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EXTREMES OF MULTIDIMENSIONAL STATIONARY GAUSSIAN RANDOM FIELDS

BY

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Abstract. Let $\{X(\mathbf{t}) : \mathbf{t} = (t_1, t_2, \ldots, t_d) \in [0,\infty)^d\}$ be a centered stationary Gaussian field with almost surely continuous sample paths, unit variance and correlation function r satisfying $r(\mathbf{t}) < 1$ for every $\mathbf{t} \neq \mathbf{0}$ and $r(\mathbf{t}) = 1 - \sum_{i=1}^d |t_i|^{\alpha_i} + o\big(\sum_{i=1}^d |t_i|^{\alpha_i}\big)$, as $\mathbf{t} \to \mathbf{0}$, with some $\alpha_1, \alpha_2, \ldots, \alpha_d \in (0,2]$. The main result of this contribution is the description of the asymptotic behaviour of $\mathrm{P}\left(\sup\{X(\mathbf{t}) : \mathbf{t} \in \mathcal{J}_{\mathbf{m}}^{\mathbf{x}}\} \leqslant u\right)$, as $u \to \infty$, for some Jordan-measurable sets $\mathcal{J}_{\mathbf{m}}^{\mathbf{x}}$ of volume proportional to $\mathrm{P}\left(\sup\{X(\mathbf{t}) : \mathbf{t} \in [0,1]^d\} > u\right)^{-1}\big(1 + o(1)\big)$.

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1. INTRODUCTION

In extreme value theory of Gaussian processes, we have the following seminal result (see Leadbetter et al. [4], Theorem 12.3.4; Arendarczyk and Dębicki [1], Lemma 4.3; Tan and Hashorva [7], Lemma 3.3) concerning the asymptotics of the distribution of supremum of a centered stationary Gaussian process $\{X(t) : t \geqslant 0\}$ with correlation function satisfying

$$r(t) = \mathrm{Cov}\,\big(X(t), X(0)\big) = 1 - |t|^\alpha + o(|t|^\alpha), \qquad \text{as } t \to 0, \tag{1.1}$$

for some $\alpha \in (0,2]$, over intervals with length proportional to

$$\mu(u) = \mathrm{P}\,\Big(\sup_{t\in[0,1]} X(t) > u\Big)^{-1}\big(1 + o(1)\big), \qquad \text{as } u \to \infty.$$

THEOREM 1.1. *Let $\{X(t) : t \geqslant 0\}$ be a zero-mean, unit-variance stationary Gaussian process with a.s. continuous sample paths and correlation function r satisfying* (1.1) *and $r(t)\log t \to R \in [0,\infty)$ as $t \to \infty$. Let $0 < A < B < \infty$.*

Then

$$\mathrm{P}\big(\sup_{t\in[0,x\mu(u)]} X(t)\leqslant u\big)\to \mathrm{E}\exp\Big(-x\exp\big(-R+\sqrt{2R}\mathcal{W}\big)\Big),$$

as $u\to\infty$, *uniformly for* $x\in[A,B]$, *with* $\mathcal{W}$ *an* $N(0,1)$ *random variable.*

It is natural to study a similar problem in the d-dimensional setting for arbitrary $d\in\mathbb{N}$. In this case one considers a centered stationary Gaussian process $\{X(t_1,t_2,\ldots,t_d): t_1,t_2,\ldots,t_d\geqslant 0\}$ with unit variance and correlation function $r(t_1,t_2,\ldots,t_d)=\mathrm{Cov}\big(X(t_1,t_2,\ldots,t_d),X(0,0,\ldots,0)\big)$ satisfying

$$r(t_1,t_2,\ldots,t_d)=1-\sum_{i=1}^{d}|t_i|^{\alpha_i}+o\big(\sum_{i=1}^{d}|t_i|^{\alpha_i}\big), \tag{1.2}$$

as $t_1,t_2,\ldots,t_d\to 0$, with $\alpha_1,\alpha_2,\ldots,\alpha_d\in(0,2]$. The subject of interest is then the distribution of supremum of the field $\{X(t_1,t_2,\ldots,t_d)\}$ over sets of volume proportional to

$$m(u)=\mathrm{P}\big(\sup_{(t_1,t_2,\ldots,t_d)\in[0,1]^d} X(t_1,t_2,\ldots,t_d)>u\big)^{-1}\big(1+o(1)\big).$$

In this paper we investigate suprema over sets of the form

$$\mathcal{J}_{\mathbf{m}}^{\mathbf{x}}:=\Big\{(t_1,t_2,\ldots,t_d)\in\mathbb{R}^d:\Big(\frac{t_1}{x_1m_1(u)},\frac{t_2}{x_2m_2(u)},\ldots,\frac{t_d}{x_dm_d(u)}\Big)\in\mathcal{J}\Big\},$$

where $\mathcal{J}\subset\mathbb{R}^d$ is a Jordan-measurable set with Lebesgue measure $\lambda(\mathcal{J})>0$, $\mathbf{x}=(x_1,x_2,\ldots,x_d)\in(0,\infty)^d$ and $\mathbf{m}=(m_1,m_2,\ldots,m_d)$ with $m_1,m_2,\ldots,m_d$ some positive functions satisfying $m_1(u)m_2(u)\ldots m_d(u)=m(u)$. Let us put $\mathcal{J}_{\mathbf{m}}:=\mathcal{J}_{\mathbf{m}}^{(1,\ldots,1)}$. One interesting case is $\mathcal{J}=[0,1]^d$ with $\mathcal{J}_{\mathbf{m}}^{\mathbf{x}}=\prod_{i=1}^{d}[0,x_im_i(u)]$.

In a recent paper Dębicki et al. [3] consider the case $d=2$. They assume that the functions m_1 and m_2 tend to infinity and satisfy

$$\frac{\log m_1(u)}{\log m_2(u)}\to 1\quad \text{as } u\to\infty. \tag{1.3}$$

The authors establish the following two-dimensional counterpart ([3], Theorem 2) of Theorem 1.1.

THEOREM 1.2. *Let* $\{X(t_1,t_2): t_1,t_2\geqslant 0\}$ *be a zero-mean, unit-variance stationary Gaussian field with a.s. continuous sample paths and correlation function* r *satisfying* (1.2) *and* $r(t_1,t_2)\log\sqrt{t_1^2+t_2^2}\to R\in[0,\infty)$ *as* $t_1^2+t_2^2\to\infty$. *Let* m_1 *and* m_2 *be positive functions such that* $m_1(u)m_2(u)=m(u)$ *and* (1.3) *hold. Then*:

(i) *for each* $0 < A < B < \infty$,

$$\mathrm{P}\Big(\sup_{(t_1,t_2)\in[0,x_1m_1]\times[0,x_2m_2]} X(t_1,t_2) \leqslant u\Big) \to \mathrm{E}\, e^{-x_1x_2\exp(-2R+2\sqrt{R}\mathcal{W})},$$

as $u \to \infty$, *uniformly for* $(x_1, x_2) \in [A, B]^2$, *with* $\mathcal{W}$ *an* $N(0,1)$ *random variable*;

(ii) *for every Jordan-measurable set* $\mathcal{J} \subset \mathbb{R}^2$ *with Lebesgue measure* $\lambda(\mathcal{J}) > 0$,

$$\mathrm{P}\Big(\sup_{(t_1,t_2)\in\mathcal{J}_{\mathbf{m}}} X(t_1,t_2) \leqslant u\Big) \to \mathrm{E}\, e^{-\lambda(\mathcal{J})\exp(-2R+2\sqrt{R}\mathcal{W})},$$

as $u \to \infty$, *with* $\mathcal{W}$ *an* $N(0,1)$ *random variable.*

Our goal is to derive a general limit theorem for the distribution of supremum of the field $\{X(t_1, t_2, \ldots, t_d)\}$ over sets $\mathcal{J}^{\mathbf{x}}_{\mathbf{m}}$, for arbitrary $d \in \mathbb{N}$ and for a wide class of families $\{m_1, m_2, \ldots, m_d\}$ of functions, uniform for $\mathbf{x} \in [A, B]^d$, for all $0 < A < B < \infty$. The main result is Theorem 3.1. In the paper we do not assume that every m_i tends to infinity like Dębicki et al. [3] do. We fully explain the case when all m_is are separated from zero (see Theorem 3.1 and Remark 3.1) and give some partial results in the case when some of m_is tend to zero (see Corollaries 3.4 and 3.5).

2. PRELIMINARIES

We consider $\mathbb{R}^d$ with coordinatewise order $\leqslant$, write $\mathbf{t} = (t_1, t_2, \ldots, t_d)$ for an element $\mathbf{t} \in \mathbb{R}^d$, put $\mathbf{0} := (0, 0, \ldots, 0)$ and $\mathbf{1} := (1, 1, \ldots, 1)$, and denote by $\|\cdot\|_\infty$ the sup-norm in $\mathbb{R}^d$, i.e., $\|\mathbf{t}\|_\infty = \max\{|t_1|, |t_2|, \ldots, |t_d|\}$ for any $\mathbf{t} \in \mathbb{R}^d$.

Let $\{X(\mathbf{t}) : \mathbf{t} \in [0,\infty)^d\}$ be a centered stationary Gaussian field with a.s. continuous sample paths, unit variance and correlation function

$$r(\mathbf{t}) = \mathrm{Cov}\big(X(\mathbf{t}), X(\mathbf{0})\big).$$

We will often assume that the correlation function satisfies:

A1: $r(\mathbf{t}) = 1 - \sum_{i=1}^d |t_i|^{\alpha_i} + o\big(\sum_{i=1}^d |t_i|^{\alpha_i}\big)$ as $t_1, t_2, \ldots, t_d \to 0$;

A2: $r(\mathbf{t}) < 1$ for $\mathbf{t} \neq \mathbf{0}$;

A3: $r(\mathbf{t}) \log \sqrt{t_1^2 + t_2^2 + \ldots + t_d^2} \to R$ as $t_1^2 + t_2^2 + \ldots + t_d^2 \to \infty$,

with some constants $\alpha_1, \alpha_2, \ldots, \alpha_d \in (0, 2]$ and $R \in [0, \infty)$.

The above conditions are analogous to the ones given in [4], [1], [7], [3].

Condition **A1** implies that the correlation function r is continuous. **A1** and **A2** give $|r(\mathbf{t})| < 1$ for $\mathbf{t} \neq \mathbf{0}$. Moreover, condition **A2** follows from **A1** and **A3**. Notice that we study both *weakly dependent* fields, satisfying **A3** with $R = 0$, and *strongly dependent* fields, satisfying **A3** with $R \in (0, \infty)$.

For every $\alpha \in (0, 2]$, we denote by $\mathcal{H}_\alpha$ the Pickands constant (see [5]), i.e.,

$$\mathcal{H}_\alpha := \lim_{T\to\infty} \frac{\mathrm{E}\exp\big(\max_{0\leqslant t\leqslant T} B_{\alpha/2}(t) - |t|^\alpha\big)}{T},$$

where $\{B_{\alpha/2}(t) : t \geqslant 0\}$ is a fractional Brownian motion with Hurst index $\alpha/2$.

Let $\mathcal{W}$ be a standard normal random variable and let $\Phi(u) := \mathrm{P}(\mathcal{W} \leqslant u)$, $\Psi(u) := \mathrm{P}(\mathcal{W} > u)$. We recall that

$$\Psi(u) = \frac{1}{\sqrt{2\pi}u} \exp\left(-\frac{u^2}{2}\right)\big(1 + o(1)\big) \quad \text{as } u \to \infty.$$

If the considered field $\{X(\mathbf{t})\}$ satisfies **A1** and **A2**, then, for arbitrary Jordan-measurable set $\mathcal{J} \subset \mathbb{R}^d$ with Lebesgue measure $\lambda(\mathcal{J}) > 0$, we have

$$\text{(2.1)} \qquad \mathrm{P}\big(\max_{\mathbf{t}\in\mathcal{J}} X(\mathbf{t}) > u\big) = \lambda(\mathcal{J}) \prod_{i=1}^{d} (\mathcal{H}_{\alpha_i} u^{2/\alpha_i}) \Psi(u)\big(1 + o(1)\big),$$

as $u \to \infty$, due to Piterbarg [6], Theorem 7.1. Thus

$$m(u) := \big(\prod_{i=1}^{d} (\mathcal{H}_{\alpha_i} u^{2/\alpha_i}) \Psi(u)\big)^{-1} = \mathrm{P}\big(\max_{\mathbf{t}\in[0,1]^d} X(\mathbf{t}) > u\big)^{-1}\big(1 + o(1)\big).$$

Let $m_1, m_2, \ldots, m_d$ be positive functions such that

$$m_1(u)m_2(u)\ldots m_d(u) = m(u)$$

and for some $k \in \{0, 1, \ldots, d-1\}$:

1. for every $i \in \{1, 2, \ldots, k\}$ there exists an $M_i \in (0, \infty)$ such that

$$m_i(u) \to M_i \quad \text{as } u \to \infty;$$

2. for every $i \in \{k+1, k+2, \ldots, d\}$ we have

$$m_i(u) \to \infty \text{ (as } u \to \infty) \quad \text{and} \quad m_i(u) = \exp(\gamma_i u^2) c_i(u),$$

for some constant $\gamma_i \in [0, 1/2]$ and positive function c_i with $\log c_i(u) = o(u^2)$. Then $\gamma_{k+1} + \gamma_{k+2} + \ldots + \gamma_d = 1/2$. We put $\gamma := \max_i \gamma_i$.

For arbitrary $\mathbf{x} \in (0, \infty)^d$, we define $\mathcal{R}^{\mathbf{x}} := [0, x_1] \times [0, x_2] \times \ldots \times [0, x_d]$ and $\mathcal{R}^{\mathbf{x}}_{\mathbf{m}} := [0, x_1 m_1(u)] \times [0, x_2 m_2(u)] \times \ldots \times [0, x_d m_d(u)]$ for each $u \in \mathbb{R}$. Note that $\mathcal{R}^{\mathbf{x}}_{\mathbf{m}} = \mathcal{J}^{\mathbf{x}}_{\mathbf{m}}$ for $\mathcal{J} = [0, 1]^d$.

3. RESULTS

Below, in Section 3.1, we present Theorem 3.1, which is the main result. Its proof is given in Sections 3.3 and 3.4. Some consequences of Theorem 3.1 can be found in Sections 3.1 and 3.2.

3.1. Main theorem. The following theorem describes the asymptotic behaviour of $\mathrm{P}\left(\sup\{X(\mathbf{t}) : \mathbf{t} \in \mathcal{J}_{\mathbf{m}}^{\mathbf{x}}\} \leqslant u\right)$, as $u \to \infty$, for Jordan-measurable sets $\mathcal{J}_{\mathbf{m}}^{\mathbf{x}}$ of volume proportional to $m(u)$.

THEOREM 3.1. *Let* $\{X(\mathbf{t}) : \mathbf{t} \in [0, \infty)^d\}$ *be a centered stationary Gaussian field with a.s. continuous sample paths, unit variance and correlation function* r *that satisfies* **A1** *and* **A3** *with some* $R \in [0, \infty)$. *Then, for every Jordan-measurable set* $\mathcal{J} \subset \mathbb{R}^d$ *with* $\lambda(\mathcal{J}) > 0$, *for each* $0 < A < B < \infty$,

$$\mathrm{P}\Big(\sup_{\mathbf{t}\in\mathcal{J}_{\mathbf{m}}^{\mathbf{x}}} X(\mathbf{t}) \leqslant u\Big) \to \mathrm{E}\exp\bigg(-x_1x_2\ldots x_d\lambda(\mathcal{J})\exp\bigg(-\frac{R}{2\gamma} + \sqrt{\frac{R}{\gamma}}\mathcal{W}\bigg)\bigg),$$

as $u \to \infty$, *uniformly for* $\mathbf{x} \in [A, B]^d$.

Applying the above theorem for $\mathcal{J} = [0, 1]^d$, we obtain the following result.

COROLLARY 3.1. *Let* $\{X(\mathbf{t})\}$ *satisfy the assumptions of Theorem* 3.1. *Then, for each* $0 < A < B < \infty$,

$$\mathrm{P}\Big(\sup_{\mathbf{t}\in\mathcal{R}_{\mathbf{m}}^{\mathbf{x}}} X(\mathbf{t}) \leqslant u\Big) \to \mathrm{E}\exp\bigg(-x_1x_2\ldots x_d\exp\bigg(-\frac{R}{2\gamma} + \sqrt{\frac{R}{\gamma}}\mathcal{W}\bigg)\bigg),$$

as $u \to \infty$, *uniformly for* $\mathbf{x} \in [A, B]^d$.

In the special case, when $k = 0$ and the functions $m_1, m_2, \ldots, m_d$ are chosen so that $\gamma_1 = \gamma_2 = \ldots = \gamma_d$ (and thus a d-dimensional analog of (1.3) holds), we have the following corollary. Note that for $d = 2$ it coincides with Theorem 1.2.

COROLLARY 3.2. *Let the assumptions of Theorem* 3.1 *be satisfied and let*

$$\frac{\log m_i(u)}{\log m_j(u)} \to 1 \text{ as } u \to \infty, \quad \text{for } i, j \in \{1, 2, \ldots, d\}. \tag{3.1}$$

Then, for every Jordan-measurable set $\mathcal{J} \subset \mathbb{R}^d$ *with* $\lambda(\mathcal{J}) > 0$,

$$\mathrm{P}\Big(\sup_{\mathbf{t}\in\mathcal{J}_{\mathbf{m}}^{\mathbf{x}}} X(\mathbf{t}) \leqslant u\Big) \to \mathrm{E}\exp\Big(-x_1x_2\ldots x_d\lambda(\mathcal{J})\exp\big(-dR + \sqrt{2dR}\mathcal{W}\big)\Big),$$

as $u \to \infty$, *uniformly for* $\mathbf{x} \in [A, B]^d$, *for each* $0 < A < B < \infty$.

3.2. Some consequences of the main theorem. Let the field $\{X(\mathbf{t})\}$ satisfy the assumptions of Theorem 3.1. In this section we ask for the asymptotic behaviour of the supremum of $\{X(\mathbf{t})\}$ over sets $\mathcal{J}_{\bar{\mathbf{m}}}^{\mathbf{x}}$, for $\mathcal{J} \subset \mathbb{R}^d$ a Jordan-measurable set with $\lambda(\mathcal{J}) > 0$, $\mathbf{x} \in (0, \infty)^d$, $\bar{\mathbf{m}} = (\bar{m}_1, \bar{m}_2, \ldots, \bar{m}_d)$ and $\bar{m}_1, \bar{m}_2, \ldots, \bar{m}_d$ some positive functions with $\bar{m}_1(u)\bar{m}_2(u)\ldots\bar{m}_d(u) = m(u)$. Note, we do not assume that $\bar{m}_1, \bar{m}_2, \ldots, \bar{m}_d$ fulfill all the conditions, which have to be satisfied by the functions $m_1, m_2, \ldots, m_d$ introduced in Section 2.

First, we consider the case when the functions $\bar{m}_1, \bar{m}_2, \ldots, \bar{m}_d$ are separated from zero, i.e., $\bar{m}_1(u), \bar{m}_2(u), \ldots, \bar{m}_d(u) > \varepsilon$ for some $\varepsilon > 0$. Then, it is easy to show that every sequence $\{u_n\}_{n\in\mathbb{N}}$ tending to infinity contains a subsequence $\{u_{n_j}\}_{j\in\mathbb{N}}$ such that for each $i \in \{1, 2, \ldots, d\}$ we have $\bar{m}_i(u_{n_j}) \to \bar{M}_i \in [\varepsilon, \infty)$, as $j \to \infty$, or, alternatively, $\bar{m}_i(u_{n_j}) = \exp(\bar{\gamma}_i u_{n_j}^2)\bar{c}_i(u_{n_j}) \to \infty$, as $j \to \infty$, for some constant $\bar{\gamma}_i \in [0, 1/2]$ and some function $\bar{c}_i$ with $\log \bar{c}_i(u_{n_j}) = o(u_{n_j}^2)$. We can apply Theorem 3.1 for such subsequences. This justifies the following remark.

REMARK 3.1. *Theorem* 3.1 *fully explains the case when* $\bar{m}_1, \bar{m}_2, \ldots, \bar{m}_d$ *are positive functions separated from zero*, *such that* $\bar{m}_1(u)\bar{m}_2(u)\ldots\bar{m}_d(u) = m(u)$. *It gives the asymptotics for convergent subsequences.*

Since for weakly dependent Gaussian fields the limit in Theorem 3.1 does not depend on γ, the above considerations entail a concise corollary.

COROLLARY 3.3. *Let* $\{X(\mathbf{t})\}$ *satisfy the assumptions of Theorem* 3.1 *with* $R = 0$ *and let* $\bar{m}_1, \bar{m}_2, \ldots, \bar{m}_d$ *be positive functions separated from zero*, *such that* $\bar{m}_1(u)\bar{m}_2(u)\ldots\bar{m}_d(u) = m(u)$. *Then*, *for each* $0 < A < B < \infty$,

$$\mathrm{P}\big(\sup_{\mathbf{t}\in\mathcal{J}_{\bar{\mathbf{m}}}^{\mathbf{x}}} X(\mathbf{t}) \leqslant u\big) \to \exp\big(-x_1x_2\ldots x_d\lambda(\mathcal{J})\big),$$

as $u \to \infty$, *uniformly for* $\mathbf{x} \in [A, B]^d$.

Next, we focus on the case when $\bar{m}_i$s are allowed to tend to zero. In general, such weakening of the assumptions enforces a different approach. However, basing on Theorem 3.1, we can give the limit theorems in two special opposite cases: when $\bar{m}_i \to 0$ sufficiently fast and when $\bar{m}_i \to 0$ sufficiently slow.

Suppose that for some $0 \leqslant j \leqslant k < d$:

0. for every $i \in \{1, 2, \ldots, j\}$ we have

$$\bar{m}_i(u) \to 0 \quad \text{as } u \to \infty;$$

1. for every $i \in \{j+1, j+2, \ldots, k\}$ there exists an $\bar{M}_i \in (0, \infty)$ such that

$$\bar{m}_i(u) \to \bar{M}_i \quad \text{as } u \to \infty;$$

2. for every $i \in \{k+1, k+2, \ldots, d\}$

$$\bar{m}_i(u) \to \infty \text{ (as } u \to \infty) \quad \text{and} \quad \bar{m}_i(u) = \exp(\bar{\gamma}_i u^2)\bar{c}_i(u)$$

hold for some constant $\bar{\gamma}_i \geqslant 0$ and function $\bar{c}_i$ such that $\log \bar{c}_i(u) = o(u^2)$. Then $\bar{\gamma}_{k+1} + \bar{\gamma}_{k+2} + \ldots + \bar{\gamma}_d \geqslant 1/2$. We put $\bar{\gamma} := \max_i \bar{\gamma}_i$.

Note that the above conditions are very similar to the conditions given in Section 2 for the functions $m_1, m_2, \ldots, m_d$. Under these assumptions (and some extra ones) we can prove the following results.

COROLLARY 3.4. *Assume that* $\bar{m}_1, \bar{m}_2, \ldots, \bar{m}_d$ *satisfy the above conditions and, moreover,*

$$\bar{m}_1(u) = \exp(-\kappa u^2) c(u)$$

for some constant $\kappa > 0$ *and function* c *satisfying* $\log c(u) = o(u^2)$. *Then,*

$$\mathrm{P}\big(\sup_{\mathbf{t}\in\mathcal{J}^{\mathbf{x}}_{\bar{\mathbf{m}}}} X(\mathbf{t}) \leqslant u\big) \to 0, \quad \text{as } u \to \infty,$$

uniformly for $\mathbf{x} \in [A, \infty)^d$, *for each* $A > 0$.

Proof. Let $\mathbf{x} \in (0,\infty)^d$. Since the set $\mathcal{J} \subset \mathbb{R}^d$ is Jordan-measurable and $\lambda(\mathcal{J}) > 0$, there exist $\mathbf{y} \in \mathbb{R}^d$ and $\mathbf{z} \in (0,\infty)^d$ such that $\mathbf{y} + \mathcal{R}^{\mathbf{z}} \subset \mathcal{J}$. Thus

$$\mathrm{P}\big(\sup_{\mathbf{t}\in\mathcal{J}^{\mathbf{x}}_{\bar{\mathbf{m}}}} X(\mathbf{t}) \leqslant u\big) \leqslant \mathrm{P}\big(\sup_{\mathbf{t}\in(\mathbf{y}+\mathcal{R}^{\mathbf{z}})^{\mathbf{x}}_{\bar{\mathbf{m}}}} X(\mathbf{t}) \leqslant u\big) = \mathrm{P}\big(\sup_{\mathbf{t}\in\mathcal{R}^{\mathbf{zx}}_{\bar{\mathbf{m}}}} X(\mathbf{t}) \leqslant u\big),$$

with $\mathbf{zx} := (z_1x_1, z_2x_2, \ldots, z_dx_d)$, where the last equality is a consequence of stationarity. Furthermore,

$$\mathrm{P}\big(\sup_{\mathbf{t}\in\mathcal{R}^{\mathbf{zx}}_{\bar{\mathbf{m}}}} X(\mathbf{t}) \leqslant u\big) \leqslant \mathrm{P}\big(\sup_{0\leqslant t_i\leqslant z_ix_i\bar{m}_i} X(0,\ldots,0,t_{k+1},t_{k+2},\ldots,t_d) \leqslant u\big).$$

We will show that the right-hand side of the above inequality tends to zero, using Theorem 3.1 for the field $\hat{X}(t_{k+1}, t_{k+2}, \ldots, t_d) := X(0,\ldots,0,t_{k+1},t_{k+2},\ldots,t_d)$, $t_{k+1}, t_{k+2}, \ldots, t_d \geqslant 0$, that satisfies $(d-k)$-dimensional conditions **A1** and **A3**.

Since $\kappa > 0$, we have $\sigma := \bar{\gamma}_{k+1} + \bar{\gamma}_{k+2} + \ldots + \bar{\gamma}_d > 1/2$. Hence

$$\frac{\bar{m}_{k+1}(u)\bar{m}_{k+2}(u)\ldots\bar{m}_d(u)}{\hat{m}(u)} \to \infty, \quad \text{as } u \to \infty,$$

where

$$\hat{m}(u) := \Big(\prod_{i=k+1}^{d} (\mathcal{H}_{\alpha_i} u^{2/\alpha_i})\Psi(u)\Big)^{-1}.$$

For every $i \in \{k+1, k+2, \ldots, d\}$, we put

$$\hat{m}_i(u) := \exp(\hat{\gamma}_i u^2)\hat{c}_i(u),$$

with $\hat{\gamma}_i := (2\sigma)^{-1}\bar{\gamma}_i$ and $\hat{c}_i(u) := \big(\hat{m}(u)\exp(-u^2/2)\big)^{1/(d-k)}$. Then $\hat{\gamma}_i \in [0, 1/2]$, $\log \hat{c}_i(u) = o(u^2)$ and $\hat{\gamma}_{k+1} + \hat{\gamma}_{k+2} + \ldots + \hat{\gamma}_d = 1/2$. Moreover, the functions $\hat{m}_i$ satisfy $\hat{m}_{k+1}(u)\hat{m}_{k+2}(u)\ldots\hat{m}_d(u) = \hat{m}(u)$ and we have

$$\frac{\bar{m}_i(u)}{\hat{m}_i(u)} \to \infty \quad \text{as } u \to \infty.$$

Let $C>0$ be arbitrary. Since $\tilde{m}_i(u)/\hat{m}_i(u)>C$ for all sufficiently large u, we obtain

$$\begin{aligned}\limsup_{u\to\infty} \mathrm{P}\big(\sup_{0\leqslant t_i\leqslant x_i\tilde{m}_i} X(0,\dots,0,t_{k+1},t_{k+2},\dots,t_d)\leqslant u\big)\\ =\limsup_{u\to\infty} \mathrm{P}\big(\sup_{0\leqslant t_i\leqslant x_i\tilde{m}_i} \hat{X}(t_{k+1},t_{k+2},\dots,t_d)\leqslant u\big)\\ \leqslant\limsup_{u\to\infty} \mathrm{P}\big(\sup_{0\leqslant t_i\leqslant Cx_i\hat{m}_i} \hat{X}(t_{k+1},t_{k+2},\dots,t_d)\leqslant u\big)\\ =\mathrm{E}\exp\bigg(-C^{d-k}x_{k+1}x_{k+2}\dots x_d\exp\bigg(-\frac{R}{2\hat{\gamma}}+\sqrt{\frac{R}{\hat{\gamma}}}\mathcal{W}\bigg)\bigg),\end{aligned}$$

with $\hat{\gamma}:=\max_i\hat{\gamma}_i$, due to Theorem 3.1. Since the right-hand side tends to zero as $C\to\infty$, the proof of pointwise convergence is complete. Uniform convergence simply follows from the monoticity of $\mathbf{x}\mapsto \mathrm{P}\big(\sup\{X(\mathbf{t})\leqslant u:\mathbf{t}\in\mathcal{J}_{\mathbf{m}}^{\mathbf{x}}\}\big)$. ■

COROLLARY 3.5. *Suppose that $m_1,m_2,\dots,m_d$ are positive functions such that $m_1(u)m_2(u)\dots m_d(u)=m(u)$ holds and, moreover, assume that*

$$m_i(u)=1,\quad \textit{for } i\leqslant j, \textit{ for some } j\in\{1,2,\dots,d-1\},$$

$$m_i(u)=\exp(\gamma_i u^2)c_i(u)\to M_i\in(0,\infty],\ \textit{as } u\to\infty,\quad \textit{for } i>j,$$

where $\gamma_i\in[0,1/2]$ and $\log c_i(u)=o(u^2)$. There exist some positive functions $\nu_1,\nu_2,\dots,\nu_j$ satisfying $\nu_i(u)\to 0$, such that for all $\bar{m}_1,\bar{m}_2,\dots,\bar{m}_d$ satisfying the conditions: $\nu_i(u)=o\big(\bar{m}_i(u)\big)$ for each $i\in\{1,2,\dots,j\}$, $\bar{m}_i(u)=m_i(u)$ for each $i\in\{j+1,j+2,\dots,d-1\}$ and $\bar{m}_d(u)=m_d(u)\cdot\prod_{i=1}^j\bar{m}_i(u)^{-1}$, we have

$$\mathrm{P}\big(\sup_{\mathbf{t}\in\mathcal{J}_{\bar{\mathbf{m}}}} X(\mathbf{t})\leqslant u\big)\to\mathrm{E}\exp\bigg(-\lambda(\mathcal{J})\exp\bigg(-\frac{R}{2\gamma}+\sqrt{\frac{R}{\gamma}}\mathcal{W}\bigg)\bigg),$$

as $u\to\infty$.

Proof. Let $\varepsilon_1,\varepsilon_2,\dots,\varepsilon_j>0$ and $\boldsymbol{\varepsilon}:=\big(\varepsilon_1,\varepsilon_2,\dots,\varepsilon_j,1,\dots,1,\prod_{i=1}^j\varepsilon_i^{-1}\big)$. By application of Theorem 3.1, we obtain

$$\mathrm{P}\big(\sup_{\mathbf{t}\in\mathcal{J}_{\mathbf{m}}^{\boldsymbol{\varepsilon}}} X(\mathbf{t})\leqslant u\big)\to\mathrm{E}\exp\bigg(-\lambda(\mathcal{J})\exp\bigg(-\frac{R}{2\gamma}+\sqrt{\frac{R}{\gamma}}\mathcal{W}\bigg)\bigg),$$

as $u\to\infty$, uniformly for $\boldsymbol{\varepsilon}\in[A,B]^d$, for all $0<A<B<\infty$. Note that the above limit does not depend on the choice of $\varepsilon_1,\varepsilon_2,\dots,\varepsilon_j$. It is not difficult to show that there exist some functions ν_i, $i\in\{1,2,\dots,j\}$, tending to zero, such that for positive functions $\varepsilon_i = \varepsilon_i(u)$, $i \in \{1, 2, \dots, j\}$, tending to zero, and for $\boldsymbol{\varepsilon}(u) := \big(\varepsilon_1(u), \varepsilon_2(u), \dots, \varepsilon_j(u), 1, \dots, 1, \prod_{i=1}^{j} \varepsilon_i^{-1}(u)\big)$, we have

$$\mathrm{P}\Big(\sup_{\mathbf{t}\in\mathcal{J}_{\mathbf{m}}^{\boldsymbol{\varepsilon}(u)}} X(\mathbf{t}) \leqslant u\Big) \to \mathrm{E}\exp\left(-\lambda(\mathcal{J})\exp\left(-\frac{R}{2\gamma} + \sqrt{\frac{R}{\gamma}}\mathcal{W}\right)\right),$$

whenever $\nu_i(u) = o\big(\varepsilon_i(u)\big)$. We shall put $\varepsilon_i(u) = \bar{m}_i(u)$ for $i \in \{1, 2, \dots, j\}$. ■

REMARK 3.2. *We do not know the form of the functions $\nu_1, \nu_2, \dots, \nu_j$ from Corollary* 3.5. *Our conjecture is that $\nu_i(u) = u^{-2/\alpha_i}$ for $i \in \{1, 2, \dots, j\}$.*

3.3. Lemmas. The lemmas formulated in this section are crucial in the proof of Theorem 3.1 (see Section 3.4). They are d-dimensional counterparts of known results: Lemma 3.1 generalizes Lemma 12.2.11 in [4] and Lemma 1 in [3]; Lemma 3.3 combines d-dimensional analogs of Lemma 12.3.1 in [4] (for weakly dependent fields) and Lemma 3.1 in [7] (for strongly dependent fields), it is a generalization of Lemma 2 in [3]. Since the argumentation for Lemmas 3.1 and 3.2 mimics the one given in [4] and expanded in [7], [2], the proofs are skipped. We present the proof of Lemma 3.3, which improves the lemma given by Dębicki et al. [3], [2] and enables us to establish far more general results than the ones in [3].

Let $a > 0$. Put $q_i = q_i(u) := au^{-2/\alpha_i}$ for $i \in \{1, 2, \dots, d\}$. Moreover, define $\mathbf{jq} = \mathbf{jq}(u) := \big(j_1 q_1(u), j_2 q_2(u), \dots, j_d q_d(u)\big)$ for $\mathbf{j} = (j_1, j_2, \dots, j_d) \in \mathbb{Z}^d$.

LEMMA 3.1. *Assume that conditions* **A1** *and* **A2** *hold. Then there exists a function ϑ satisfying $\vartheta(a) \to 0$, as $a \to 0$, such that for every $a > 0$ we have*

$$\mathrm{P}\Big(\sup_{\mathbf{jq}\in\mathbf{y}+\mathcal{R}^{\mathbf{x}}} X(\mathbf{jq}) \leqslant u\Big) - \mathrm{P}\Big(\sup_{\mathbf{t}\in\mathbf{y}+\mathcal{R}^{\mathbf{x}}} X(\mathbf{t}) \leqslant u\Big) \leqslant \frac{x_1 x_2 \dots x_d}{m}\vartheta(a) + o\left(\frac{1}{m}\right),$$

as $u \to \infty$, uniformly for $\mathbf{y} \in [0, \infty)^d$ and $\mathbf{x} \in [A, B]^d$, for all $0 < A < B < \infty$.

REMARK 3.3. *An explicit formula for ϑ from Lemma* 3.1 *can be found in* [2].

LEMMA 3.2. *Suppose that $T = T(u) \to \infty$ as $u \to \infty$. Then, providing that conditions* **A1** *and* **A2** *are fulfilled, there exists an $\varepsilon > 0$ such that for all $R \geqslant 0$*

$$\frac{m}{q_1 q_2 \cdots q_d} \sum_{\substack{\mathbf{jq}\in(-\varepsilon,\varepsilon)^d \\ \mathbf{jq}\neq(0,0,\dots,0)}} \left[\big(1-r(\mathbf{jq})\big)\frac{R}{\log T}\left(1-\left(r(\mathbf{jq})+\big(1-r(\mathbf{jq})\big)\frac{R}{\log T}\right)^2\right)^{-1/2} \right.$$
$$\left. \times \exp\left(-\frac{u^2}{1+r(\mathbf{jq})+\big(1-r(\mathbf{jq})\big)R/\log T}\right)\right] \to 0,$$

as $u \to \infty$.

Let $R \geqslant 0$ be fixed. The last lemma concerns functions ρ_T and ϱ_T defined for an arbitrary $T > 1$ and for $\mathbf{t} \in \mathbb{R}^d$ as follows:

$$\rho_T(\mathbf{t}) := \begin{cases} 1, & \max\{|t_{k+1}|, |t_{k+2}|, \ldots, |t_d|\} < 1, \\ \left|r(\mathbf{t}) - \frac{R}{\log T}\right|, & \text{otherwise;} \end{cases}$$

(3.2)

$$\varrho_T(\mathbf{t}) := \begin{cases} |r(\mathbf{t})| + \big(1 - r(\mathbf{t})\big)\frac{R}{\log T}, & \max\{|t_{k+1}|, |t_{k+2}|, \ldots, |t_d|\} < 1, \\ \frac{R}{\log T}, & \text{otherwise.} \end{cases}$$

LEMMA 3.3. *Assume that $T_i = T_i(u) \sim \tau_i m_i(u)$, as $u \to \infty$, for some $\tau_i > 0$ and every $i \in \{1, 2, \ldots, d\}$. Let $\varepsilon > 0$. Then, providing that conditions* **A1** *and* **A3** *with $R \in [0, \infty)$ are fulfilled,*

$$\frac{T_1 T_2 \ldots T_d}{q_1 q_2 \ldots q_d} \sum_{\substack{\mathbf{jq} \in \prod_{i=1}^d [-T_i, T_i] \\ \mathbf{jq} \notin (-\varepsilon, \varepsilon)^d}} \rho_T(\mathbf{jq}) \exp\left(-\frac{u^2}{1 + \max\{|r(\mathbf{jq})|, \varrho_T(\mathbf{jq})\}}\right) \to 0,$$

as $u \to \infty$, with $T := \max\{T_1, T_2, \ldots, T_d\}$.

P r o o f. We present the proof in the case $d = 2$. The argumentation for other dimensions is analogous. We follow the reasoning from Lemma 2 in [2], making modifications and skipping some details, which can be found in [2].

Since $T_1(u) T_2(u) \sim \tau_1 \tau_2 m(u)$, as $u \to \infty$, we get

$$u^2 = 2 \log(T_1 T_2) + \left(\frac{2}{\alpha_1} + \frac{2}{\alpha_2} - 1\right) \log\log(T_1 T_2) + O(1). \tag{3.3}$$

It is not difficult to see that there exists a constant $\delta \in (0, 1)$ such that for all sufficiently large L

$$\sup_{\varepsilon \leqslant \|\mathbf{t}\|_\infty \leqslant L} \max\{|r(\mathbf{t})|, \varrho_L(\mathbf{t})\} < \delta.$$

Denote by β a constant satisfying $0 < \beta < (1 - \delta)/(1 + \delta)$ and divide the set $\mathcal{Q} := [-T_1, T_1] \times [-T_2, T_2] - (-\varepsilon, \varepsilon)^2$ into two subsets:

$$\mathcal{S}^* := \{\mathbf{t} \in \mathcal{Q} : |t_1| \leqslant m(u)^{\beta/2}, |t_2| \leqslant m(u)^{\beta/2}\},$$
$$\mathcal{S} := \mathcal{Q} - \mathcal{S}^*.$$

Observe that the shape of the set $\mathcal{S}^*$ of volume $m(u)^\beta\big(1 + o(1)\big)$ does not depend on the choice of m_1 and m_2.

Following line-by-line the arguments from [2], thanks to the proper choice of β, we obtain

$$\frac{T_1T_2}{q_1q_2}\sum_{\mathbf{jq}\in\mathcal{S}^*}\rho_T(\mathbf{jq})\exp\left(-\frac{u^2}{1+\max\{|r(\mathbf{jq})|,\varrho_T(\mathbf{jq})\}}\right)\to 0, \tag{3.4}$$

as $u\to\infty$.

To complete the proof, it suffices to show that

$$\frac{T_1T_2}{q_1q_2}\sum_{\mathbf{jq}\in\mathcal{S}}\rho_T(\mathbf{jq})\exp\left(-\frac{u^2}{1+\max\{|r(\mathbf{jq})|,\varrho_T(\mathbf{jq})\}}\right)\to 0, \tag{3.5}$$

as $u\to\infty$. By an argument from [2] and the fact that $m(u)^{\beta/2}\to\infty$, we get

$$\max\{|r(\mathbf{jq})|,\varrho_T(\mathbf{jq})\}\leqslant\frac{C}{\log m(u)^{\beta/2}},$$

for sufficiently large u, some constant $C>0$ and all points $\mathbf{jq}\in\mathcal{Q}$ satisfying $\|\mathbf{jq}\|_\infty\geqslant m(u)^{\beta/2}$. Hence we have

$$\begin{aligned}\frac{T_1T_2}{q_1q_2}&\sum_{\mathbf{jq}\in\mathcal{S}}\rho_T(\mathbf{jq})\exp\left(-\frac{u^2}{1+\max\{|r(\mathbf{jq})|,\varrho_T(\mathbf{jq})\}}\right)\\ &\leqslant 4\frac{T_1^2T_2^2}{q_1^2q_2^2}\exp\left(-u^2\left(1-\frac{C}{\log m^{\beta/2}}\right)\right)\frac{1}{\log m^{\beta/2}}\\ &\qquad\times\frac{q_1q_2\log m^{\beta/2}}{T_1T_2}\sum_{\mathbf{jq}\in\mathcal{S}}\left|r(\mathbf{jq})-\frac{R}{\log T}\right|\\ &=:I_1(u)\times I_2(u).\end{aligned}$$

Applying the equality (3.3), the definition of the functions q_1 and q_2 and the convergence $\log\big(T_1(u)T_2(u)\big)/\log m(u)^{\beta/2}\to 2/\beta$, as $u\to\infty$,we conclude that I_1 is bounded. Our argumentation is analogous to the one given in [2]. The strong condition (1.3) turns out not to be necessary.

In the next step we prove that $I_2(u)\to 0$ as $u\to\infty$. Observe that we have

$$\begin{aligned}I_2(u)&\leqslant\frac{q_1q_2}{T_1T_2}\sum_{\mathbf{jq}\in\mathcal{S}}\left|r(\mathbf{jq})\log\sqrt{(j_1q_1)^2+(j_2q_2)^2}-R\right|\big(1+o(1)\big)\\ &\quad+\beta R\frac{q_1q_2}{T_1T_2}\sum_{\mathbf{jq}\in\mathcal{S}}\left|1-\frac{\log T}{\log\sqrt{(j_1q_1)^2+(j_2q_2)^2}}\right|\big(1+o(1)\big)\\ &=:J_1(u)+J_2(u).\end{aligned}$$

We need to show that both J_1 and J_2 tend to zero. Note that $J_1(u) \to 0$ as $u \to \infty$, due to **A3**. Additionally,

$$J_2(u) \leqslant \frac{2R}{\log m} \frac{q_1 q_2}{T_1 T_2} \sum_{\mathbf{jq} \in \mathcal{S}} \left| \log \left(\frac{\sqrt{(j_1 q_1)^2 + (j_2 q_2)^2}}{T} \right) \right|,$$

and hence

$$J_2(u) = \frac{2R}{\log m} \cdot O\Big(\int_0^1 \int_0^1 \big| \log \big(\sqrt{x^2 + y^2} \big) \big| dx dy + \int_0^1 \big| \log |x| \big| dx \Big).$$

Thus (3.5) holds. The combination of (3.4) and (3.5) completes the proof. ■

3.4. Proof of Theorem 3.1. To establish the main result, we develop the ideas given in [4], [1], [7], [3]. The following proof of Theorem 3.1 combines the method of proof of Theorem 1.2 for $d = 2$ and $\gamma_1 = \gamma_2 = 1/4$ (see [3], Theorem 2), the lemmas from Section 3.3 and some new observations.

The proof consists of two parts. In (i), we present a complete argumentation for the special case $\mathcal{J} = [0,1]^d$. In (ii), we explain how to apply the first part of the proof to obtain the limit theorem for arbitrary $\mathcal{J}$.

(i) Let us consider $\mathcal{J} = [0,1]^d$. Then $\mathcal{J}_{\mathbf{m}}^{\mathbf{x}} = \mathcal{R}_{\mathbf{m}}^{\mathbf{x}}$ for $\mathbf{x} \in (0,\infty)^d$. Let $\{X^{\mathbf{k}}(\mathbf{t})\}$, for $\mathbf{k} \in \mathbb{N}^{d-k}$, be independent copies of $\{X(\mathbf{t})\}$ and let

$$\eta(\mathbf{t}) := X^{\mathbf{k}(\mathbf{t})}(\mathbf{t}) \quad \text{for } \mathbf{t} \in [0,\infty)^d,$$

with $\mathbf{k}(\mathbf{t}) = (\lfloor t_{k+1} \rfloor + 1, \lfloor t_{k+2} \rfloor + 1, \ldots, \lfloor t_d \rfloor + 1)$. For any $T > 0$, we define a Gaussian random field $\{Y_T(\mathbf{t}) : \mathbf{t} \in [0,T]^d\}$ by

$$Y_T(\mathbf{t}) := \left(1 - \frac{R}{\log T}\right)^{1/2} \eta(\mathbf{t}) + \left(\frac{R}{\log T}\right)^{1/2} \mathcal{W},$$

where $\mathcal{W}$ denotes an $N(0,1)$ random variable independent of $\{\eta(\mathbf{t})\}$. Then the covariance $C_T(\mathbf{t}, \mathbf{t} + \mathbf{s}) := \mathrm{Cov}\big(Y_T(\mathbf{t}), Y_T(\mathbf{t} + \mathbf{s})\big)$ equals

$$C_T(\mathbf{t}, \mathbf{t} + \mathbf{s}) = \begin{cases} r(\mathbf{s}) + \big(1 - r(\mathbf{s})\big) \frac{R}{\log T} & \text{if } \lfloor s_i + t_i \rfloor = \lfloor t_i \rfloor \text{ for } k < i \leqslant d, \\ \frac{R}{\log T} & \text{otherwise.} \end{cases}$$

For $\mathbf{x} \in (0,\infty)^d$ we define $\mathbf{n}(\mathbf{x}, \mathbf{m}) := (n_1^{\mathbf{x}}, n_2^{\mathbf{x}}, \ldots, n_d^{\mathbf{x}})$ with $n_i^{\mathbf{x}} := x_i M_i$ for $i \in \{1, 2, \ldots, k\}$ and $n_i^{\mathbf{x}} = n_i^{\mathbf{x}}(u) := \lfloor x_i m_i(u) \rfloor$ for $i \in \{k+1, k+2, \ldots, d\}$. Since

$$\mathrm{P}\big(\sup_{\mathbf{t} \in \mathcal{R}_{\mathbf{m}}^{\mathbf{x}}} X(\mathbf{t}) \leqslant u \big) - \mathrm{P}\big(\sup_{\mathbf{t} \in \mathcal{R}^{\mathbf{n}(\mathbf{x},\mathbf{m})}} X(\mathbf{t}) \leqslant u \big) = o(1), \quad \text{as } u \to \infty,$$

we may focus on the asymptotics of the right-hand side of the above equality.

S t e p 1. Let $\varepsilon > 0$ be fixed. We divide the set $\mathcal{R}^{\mathbf{n}(\mathbf{x},\mathbf{m})}$ into $n_{k+1}^{\mathbf{x}} n_{k+2}^{\mathbf{x}} \dots n_d^{\mathbf{x}}$ boxes

$$\mathcal{G}_{\mathbf{l}} := \prod_{i=1}^{k} [0, x_i M_i] \times \prod_{i=k+1}^{d} [l_i - 1, l_i],$$

indexed by $\mathbf{l} = (l_{k+1}, l_{k+2}, \dots, l_d) \in \mathbb{N}^{d-k}$ such that $1 \leqslant l_i \leqslant n_i^{\mathbf{x}}$. Next we split each box $\mathcal{G}_{\mathbf{l}}$ into two subsets $\mathcal{I}_{\mathbf{l}}$ and $\mathcal{I}_{\mathbf{l}}^*$ as follows:

$$\begin{aligned} \mathcal{I}_{\mathbf{l}} &:= \prod_{i=1}^{k} [0, x_i M_i] \times \prod_{i=k+1}^{d} [(l_i - 1) + \varepsilon, l_i], \\ \mathcal{I}_{\mathbf{l}}^* &:= \mathcal{G}_{\mathbf{l}} - \mathcal{I}_{\mathbf{l}}. \end{aligned}$$

To simplify the notation, we will write

$$\mathcal{I} := \bigcup \{ \mathcal{I}_{\mathbf{l}} : \mathbf{1} \leqslant \mathbf{l} \leqslant (n_{k+1}^{\mathbf{x}}, n_{k+2}^{\mathbf{x}}, \dots, n_d^{\mathbf{x}}) \}.$$

Applying the Bonferroni inequality, stationarity and the asymptotics (2.1), we get

$$\begin{aligned} \limsup_{u \to \infty} \Big| \mathrm{P}\big(\sup_{\mathbf{t} \in \mathcal{R}^{\mathbf{n}(\mathbf{x},\mathbf{m})}} X(\mathbf{t}) \leqslant u \big) - \mathrm{P}\big(\sup_{\mathbf{t} \in \mathcal{I}} X(\mathbf{t}) \leqslant u \big) \Big| \\ \leqslant \limsup_{u \to \infty} n_{k+1}^{\mathbf{x}} n_{k+2}^{\mathbf{x}} \dots n_d^{\mathbf{x}} \, \mathrm{P}\big(\sup_{\mathbf{t} \in \mathcal{I}_{\mathbf{1}}^*} X(\mathbf{t}) > u \big) \leqslant \zeta_1(\varepsilon), \end{aligned}$$

uniformly for $\mathbf{x} \in [A, B]^d$, with $\zeta_1(\varepsilon) \to 0$ as $\varepsilon \to 0$.

S t e p 2. Let $a > 0$ be fixed and let $q_1, q_2, \dots, q_d$ be defined as at the beginning of Section 3.3. Then we have

$$\begin{aligned} &\limsup_{u \to \infty} \Big| \mathrm{P}\big(\sup_{\mathbf{t} \in \mathcal{I}} X(\mathbf{t}) \leqslant u \big) - \mathrm{P}\big(\sup_{\mathbf{j}\mathbf{q} \in \mathcal{I}} X(\mathbf{j}\mathbf{q}) \leqslant u \big) \Big| \\ &\leqslant \limsup_{u \to \infty} n_{k+1}^{\mathbf{x}} n_{k+2}^{\mathbf{x}} \dots n_d^{\mathbf{x}} \Big| \mathrm{P}\big(\sup_{\mathbf{t} \in \mathcal{I}_{\mathbf{1}}} X(\mathbf{t}) \leqslant u \big) - \mathrm{P}\big(\sup_{\mathbf{j}\mathbf{q} \in \mathcal{I}_{\mathbf{1}}} X(\mathbf{j}\mathbf{q}) \leqslant u \big) \Big| \\ &\leqslant \zeta_2(a), \end{aligned}$$

uniformly for $\mathbf{x} \in [A, B]^d$, with $\zeta_2(a) \to 0$ as $a \to 0$, due to the Bonferroni inequality and Lemma 3.1.

S t e p 3. Let T be a function defined as follows:

$$T(u) := B \max\{ m_1(u), m_2(u), \dots, m_d(u) \}.$$

Note that if $T = T(u)$ is sufficiently large (and thus, if u is sufficiently large), then

$$\begin{aligned} \big| r\big((\mathbf{j} - \mathbf{j}')\mathbf{q} \big) - C_T(\mathbf{j}\mathbf{q}, \mathbf{j}'\mathbf{q}) \big| &\leqslant \rho_T\big((\mathbf{j} - \mathbf{j}')\mathbf{q} \big), \\ |C_T(\mathbf{j}\mathbf{q}, \mathbf{j}'\mathbf{q})| &\leqslant \varrho_T\big((\mathbf{j} - \mathbf{j}')\mathbf{q} \big), \end{aligned}$$

where the functions ρ_T and ϱ_T are defined by (3.2). Moreover, for all pairs of points $\mathbf{jq}, \mathbf{j}'\mathbf{q} \in \mathcal{I}$ satisfying $\|\mathbf{j}-\mathbf{j}'\|_\infty < \varepsilon$, provided that ε is sufficiently small, we obtain

$$\big|r\big((\mathbf{j}-\mathbf{j}')\mathbf{q}\big) - C_T(\mathbf{jq},\mathbf{j}'\mathbf{q})\big| = \frac{R\cdot\big(1-r\big((\mathbf{j}-\mathbf{j}')\mathbf{q}\big)\big)}{\log T},$$

$$\max\big\{\big|r\big((\mathbf{j}-\mathbf{j}')\mathbf{q}\big)\big|, |C_T(\mathbf{jq},\mathbf{j}'\mathbf{q})|\big\} = r\big((\mathbf{j}-\mathbf{j}')\mathbf{q}\big) + \frac{R\cdot\big(1-r\big((\mathbf{j}-\mathbf{j}')\mathbf{q}\big)\big)}{\log T}.$$

Combining the above properties, the normal comparison lemma ([4], Theorem 4.2.1) and Lemmas 3.2 and 3.3 in the same way as in [3], we conclude that

$$\lim_{u\to\infty}\big|\,\mathrm{P}\big(\sup_{\mathbf{jq}\in\mathcal{I}} X(\mathbf{jq}) \leqslant u\big) - \mathrm{P}\big(\sup_{\mathbf{jq}\in\mathcal{I}} Y_T(\mathbf{jq}) \leqslant u\big)\big| = 0,$$

uniformly for $\mathbf{x} \in [A,B]^d$.

S t e p 4. By the definition of the random field $\{Y_T(\mathbf{t})\}$, we have

$$\mathrm{P}\big(\sup_{\mathbf{jq}\in\mathcal{I}} Y_T(\mathbf{jq}) \leqslant u\big) = \int_{-\infty}^{\infty} \mathrm{P}\left(\eta(\mathbf{jq}) \leqslant \frac{u-(R/\log T)^{1/2}z}{(1-R/\log T)^{1/2}}\,;\, \mathbf{jq}\in\mathcal{I}\right) d\Phi(z).$$

Since $T = T(u) = \exp(\gamma u^2)c(u)$ for some function c satisfying $\log c(u) = o(u^2)$, the condition

$$\begin{aligned} u_z &:= \frac{u-(R/\log T)^{1/2}z}{(1-R/\log T)^{1/2}} \\ &= \left(u - \left(\frac{R}{\log T}\right)^{1/2} z\right)\left(1+\frac{R}{2\log T} + o\left(\frac{R}{\log T}\right)\right) \\ &= u + \frac{1}{u}\left(-\sqrt{\frac{R}{\gamma}}z + \frac{R}{2\gamma}\right) + o\left(\frac{1}{u}\right) \end{aligned}$$

holds for every $z \in \mathbb{R}$. Moreover, as $u \to \infty$,

$$\frac{m(u)}{m(u_z)} = \frac{u_z^{2/\alpha_1}u_z^{2/\alpha_2}\ldots u_z^{2/\alpha_d}\Psi(u_z)}{u^{2/\alpha_1}u^{2/\alpha_2}\ldots u^{2/\alpha_d}\Psi(u)} \to \exp\left(-\frac{R}{2\gamma}+\sqrt{\frac{R}{\gamma}}z\right),$$

and thus

$$n^{\mathbf{x}}_{k+1}n^{\mathbf{x}}_{k+2}\ldots n^{\mathbf{x}}_{d} = \frac{x_{k+1}\ldots x_d}{M_1\ldots M_k}\exp\left(-\frac{R}{2\gamma}+\sqrt{\frac{R}{\gamma}}z\right) m(u_z)\big(1+o(1)\big). \tag{3.6}$$

Applying the dependence structure of $\{\eta(\mathbf{t})\}$ and stationarity of $\{X(\mathbf{t})\}$, we obtain

$$\mathrm{P}\big(\sup_{\mathbf{jq}\in\mathcal{I}} \eta(\mathbf{jq}) \leqslant u_z\big) = \mathrm{P}\big(\sup_{\mathbf{jq}\in\mathcal{I}_1} X(\mathbf{jq}) \leqslant u_z\big)^{n^{\mathbf{x}}_{k+1}n^{\mathbf{x}}_{k+2}\cdots n^{\mathbf{x}}_{d}} + o(1).$$

By Lemma 3.1, the definition of $m(u_z)$ and properties (2.1) and (3.6), we get

$$\begin{aligned}
&\mathrm{P}\big(\sup_{\mathbf{jq}\in\mathcal{I}_1} X(\mathbf{jq}) \leqslant u_z\big)^{n^{\mathbf{x}}_{k+1}n^{\mathbf{x}}_{k+2}\cdots n^{\mathbf{x}}_d}\\
&\leqslant \bigg(\mathrm{P}\big(\sup_{\mathbf{t}\in\mathcal{G}_1} X(\mathbf{t}) \leqslant u_z\big) + \frac{\prod_{i=1}^k M_i x_i \cdot \big(\vartheta(a)+2\varepsilon+o(1)\big)}{m(u_z)}\bigg)^{n^{\mathbf{x}}_{k+1}n^{\mathbf{x}}_{k+2}\cdots n^{\mathbf{x}}_d}\\
&= \bigg(1-\frac{\prod_{i=1}^k M_i x_i \cdot \big(1-\vartheta(a)-2\varepsilon+o(1)\big)}{m(u_z)}\bigg)^{\frac{x_{k+1}\cdots x_d}{M_1\cdots M_k}\exp\left(-\frac{R}{2\gamma}+\sqrt{\frac{R}{\gamma}}z\right)m(u_z)} + o(1)\\
&\xrightarrow[u\to\infty]{} \exp\bigg(-\big(1-\vartheta(a)-2\varepsilon\big)x_1x_2\ldots x_d\exp\bigg(-\frac{R}{2\gamma}+\sqrt{\frac{R}{\gamma}}z\bigg)\bigg),
\end{aligned}$$

where $\vartheta(a)\to 0$ as $a\to 0$. Thus

$$\begin{aligned}
&\limsup_{u\to\infty}\int_{-\infty}^{\infty}\mathrm{P}\big(\sup_{\mathbf{jq}\in\mathcal{I}_1} X(\mathbf{jq}) \leqslant u_z\big)^{n^{\mathbf{x}}_{k+1}n^{\mathbf{x}}_{k+2}\cdots n^{\mathbf{x}}_d} d\Phi(z)\\
&\qquad\leqslant \mathrm{E}\exp\bigg(-\big(1-\vartheta(a)-2\varepsilon\big)x_1x_2\ldots x_d\exp\bigg(-\frac{R}{2\gamma}+\sqrt{\frac{R}{\gamma}}\mathcal{W}\bigg)\bigg).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\mathrm{P}\big(\sup_{\mathbf{jq}\in\mathcal{I}_1} X(\mathbf{jq}) \leqslant u_z\big)^{n^{\mathbf{x}}_{k+1}n^{\mathbf{x}}_{k+2}\cdots n^{\mathbf{x}}_d}\\
&\qquad\geqslant \mathrm{P}\big(\sup_{\mathbf{t}\in\mathcal{G}_1} X(\mathbf{t}) \leqslant u_z\big)^{n^{\mathbf{x}}_{k+1}n^{\mathbf{x}}_{k+2}\cdots n^{\mathbf{x}}_d}\\
&\qquad\geqslant \bigg(1-\frac{\prod_{i=1}^k M_i x_i}{m(u_z)}\bigg)^{\frac{x_{k+1}x_{k+2}\cdots x_d}{M_1M_2\cdots M_k}\exp\left(-\frac{R}{2\gamma}+\sqrt{\frac{R}{\gamma}}z\right)m(u_z)} + o(1)\\
&\qquad\xrightarrow[u\to\infty]{} \exp\bigg(-x_1x_2\ldots x_d\exp\bigg(-\frac{R}{2\gamma}+\sqrt{\frac{R}{\gamma}}z\bigg)\bigg),
\end{aligned}$$

and thus

$$\begin{aligned}
&\liminf_{u\to\infty}\int_{-\infty}^{\infty}\mathrm{P}\big(\sup_{\mathbf{jq}\in\mathcal{I}_1} X(\mathbf{jq}) \leqslant u_z\big)^{n^{\mathbf{x}}_{k+1}n^{\mathbf{x}}_{k+2}\cdots n^{\mathbf{x}}_d} d\Phi(z)\\
&\qquad\geqslant \mathrm{E}\exp\bigg(-x_1x_2\ldots x_d\exp\bigg(-\frac{R}{2\gamma}+\sqrt{\frac{R}{\gamma}}\mathcal{W}\bigg)\bigg).
\end{aligned}$$

Summarizing, we obtain

$$
\begin{aligned}
(3.7)\quad & \mathrm{E}\exp\left(-x_1x_2\dots x_d\exp\left(-\frac{R}{2\gamma}+\sqrt{\frac{R}{\gamma}}\mathcal{W}\right)\right) \\
&\leqslant \liminf_{u\to\infty}\mathrm{P}\left(\sup_{\mathbf{jq}\in\mathcal{I}}Y_T(\mathbf{jq})\leqslant u\right)\leqslant \limsup_{u\to\infty}\mathrm{P}\left(\sup_{\mathbf{jq}\in\mathcal{I}}Y_T(\mathbf{jq})\leqslant u\right) \\
&\leqslant \mathrm{E}\exp\left(-\big(1-\vartheta(a)-2\varepsilon\big)x_1x_2\dots x_d\exp\left(-\frac{R}{2\gamma}+\sqrt{\frac{R}{\gamma}}\mathcal{W}\right)\right),
\end{aligned}
$$

uniformly for $\mathbf{x}\in[A,B]^d$.

Step 5. From Steps 1–3 of the proof we know that

$$
\limsup_{u\to\infty}\left|\mathrm{P}\left(\sup_{\mathbf{t}\in\mathcal{R}^{\mathbf{n}(\mathbf{x},\mathbf{m})}}X(\mathbf{t})\leqslant u\right)-\mathrm{P}\left(\sup_{\mathbf{jq}\in\mathcal{I}}Y_T(\mathbf{jq})\leqslant u\right)\right|\leqslant\zeta_1(\varepsilon)+\zeta_2(a),
$$

uniformly for $\mathbf{x}\in[A,B]^d$, with $\zeta_1(\varepsilon)\to 0$ as $\varepsilon\to 0$ and $\zeta_2(a)\to 0$ as $a\to 0$. Combining it with the inequalities (3.7) and passing with $\varepsilon\to 0$ and $a\to 0$, we finish the first part of the proof.

(ii) Let $\mathcal{J}\subset\mathbb{R}^d$ be an arbitrary Jordan-measurable set with Lebesgue measure $\lambda(\mathcal{J})>0$. We follow the argumentation from [3], Theorem 2 (ii). Observe that for every $\varepsilon>0$ there exist some positive constants $z_1,z_2,\dots,z_d$ and some sets $\mathcal{L}_\varepsilon,\mathcal{U}_\varepsilon\subset\mathbb{R}^d$ being finite sums of disjoint closed hyperrectangles with dimensions $z_1\times z_2\times\dots\times z_d$, such that $\mathcal{L}_\varepsilon\subset\mathcal{J}\subset\mathcal{U}_\varepsilon$ and $\lambda(\mathcal{L}_\varepsilon)+\varepsilon>\lambda(\mathcal{J})>\lambda(\mathcal{U}_\varepsilon)-\varepsilon$. Then, following nearly line-by-line the arguments given in the proof of part (i), we obtain

$$
\mathrm{P}\left(\sup_{\mathbf{t}\in(\mathcal{L}_\varepsilon)^{\mathbf{x}}_{\mathbf{m}}}X(\mathbf{t})\leqslant u\right)\to\mathrm{E}\exp\left(-x_1x_2\dots x_d\lambda(\mathcal{L}_\varepsilon)\exp\left(-\frac{R}{2\gamma}+\sqrt{\frac{R}{\gamma}}\mathcal{W}\right)\right)
$$

and

$$
\mathrm{P}\left(\sup_{\mathbf{t}\in(\mathcal{U}_\varepsilon)^{\mathbf{x}}_{\mathbf{m}}}X(\mathbf{t})\leqslant u\right)\to\mathrm{E}\exp\left(-x_1x_2\dots x_d\lambda(\mathcal{U}_\varepsilon)\exp\left(-\frac{R}{2\gamma}+\sqrt{\frac{R}{\gamma}}\mathcal{W}\right)\right),
$$

as $u\to\infty$, uniformly for $\mathbf{x}\in[A,B]^d$. Since $\varepsilon>0$ is arbitrarily small, it gives

$$
\mathrm{P}\left(\sup_{\mathbf{t}\in\mathcal{J}^{\mathbf{x}}_{\mathbf{m}}}X(\mathbf{t})\leqslant u\right)\to\mathrm{E}\exp\left(-x_1x_2\dots x_d\lambda(\mathcal{J})\exp\left(-\frac{R}{2\gamma}+\sqrt{\frac{R}{\gamma}}\mathcal{W}\right)\right),
$$

as $u\to\infty$, uniformly for $\mathbf{x}\in[A,B]^d$, which completes the proof.

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