hat{m}_i(u)/\tilde{m}_i(u) > C$ for all sufficiently large $u$, we obtain

$$
\limsup_{u \to \infty} \mathbb{P} \left( \sup_{0 \leq t_i \leq x_i \tilde{m}_i} X(0, \ldots, 0, t_{k+1}, t_{k+2}, \ldots, t_d) \leq u \right)
\leq \limsup_{u \to \infty} \mathbb{P} \left( \sup_{0 \leq t_i \leq C_j \tilde{m}_i} \tilde{X}(t_{k+1}, t_{k+2}, \ldots, t_d) \leq u \right)
= \mathbb{E} \exp \left( -C_d - k \int_{k+1}^{x_{k+2}} \cdots \exp \left( -R \frac{d}{2g} + \sqrt{\frac{R}{\gamma W}} \right) \right),
$$

with $\hat{\gamma} := \max \{ \hat{\gamma}_i \}$ due to Theorem 3.1. Since the right-hand side tends to zero as $C \to \infty$, the proof of pointwise convergence is complete. Uniform convergence simply follows from the monotonicity of $x \to \mathbb{P} \left( \sup_{t \in J_m} X(t) \leq u : t \in J_{m} \right)$. 

**Corollary 3.5.** Suppose that $m_1, m_2, \ldots, m_d$ are positive functions such that $m_1(u) m_2(u) \cdots m_d(u) = m(u)$ holds and, moreover, assume that

$$m_i(u) = 1, \quad \text{for } i \leq j, \text{ for some } j \in \{1, 2, \ldots, d-1\},$$

$$m_i(u) = \exp(\gamma_i u^2) c_i(u) \to M_i \in (0, \infty], \text{ as } u \to \infty, \quad \text{for } i > j,$$

where $\gamma_i \in [0, 1/2]$ and $\log c_i(u) = o(u^2)$. There exist some positive functions $\nu_1, \nu_2, \ldots, \nu_j$ satisfying $\nu_i(u) \to 0$, such that for all $\tilde{m}_1, \tilde{m}_2, \ldots, \tilde{m}_d$ satisfying the conditions: $\nu_i(u) = o(\tilde{m}_i(u))$ for each $i \in \{1, 2, \ldots, j\}$, $\tilde{m}_i(u) = m_i(u)$ for each $i \in \{j+1, j+2, \ldots, d-1\}$ and $m_d(u) = m_d(u) \cdot \prod_{i=1}^{j} \tilde{m}_i(u)^{-1}$, we have

$$
\mathbb{P} \left( \sup_{t \in J_m} X(t) \leq u \right) \to \mathbb{E} \exp \left( -\lambda(J) \exp \left( -R \frac{d}{2g} + \sqrt{\frac{R}{\gamma W}} \right) \right),
$$

as $u \to \infty$.

**Proof.** Let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_j > 0$ and $\varepsilon := (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_j, 1, \ldots, 1, \prod_{i=1}^{j} \varepsilon_i^{-1})$. By application of Theorem 3.1, we obtain

$$
\mathbb{P} \left( \sup_{t \in J_m} X(t) \leq u \right) \to \mathbb{E} \exp \left( -\lambda(J) \exp \left( -R \frac{d}{2g} + \sqrt{\frac{R}{\gamma W}} \right) \right),
$$

as $u \to \infty$, uniformly for $\varepsilon \in [A, B]^d$, for all $0 < A < B < \infty$. Note that the above limit does not depend on the choice of $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_j$. It is not difficult to show that there exist some functions $\nu_i$, $i \in \{1, 2, \ldots, j\}$, tending to zero, such
that for positive functions $\varepsilon_i = \varepsilon_i(u)$, $i \in \{1, 2, \ldots, j\}$, tending to zero, and for $\varepsilon(u) := (\varepsilon_1(u), \varepsilon_2(u), \ldots, \varepsilon_j(u), 1, \ldots, 1, \prod_{i=1}^j \varepsilon_i^{-1}(u))$, we have

$$P \left( \sup_{t \in \mathbb{Z}_m^n(u)} X(t) \leq u \right) \to E \exp \left( -\lambda(J) \exp \left( -\frac{R}{2\gamma} + \sqrt{\frac{R}{\gamma}} W \right) \right),$$

whenever $\nu_i(u) = o(\varepsilon_i(u))$. We shall put $\varepsilon_i(u) = \tilde{m}_i(u)$ for $i \in \{1, 2, \ldots, j\}$. ■

**Remark 3.2.** We do not know the form of the functions $\nu_1, \nu_2, \ldots, \nu_j$ from Corollary 3.3. Our conjecture is that $\nu_i(u) = u^{-2/\alpha_i}$ for $i \in \{1, 2, \ldots, j\}$.

### 3.3. Lemmas

The lemmas formulated in this section are crucial in the proof of Theorem 3.1 (see Section 3.2). They are $d$-dimensional counterparts of known results: Lemma 3.1 generalizes Lemma 12.2.11 in [3] and Lemma 1 in [4]; Lemma 3.2 combines $d$-dimensional analogs of Lemma 12.3.1 in [3] (for weakly dependent fields) and Lemma 3.1 in [7] (for strongly dependent fields), it is a generalization of Lemma 2 in [4]. Since the argumentation for Lemmas 3.1 and 3.2 mimics the one given in [3] and expanded in [7], [2], the proofs are skipped. We present the proof of Lemma 3.3, which improves the lemma given by Dębicki et al. [3], [2] and enables us to establish far more general results than the ones in [3].

Let $a > 0$. Put $q_i = q_i(u) := au^{-2/\alpha_i}$ for $i \in \{1, 2, \ldots, d\}$. Moreover, define $j\mathbf{q} = j\mathbf{q}(u) := (j_1q_1(u), j_2q_2(u), \ldots, j_dq_d(u))$ for $j = (j_1, j_2, \ldots, j_d) \in \mathbb{Z}^d$.

**Lemma 3.1.** Assume that conditions A1 and A2 hold. Then there exists a function $\vartheta$ satisfying $\vartheta(a) \to 0$, as $a \to 0$, such that for every $a > 0$ we have

$$P \left( \sup_{j\mathbf{q} \in y + \mathbb{R}^x} X(j\mathbf{q}) \leq u \right) - P \left( \sup_{t \in y + \mathbb{R}^x} X(t) \leq u \right) \leq \frac{x_1x_2 \cdots x_d}{m} \vartheta(a) + o \left( \frac{1}{m} \right),$$

as $u \to \infty$, uniformly for $y \in [0, \infty)^d$ and $x \in [A, B]^d$, for all $0 < A < B < \infty$.

**Remark 3.3.** An explicit formula for $\vartheta$ from Lemma 3.1 can be found in [2].

**Lemma 3.2.** Suppose that $T = T(u) \to \infty$ as $u \to \infty$. Then, providing that conditions A1 and A2 are fulfilled, there exists an $\varepsilon > 0$ such that for all $R \geq 0$

$$\frac{m}{q_1q_2 \cdots q_d} \sum_{j\mathbf{q} \in (-\varepsilon, \varepsilon)^d \setminus \{0, 0, \ldots, 0\}} \left[ (1-r(j\mathbf{q})) \frac{R}{\log T} \left( 1 - \frac{R \log T}{1 + r(j\mathbf{q})} \right)^2 \right]^{1/2} \times \exp \left( -\frac{u^2}{1 + r(j\mathbf{q}) + (1 - r(j\mathbf{q})) R/\log T} \right) \to 0,$$

as $u \to \infty$. 
Let \( R > 0 \) be fixed. The last lemma concerns functions \( \rho_T \) and \( \varrho_T \) defined for an arbitrary \( T > 1 \) and for \( t \in \mathbb{R}^d \) as follows:

\[
\rho_T(t) := \begin{cases} 
1, & \max\{|t_{k+1}|, |t_{k+2}|, \ldots, |t_d|\} < 1, \\
|r(t) - \frac{R}{\log T}|, & \text{otherwise}; 
\end{cases}
\]

(3.2)

\[
\varrho_T(t) := \begin{cases} 
|r(t)| + \left(1 - r(t)\right) \frac{R}{\log T}, & \max\{|t_{k+1}|, |t_{k+2}|, \ldots, |t_d|\} < 1, \\
\frac{R}{\log T}, & \text{otherwise}; 
\end{cases}
\]

Lemma 3.3. Assume that \( T_i = T_i(u) \sim \tau_i m_i(u), \) as \( u \to \infty, \) for some \( \tau_i > 0 \) and every \( i \in \{1, 2, \ldots, d\} \). Let \( \varepsilon > 0. \) Then, providing that conditions A1 and A3 with \( R \in [0, \infty) \) are fulfilled,

\[
\frac{T_1 T_2 \ldots T_d}{q_1 q_2 \ldots q_d} \sum_{\mathbf{j} \in \prod_{i=1}^{d} [-T_i, T_i]} \rho_T(q \mathbf{j}) \exp\left(-\frac{u^2}{1+\max\{|r(q \mathbf{j})|, \varrho_T(q \mathbf{j})|\}}\right) \to 0,
\]

as \( u \to \infty, \) with \( T := \max\{T_1, T_2, \ldots, T_d\}. \)

Proof. We present the proof in the case \( d = 2. \) The argumentation for other dimensions is analogous. We follow the reasoning from Lemma 2 in [2], making modifications and skipping some details, which can be found in [2].

Since \( T_1(u) T_2(u) \sim \tau_1 \tau_2 m(u), \) as \( u \to \infty, \) we get

(3.3) \[ u^2 = 2 \log(T_1 T_2) + \left(\frac{2}{\alpha_1} + \frac{2}{\alpha_2} - 1\right) \log \log(T_1 T_2) + O(1). \]

It is not difficult to see that there exists a constant \( \delta \in (0, 1) \) such that for all sufficiently large \( L \)

\[
\sup_{\varepsilon \leq ||t||_{\infty} \leq L} \max\{|r(t)|, \varrho_L(t)|\} < \delta.
\]

Denote by \( \beta \) a constant satisfying \( 0 < \beta < (1 - \delta)/(1 + \delta) \) and divide the set \( \mathcal{Q} := [-T_1, T_1] \times [-T_2, T_2] - (-\varepsilon, \varepsilon)^2 \) into two subsets:

\[
\mathcal{S}^* := \{ t \in \mathcal{Q} : |t_1| \leq m(u)^{\beta/2}, |t_2| \leq m(u)^{\beta/2} \},
\]

\[
\mathcal{S} := \mathcal{Q} - \mathcal{S}^*.
\]

Observe that the shape of the set \( \mathcal{S}^* \) of volume \( m(u)^{\beta}(1 + o(1)) \) does not depend on the choice of \( m_1 \) and \( m_2. \)
Following line-by-line the arguments from [2], thanks to the proper choice of \( \beta \), we obtain

\[
(3.4) \quad \frac{T_1 T_2}{q_1 q_2} \sum_{j_1 \in S^*} \rho_T(j_1) \exp \left( - \frac{u^2}{1 + \max\{|r(j_1)|, \varrho_T(j_1)|\}} \right) \to 0,
\]
as \( u \to \infty \).

To complete the proof, it suffices to show that

\[
(3.5) \quad \frac{T_1 T_2}{q_1 q_2} \sum_{j_1 \in S} \rho_T(j_1) \exp \left( - \frac{u^2}{1 + \max\{|r(j_1)|, \varrho_T(j_1)|\}} \right) \to 0,
\]
as \( u \to \infty \). By an argument from [2] and the fact that \( m(u)^{\beta/2} \to \infty \), we get

\[
\max\{|r(j_1)|, \varrho_T(j_1)|\} \leq \frac{C}{\log m(u)^{\beta/2}},
\]
for sufficiently large \( u \), some constant \( C > 0 \) and all points \( j_1 \in \mathcal{Q} \) satisfying \( ||j_1||\infty \geq m(u)^{\beta/2} \). Hence we have

\[
\frac{T_1 T_2}{q_1 q_2} \sum_{j_1 \in S} \rho_T(j_1) \exp \left( - \frac{u^2}{1 + \max\{|r(j_1)|, \varrho_T(j_1)|\}} \right) \leq 4 \frac{T_1^2 T_2^2}{q_1 q_2} \exp \left( -u^2 \left( 1 - \frac{C}{\log m^{\beta/2}} \right) \right) \frac{1}{\log m^{\beta/2}}
\times \frac{q_1 q_2 \log m^{\beta/2}}{T_1 T_2} \sum_{j_1 \in S} \left| r(j_1) - R \frac{\log T}{\log T} \right|
=: I_1(u) \times I_2(u).
\]

Applying the equality (3.3), the definition of the functions \( q_1 \) and \( q_2 \) and the convergence \( \log \left( \frac{T_1(u) T_2(u)}{\log m(u)^{\beta/2}} \right) \to 2/\beta \), as \( u \to \infty \), we conclude that \( I_1 \) is bounded. Our argumentation is analogous to the one given in [2]. The strong condition (1.3) turns out not to be necessary.

In the next step we prove that \( I_2(u) \to 0 \) as \( u \to \infty \). Observe that we have

\[
I_2(u) \leq \frac{q_1 q_2}{T_1 T_2} \sum_{j_1 \in S} \left| r(j_1) \log \sqrt{(j_1 q_1)^2 + (j_2 q_2)^2} - R \right| \left| 1 + o(1) \right|
+ \beta R \frac{q_1 q_2}{T_1 T_2} \sum_{j_1 \in S} \left| 1 - \frac{\log T}{\log \sqrt{(j_1 q_1)^2 + (j_2 q_2)^2}} \right| \left( 1 + o(1) \right)
=: J_1(u) + J_2(u).
\]
We need to show that both $J_1$ and $J_2$ tend to zero. Note that $J_1(u) \to 0$ as $u \to \infty$, due to A3. Additionally,

$$J_2(u) \leq \frac{2R}{\log m} \frac{q_1q_2}{T_1T_2} \sum_{j \in S} \log \left( \frac{\sqrt{(j_1q_1)^2 + (j_2q_2)^2}}{T} \right),$$

and hence

$$J_2(u) = \frac{2R}{\log m} \cdot O\left( \int_0^1 \int_0^1 \log \left( \sqrt{x^2 + y^2} \right) \, dx \, dy + \int_0^1 \log |x| \, dx \right).$$

Thus (5.5) holds. The combination of (5.4) and (5.5) completes the proof. 

3.4. Proof of Theorem 5.1. To establish the main result, we develop the ideas given in [3], [11], [7], [5]. The following proof of Theorem 5.1 combines the method of proof of Theorem 1.2 for $d = 2$ and $\gamma_1 = \gamma_2 = 1/4$ (see [4], Theorem 2), the lemmas from Section 3.3 and some new observations.

The proof consists of two parts. In (i), we present a complete argumentation for the special case $J = [0, 1]^d$. In (ii), we explain how to apply the first part of the proof to obtain the limit theorem for arbitrary $J$.

(i) Let us consider $J = [0, 1]^d$. Then $J_m^x = R_m^x$ for $x \in (0, \infty)^d$. Let $\{X^k(t)\}$, for $k \in \mathbb{N}^{d-k}$, be independent copies of $\{X(t)\}$ and let

$$\eta(t) := X^k(t) \quad \text{for } t \in [0, \infty)^d,$$

with $k(t) = (\lfloor t_{k+1} \rfloor + 1, \lfloor t_{k+2} \rfloor + 1, \ldots, \lfloor t_d \rfloor + 1)$. For any $T > 0$, we define a Gaussian random field $\{Y_T(t) : t \in [0, T]^d\}$ by

$$Y_T(t) := \left( 1 - \frac{R}{\log T} \right)^{1/2} \eta(t) + \left( \frac{R}{\log T} \right)^{1/2} \mathcal{W},$$

where $\mathcal{W}$ denotes an $N(0, 1)$ random variable independent of $\{\eta(t)\}$. Then the covariance $C_T(t, t + s) := \text{Cov} (Y_T(t), Y_T(t + s))$ equals

$$C_T(t, t + s) = \begin{cases} \frac{r(s) + (1 - r(s)) R}{\log T} & \text{if } \lfloor s_i + t_i \rfloor = \lfloor t_i \rfloor \text{ for } k < i \leq d, \\ \frac{R}{\log T} & \text{otherwise.} \end{cases}$$

For $x \in (0, \infty)^d$ we define $\mathbf{n}(x, m) := (n_1^x, n_2^x, \ldots, n_d^x)$ with $n_i^x := x_i M_i$ for $i \in \{1, 2, \ldots, k\}$ and $n_i^x = n_i^x(u) := \lfloor x_i m_i(u) \rfloor$ for $i \in \{k+1, k+2, \ldots, d\}$. Since

$$\mathbb{P} \left( \sup_{t \in R_m^x} X(t) \leq u \right) - \mathbb{P} \left( \sup_{t \in R_m^{x}(m)} X(t) \leq u \right) = o(1), \quad \text{as } u \to \infty,$$

we may focus on the asymptotics of the right-hand side of the above equality.
Applying the Bonferroni inequality, stationarity and the asymptotics (2.1), we get

\[ x \] uniformly for \( x \).

To simplify the notation, we will write each box \( G_l \) indexed by \( l \) as follows:

\[ I_l := [0, x_iM_i] \times \prod_{i=k+1}^d [l_i - 1, l_i], \]

indexed by \( l = (l_{k+1}, l_{k+2}, \ldots, l_d) \in \mathbb{N}^{d-k} \) such that \( 1 \leq l_i \leq n_i^{x} \). Next we split each box \( G_l \) into two subsets \( \mathcal{I}_1 \) and \( \mathcal{I}_1^* \) as follows:

\[ \mathcal{I}_1 := \prod_{i=1}^k [0, x_iM_i] \times \prod_{i=k+1}^d [(l_i - 1) + \varepsilon, l_i], \]

\[ \mathcal{I}_1^* := G_l - \mathcal{I}_1. \]

To simplify the notation, we will write

\[ \mathcal{I} := \bigcup \{ \mathcal{I}_1 : 1 \leq l \leq (n_{k+1}^{x}, n_{k+2}^{x}, \ldots, n_d^{x}) \}. \]

Applying the Bonferroni inequality, stationarity and the asymptotics (2.1), we get

\[
\limsup_{u \to \infty} \left| P \left( \sup_{t \in \mathcal{I}} X(t) \leq u \right) - P \left( \sup_{t \in \mathcal{I}_1} X(t) \leq u \right) \right| \\
\leq \limsup_{u \to \infty} n_{k+1}^{x} \ldots n_d^{x} P \left( \sup_{t \in \mathcal{I}_1} X(t) > u \right) \leq \zeta_1(\varepsilon),
\]

uniformly for \( x \in [A, B]^d \), with \( \zeta_1(\varepsilon) \to 0 \) as \( \varepsilon \to 0 \).

**Step 2.** Let \( a > 0 \) be fixed and let \( q_1, q_2, \ldots, q_d \) be defined as at the beginning of Section 5.3. Then we have

\[
\limsup_{u \to \infty} \left| P \left( \sup_{t \in \mathcal{I}} X(t) \leq u \right) - P \left( \sup_{j \in \mathcal{I}_1} X(jq) \leq u \right) \right| \\
\leq \limsup_{u \to \infty} n_{k+1}^{x} \ldots n_d^{x} P \left( \sup_{t \in \mathcal{I}_1} X(t) \leq u \right) - P \left( \sup_{j \in \mathcal{I}_1} X(jq) \leq u \right) \\
\leq \zeta_2(a),
\]

uniformly for \( x \in [A, B]^d \), with \( \zeta_2(a) \to 0 \) as \( a \to 0 \), due to the Bonferroni inequality and Lemma 5.1.

**Step 3.** Let \( T \) be a function defined as follows:

\[ T(u) := B \max \{ m_1(u), m_2(u), \ldots, m_d(u) \}. \]

Note that if \( T = T(u) \) is sufficiently large (and thus, if \( u \) is sufficiently large), then

\[
|v(j - j')q) - C_T(jq, j'q)| \leq \rho_T((j - j')q), \]

\[
|C_T(jq, j'q)| \leq \varrho_T((j - j')q),
\]
where the functions $\rho_T$ and $\varrho_T$ are defined by (3.7). Moreover, for all pairs of points $jq, jq' \in I$ satisfying $\|j - j'\|_\infty < \varepsilon$, provided that $\varepsilon$ is sufficiently small, we obtain
\[
|r(jq - j'q) - C_T(jq, jq')| = \frac{R \cdot (1 - r(jq - j'q))}{\log T},
\]
\[
\max \{ |r(jq - j'q)|, |C_T(jq, jq')| \} = r(jq - j'q) + \frac{R \cdot (1 - r(jq - j'q))}{\log T}.
\]
Combining the above properties, the normal comparison lemma ([3], Theorem 4.2.1) and Lemmas 3.2 and 3.3 in the same way as in [3], we conclude that
\[
\lim_{u \to \infty} |P \left( \sup_{jq \in I} X(jq) \leq u \right) - P \left( \sup_{jq \in I} Y_T(jq) \leq u \right)| = 0,
\]
uniformly for $x \in [A, B]^d$.

Step 4. By the definition of the random field $\{Y_T(t)\}$, we have
\[
P \left( \sup_{jq \in I} Y_T(jq) \leq u \right) = \int_{-\infty}^{\infty} P \left( \eta(jq) \leq u - \frac{R/\log T}{1 - (R/\log T)^{1/2}} ; jq \in I \right) d\Phi(z).
\]
Since $T = T(u) = \exp(\gamma u^2)c(u)$ for some function $c$ satisfying $\log c(u) = o(u^2)$, the condition
\[
u_z := u - \frac{(R/\log T)^{1/2} z}{1 - (R/\log T)^{1/2}}
\]
\[
= \left( u - \left( \frac{R}{\log T} \right)^{1/2} \right) \left( 1 + \frac{R}{2 \log T} + o \left( \frac{R}{\log T} \right) \right)
\]
\[
= u + \frac{1}{u} \left( 1 + \frac{R}{\gamma \log T} + R \frac{2}{\gamma \log T} \right) + o \left( \frac{1}{u} \right)
\]
holds for every $z \in \mathbb{R}$. Moreover, as $u \to \infty$,
\[
m(u) = \frac{u_z^{2/\alpha_1} u_z^{2/\alpha_2} \ldots u_z^{2/\alpha_d} \Psi(u_z)}{u^{2/\alpha_1} u^{2/\alpha_2} \ldots u^{2/\alpha_d} \Psi(u)} \to \exp \left( -\frac{R}{2 \gamma} + \sqrt{\frac{R}{\gamma}} \right),
\]
and thus
\[
(3.6) \quad n_{k+1}^x n_{k+2}^x \ldots n_d^x = \frac{x_{k+1} \ldots x_d}{M_1 \ldots M_k} \exp \left( -\frac{R}{2 \gamma} + \sqrt{\frac{R}{\gamma}} \right) \Psi(u_z)(1 + o(1)).
\]
Applying the dependence structure of $\{\eta(t)\}$ and stationarity of $\{X(t)\}$, we obtain
\[
P \left( \sup_{jq \in I} \eta(jq) \leq u_z \right) = P \left( \sup_{jq \in I} X(jq) \leq u_z \right) n_{k+1}^x n_{k+2}^x \ldots n_d^x + o(1).
\]
By Lemma 3.1, the definition of $m(u_z)$ and properties (2.1) and (3.6), we get

$$
P\left(\sup_{jq \in I_1} X(jq) \leq u_z\right) \leq \left(P\left(\sup_{t \in G_1} X(t) \leq u_z\right) + \prod_{i=1}^{k} M_i x_i \cdot \left(\vartheta(a) + 2\varepsilon + o(1)\right) \right)\frac{n_{k+1}^n u_z^{n_{k+2} \cdots n_d}}{m(u_z)}
$$

$$
= \left(1 - \prod_{i=1}^{k} M_i x_i \cdot \left(1 - \vartheta(a) - 2\varepsilon + o(1)\right) \right)\frac{n_{k+1}^n u_z^{n_{k+2} \cdots n_d}}{m(u_z)} \exp\left(- \frac{R^2}{2\gamma} + \sqrt{\frac{R}{\gamma}}\right) + o(1)
$$

$$
\overset{u \to \infty}{\longrightarrow} \exp\left(- (1 - \vartheta(a) - 2\varepsilon) x_1 x_2 \cdots x_d \exp\left(- \frac{R^2}{2\gamma} + \sqrt{\frac{R}{\gamma}}\right)\right);
$$

where $\vartheta(a) \to 0$ as $a \to 0$. Thus

$$
\lim_{u \to \infty} \int_{-\infty}^{\infty} P\left(\sup_{jq \in I_1} X(jq) \leq u_z\right) n_{k+1}^n u_z^{n_{k+2} \cdots n_d} d\Phi(z)
$$

$$
\leq \mathbb{E} \exp\left(- (1 - \vartheta(a)) x_1 x_2 \cdots x_d \exp\left(- \frac{R^2}{2\gamma} + \sqrt{\frac{R}{\gamma}}\right)\right).
$$

On the other hand, we have

$$
P\left(\sup_{jq \in I_1} X(jq) \leq u_z\right) \geq P\left(\sup_{t \in G_1} X(t) \leq u_z\right) n_{k+1}^n u_z^{n_{k+2} \cdots n_d}
$$

$$
\geq \left(1 - \prod_{i=1}^{k} M_i x_i \cdot \left(1 - \vartheta(a) - 2\varepsilon + o(1)\right) \right)\frac{n_{k+1}^n u_z^{n_{k+2} \cdots n_d}}{m(u_z)} \exp\left(- \frac{R^2}{2\gamma} + \sqrt{\frac{R}{\gamma}}\right) + o(1)
$$

$$
\overset{u \to \infty}{\longrightarrow} \exp\left(- x_1 x_2 \cdots x_d \exp\left(- \frac{R^2}{2\gamma} + \sqrt{\frac{R}{\gamma}}\right)\right),
$$

and thus

$$
\lim_{u \to \infty} \int_{-\infty}^{\infty} P\left(\sup_{jq \in I_1} X(jq) \leq u_z\right) n_{k+1}^n u_z^{n_{k+2} \cdots n_d} d\Phi(z)
$$

$$
\geq \mathbb{E} \exp\left(- x_1 x_2 \cdots x_d \exp\left(- \frac{R^2}{2\gamma} + \sqrt{\frac{R}{\gamma}}\right)\right).
$$
Summarizing, we obtain

\[ E \exp \left( -x_1 x_2 \ldots x_d \exp \left( -\frac{R}{2\gamma} + \sqrt{\frac{R}{\gamma}} W \right) \right) \]

\[ \leq \liminf_{u \to \infty} P \left( \sup_{j \in J} Y_T(j, q) \leq u \right) \leq \limsup_{u \to \infty} P \left( \sup_{j \in J} Y_T(j, q) \leq u \right) \]

\[ \leq E \exp \left( -(1 - \vartheta(a) - 2\varepsilon)x_1 x_2 \ldots x_d \exp \left( -\frac{R}{2\gamma} + \sqrt{\frac{R}{\gamma}} W \right) \right), \]

uniformly for \( x \in [A, B]^d \).

Step 5. From Steps 1–3 of the proof we know that

\[ \limsup_{u \to \infty} P \left( \sup_{t \in R^n(k, m)} X(t) \leq u \right) - P \left( \sup_{j \in J} Y_T(j, q) \leq u \right) \leq \zeta(\varepsilon) + \zeta(a), \]

uniformly for \( x \in [A, B]^d \), with \( \zeta(\varepsilon) \to 0 \) as \( \varepsilon \to 0 \) and \( \zeta(a) \to 0 \) as \( a \to 0 \). Combining it with the inequalities (3.7) and passing with \( \varepsilon \to 0 \) and \( a \to 0 \), we finish the first part of the proof.

(ii) Let \( J \subset \mathbb{R}^d \) be an arbitrary Jordan-measurable set with Lebesgue measure \( \lambda(J) > 0 \). We follow the argumentation from [2], Theorem 2 (ii). Observe that for every \( \varepsilon > 0 \) there exist some positive constants \( z_1, z_2, \ldots, z_d \) and some sets \( L_\varepsilon, U_\varepsilon \subset \mathbb{R}^d \) being finite sums of disjoint closed hyperrectangles with dimensions \( z_1 \times z_2 \times \ldots \times z_d \), such that \( L_\varepsilon \subset \mathcal{L}_\varepsilon \subset \mathcal{U}_\varepsilon \subset \mathcal{L}_\lambda(J) + \varepsilon > \lambda(J) > \lambda(U_\varepsilon) - \varepsilon \). Then, following nearly line-by-line the arguments given in the proof of part (i), we obtain

\[ P \left( \sup_{t \in \mathcal{L}_\varepsilon} X(t) \leq u \right) \to E \exp \left( -x_1 x_2 \ldots x_d \lambda(L_\varepsilon) \exp \left( -\frac{R}{2\gamma} + \sqrt{\frac{R}{\gamma}} W \right) \right) \]

and

\[ P \left( \sup_{t \in \mathcal{U}_\varepsilon} X(t) \leq u \right) \to E \exp \left( -x_1 x_2 \ldots x_d \lambda(U_\varepsilon) \exp \left( -\frac{R}{2\gamma} + \sqrt{\frac{R}{\gamma}} W \right) \right), \]

as \( u \to \infty \), uniformly for \( x \in [A, B]^d \). Since \( \varepsilon > 0 \) is arbitrarily small, it gives

\[ P \left( \sup_{t \in \mathcal{L}_\varepsilon} X(t) \leq u \right) \to E \exp \left( -x_1 x_2 \ldots x_d \lambda(J) \exp \left( -\frac{R}{2\gamma} + \sqrt{\frac{R}{\gamma}} W \right) \right), \]

as \( u \to \infty \), uniformly for \( x \in [A, B]^d \), which completes the proof.

Acknowledgments. The author would like to thank Krzysztof Dębicki and Michał Kukiela for comments and suggestions.
REFERENCES


Natalia Soja-Kukieła
Faculty of Mathematics and Computer Science
Nicolaus Copernicus University
ul. Chopina 12/18
87-100 Toruń, Poland
E-mail: natas@mat.umk.pl

Received on 11.10.2016;