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EXTREMES OF CHI-SQUARE PROCESSES WITH TREND

BY

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Abstract. This paper studies the supremum of chi-square processes with trend over a threshold-dependent-time horizon. Under the assumptions that the chi-square process is generated from a centered self-similar Gaussian process and the trend function is modeled by a polynomial function, we obtain the exact tail asymptotics of the supremum of the chi-square process with trend. These results are of interest in applications in engineering, insurance, queueing and statistics, etc. Some possible extensions of our results are also discussed.

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1. INTRODUCTION

Let $\{Y(t), t \geqslant 0\}$ be a centered self-similar Gaussian process with almost surely (a.s.) continuous sample paths and index $H \in (0,1)$, i.e., $\mathrm{Var}\big(Y(t)\big) = t^{2H}$ and for any $a > 0$ and any $s, t \geqslant 0$

$$\mathrm{Cov}\big(Y(at), Y(as)\big) = a^{2H}\mathrm{Cov}\big(Y(t), Y(s)\big).$$

It has been shown that self-similar Gaussian processes such as fractional Brownian motion (fBm), subfractional Brownian motion and bifractional Brownian motion are quite useful in applications in engineering, telecommunication, insurance, queueing, finance, etc., see [5], [14], [17], [21], [28], [36] and the references therein.

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Let β, c be two positive constants. In this paper we are interested in the tail asymptotics of the supremum of a chi-square process with trend given by

$$\psi_T(u) = \mathbb{P}\Big(\sup_{t\in[0,T]} \big(\sum_{i=1}^n b_i^2 Y_i^2(t) - ct^\beta \big) > u \Big), \quad u \to \infty, \tag{1.1}$$

where $Y_i, i = 1, \ldots, n$, are independent copies of the centered self-similar Gaussian process Y, and $1 = b_1 = \ldots = b_k > b_{k+1} \geqslant b_{k+2} \geqslant \ldots \geqslant b_n > 0$. Here $T > 0$ can be a finite constant, infinity, and eventually we allow $T = T_u, u > 0$, to be a threshold-dependent positive deterministic function.

One motivation for considering (1.1) stems from its applications in engineering sciences, see [25] and the references therein. More precisely, let

$$\boldsymbol{X}(t) = \big(X_1(t), \ldots, X_n(t)\big), \quad t \geqslant 0,$$

be a vector Gaussian load process. Of interest is the probability of exit

$$\mathbb{P}\big(\boldsymbol{X}(t) \notin \boldsymbol{S}_u(t) \text{ for some } t \in [0, T]\big)$$

where the *time-dependent safety region* $\boldsymbol{S}_u(t), t \geqslant 0$, is defined by

$$\boldsymbol{S}_u(t) = \Big\{(x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=1}^n x_i^2 \leqslant h(t, u)\Big\}$$

with $h(t, u), t, u \geqslant 0$, some positive function. Various models for $\boldsymbol{X}$ and $h(t, u)$ (especially, $h(t, u) \equiv u$) have been discussed in the literature (e.g., [4], [2], [16], [19], [30], [31]) for the case where $T \in (0, \infty)$. In this framework, $\psi_T(u)$ corresponds to the model with $\boldsymbol{X} = (b_1Y_1, \ldots, b_nY_n)$ and $h(t, u) = u + ct^\beta$. As one of the new features of this contribution, we shall deal with different types of $T = T_u, u \geqslant 0$; see Section 4.

Another motivation for considering (1.1) stems from its applications in insurance. Specifically, the surplus process of an insurance company can be modeled by

$$R_u(t) = u + ct^\beta - \sum_{i=1}^n b_i^2 Y_i^2(t), \quad t \geqslant 0, \tag{1.2}$$

where u is the initial reserve, ct^β models the total premium received up to time t, and $\sum_{i=1}^n b_i^2 Y_i^2(t)$ represents the total amount of aggregated claims up to time t from n different types of risks. In this framework, $\psi_T(u)$ is called a *ruin probability* which is the most important measure of risk of the insurance company; see, e.g., [3], [34]. Note that the model in (1.2) is also related to the framework of fluid queue; see, e.g., [10].

Finally, we remark that the study of $\psi_T(u)$ also gives some insight into the study of some limiting test statistics. In [13], it is shown that a test statistic converges weakly to

$$\sup_{t\in(0,1)} \left(\frac{U(t)^2}{2t(1-t)} - C(t) - \upsilon D(t) \right), \tag{1.3}$$

where $\{U(t), t \in [0,1]\}$ is a standard Brownian bridge, and

$$C(t) = \ln\Big(1 - \ln\big(1-(2t-1)^2\big)\Big), \quad D(t) = \ln\big(1 + C(t)^2\big), \quad \upsilon > 1.$$

Apparently, the above process involved is a chi-square process with trend. Asymptotical results for the tail probability of (1.3) are very interesting from statistical point of view; see, e.g., [20]. See also [19], [23] and the references therein for recent applications of chi-type processes in statistics.

The outline of the rest of the paper is as follows. Section 2 is concerned about some preliminary results. In Theorem 2.1 we show the tail asymptotics of the supremum of a chi-square process generated from a non-stationary Gaussian process which extends some results in [30], [16]; Lemma 2.1 derives a Fernique-type inequality for certain Gaussian random fields. In Section 3 we concentrate on the asymptotics of (1.1) over an infinite-time horizon (i.e., $T = \infty$). Under a local stationary condition on the correlation of the self-similar process Y (see (3.5)), in Theorem 3.1 we derive the asymptotics of $\psi_\infty(u)$. Section 4 is devoted to the asymptotics of (1.1) over a threshold-dependent-time horizon (i.e., $T = T_u$ a positive deterministic function). As a corollary, we also obtain approximations of the conditional first passage time of the process defined in (1.2). Finally, in Section 5 possible extensions of our results are discussed. We show that general results can also be obtained for the model where Y_i's are independent but not necessarily identical and for the model with a more general correlation structure (for Y) than that in (3.5).

2. PRELIMINARIES

Let $\{X(t), t \geqslant 0\}$ be a centered non-stationary Gaussian process with a.s. continuous sample paths. In the following, unless otherwise stated, T is considered to be a positive finite constant. We impose the following typical assumptions on the Gaussian process X (see [31]):

ASSUMPTION I. The standard deviation function $\sigma_X(\cdot) := \sqrt{\mathrm{Var}\big(X(\cdot)\big)}$ of X attains its maximum (assumed to be one) over $[0,T]$ at the unique point $t = t_0 \in [0,T]$. Further, there exist some positive constants μ, a such that

$$\sigma_X(t) = 1 - a|t-t_0|^\mu\big(1+o(1)\big), \quad t \to t_0.$$

ASSUMPTION II. There exist some $\nu \in (0,2], d > 0$ such that

$$r_X(s,t) = \mathrm{Corr}\big(X(s), X(t)\big) = 1 - d|t-s|^\nu\big(1+o(1)\big), \quad s,t \to t_0.$$

ASSUMPTION III. There exist some positive constants G, γ and ρ such that

$$\mathbb{E}\Big(\big(X(t)-X(s)\big)^2\Big) \leqslant G|t-s|^\gamma$$

holds for all $s,t \in [t_0-\rho, t_0+\rho] \cap [0,T]$.

For such a centered non-stationary Gaussian process X, it is known that (see, e.g., [32], Theorem D.3 in [31] or Theorem 2.1 in [6])

$$\begin{aligned}(2.1)\quad &\mathbb{P}\big(\sup_{t\in[0,T]} X(t) > u\big) \\ &= \mathcal{M}_{\nu,\mu,d,a}\frac{1}{\sqrt{2\pi}}u^{(2/\nu-2/\mu)_+-1}\exp\left(-\frac{u^2}{2}\right)\big(1+o(1)\big), \quad u\to\infty,\end{aligned}$$

where $(x)_+ = \max(0,x)$, and, with $I_{(\cdot)}$ denoting the indicator function,

$$(2.2)\qquad \mathcal{M}_{\nu,\mu,d,a} = \begin{cases} d^{1/\nu}a^{-1/\mu}\Gamma(1/\mu+1)(1+I_{(t_0\notin\{0,T\})})\mathcal{H}_\nu & \text{if } \nu<\mu,\\ \mathcal{P}_\nu^{a/d} & \text{if } \nu=\mu,\\ 1 & \text{if } \nu>\mu. \end{cases}$$

Here $\mathcal{H}_\nu \in (0,\infty)$ is the *Pickands constant* defined by

$$\mathcal{H}_\nu = \lim_{S\to\infty}\frac{1}{S}\mathbb{E}\Big(\exp\Big(\sup_{t\in[0,S]}\big(\sqrt{2}B_\nu(t)-t^\nu\big)\Big)\Big)$$

with $\{B_\nu(t), t\in\mathbb{R}\}$ a standard fBm defined on $\mathbb{R}$ with Hurst index $\nu/2\in(0,1]$; and $\mathcal{P}_\nu^{a/d}\in(0,\infty)$ is the *Piterbarg constant* defined by

$$(2.3)\qquad \mathcal{P}_\nu^{a/d} = \widehat{\mathcal{P}}_\nu^{a/d}I_{(t_0\in(0,T))} + \widetilde{\mathcal{P}}_\nu^{a/d}I_{(t_0\in\{0,T\})} \in (0,\infty)$$

with

$$\widehat{\mathcal{P}}_\nu^\lambda = \lim_{S_1,S_2\to\infty}\mathcal{P}_\nu^\lambda[-S_1,S_2], \qquad \widetilde{\mathcal{P}}_\nu^\lambda = \lim_{S\to\infty}\mathcal{P}_\nu^\lambda[0,S] = \lim_{S\to\infty}\mathcal{P}_\nu^\lambda[-S,0],$$
$$\mathcal{P}_\nu^\lambda[-S_1,S_2] = \mathbb{E}\Big(\exp\Big(\sup_{t\in[-S_1,S_2]}\big(\sqrt{2}B_\nu(t)-(1+\lambda)|t|^\nu\big)\Big)\Big),$$

and $\lambda>0, \max(S_1,S_2)>0$.

We refer to [31], [5], [9], [12] for the properties and generalizations of the Pickands–Piterbarg and related constants.

Let $\{\chi^2_{n,\boldsymbol{b}}(t), t\geqslant 0\}$ be a chi-square process with n degrees of freedom defined by

$$(2.4)\qquad \chi^2_{n,\boldsymbol{b}}(t) \;=\; \sum_{i=1}^n b_i^2X_i^2(t), \quad t\geqslant 0,$$

where $b_i>0$, $1\leqslant i\leqslant n$, and $\{X_i(t), t\geqslant 0\}$, $1\leqslant i\leqslant n$, are independent copies of the centered Gaussian process X satisfying assumptions I–III. As an analogue of (2.1), Hashorva and Ji [16] derived the following tail asymptotics for $\chi^2_{n,\mathbf{1}}$:

$$(2.5)\qquad \mathbb{P}\big(\sup_{t\in[0,T]}\chi^2_{n,\mathbf{1}}(t)>u\big) = \mathcal{M}_{\nu,\mu,d,a}u^{(1/\nu-1/\mu)_+}\Upsilon_n(u)\big(1+o(1)\big), \quad u\to\infty,$$

where

$$\Upsilon_n(u) := \mathbb{P}\left(\chi^2_{n,\mathbf{1}}(0) > u\right) = \frac{2^{(2-n)/2}}{\Gamma(n/2)} u^{n/2-1} \exp\left(-\frac{u}{2}\right), \quad u \geqslant 0.$$

The result in (2.5) was derived by using a similar double-sum method to that applied in [30]. As shown in [30] and [16] the usage of the double-sum method for the chi-square process is usually technical, since we have to deal with the supremum of a Gaussian random field with variance function attaining its maximum on an infinite set; see also [8] for a recent result in this direction. Below, we present a general result on the tail asymptotics of $\chi^2_{n,\boldsymbol{b}}$ allowing for different b_i's. The next result may not be surprising (see [30], [16]), but it turns out that the proof is far from trivial. As we will see, the following result is crucial when dealing with the tail asymptotics of the supremum of the chi-square process with trend; two other extensions of Theorem 2.1 will be discussed in Section 5.

THEOREM 2.1. *Let $\{\chi^2_{n,\boldsymbol{b}}(t), t \geqslant 0\}$ be a chi-square process defined as above with generic X satisfying assumptions* I–III. *If $1 = b_1 = \ldots = b_k > b_{k+1} \geqslant b_{k+2} \geqslant \ldots \geqslant b_n > 0$, then, as $u \to \infty$,*

$$\mathbb{P}\big(\sup_{t\in[0,T]} \chi^2_{n,\boldsymbol{b}}(t) > u\big) = \prod_{i=k+1}^{n} (1-b_i^2)^{-1/2} \mathcal{M}_{\nu,\mu,d,a} u^{(1/\nu-1/\mu)_+} \Upsilon_k(u)\big(1+o(1)\big). \tag{2.6}$$

We conclude this section with a Fernique-type inequality, which will be used in the proof of our main result. The proof of it is quite similar to that of the classical Fernique's inequality (see, e.g., [24]). We refer to [27] for new developments on the Fernique-type inequality.

LEMMA 2.1. *Let $\{\xi(\boldsymbol{t}), \boldsymbol{t} \in [0,1]^n\}$ be a centered Gaussian process with a.s. continuous sample paths and* $\mathrm{Var}\big(\xi(0)\big) = \sigma^2 \geqslant 0$. *Suppose that*

$$\mathbb{E}\Big(\big(\xi(\boldsymbol{t}) - \xi(\boldsymbol{s})\big)^2\Big) \leqslant \mathbb{Q} \sum_{i=1}^{n} |t_i - s_i|^{\alpha_i} \tag{2.7}$$

holds for all $\boldsymbol{t}, \boldsymbol{s} \in [0,1]^n$, with some constants $\mathbb{Q} > 0, \alpha_i > 0, 1 \leqslant i \leqslant n$. Then, for all $x > 0$,

$$\mathbb{P}\big(\sup_{\boldsymbol{t}\in[0,1]^n} \xi(\boldsymbol{t}) > x\big) \leqslant 2^{n+1} \exp\left(-\frac{c^* x^2}{\mathbb{Q}}\right) + 2^{-1} \exp\left(-\frac{x^2}{8\sigma^2}\right),$$

where $c^* = \left(2n \sum_{p=0}^{\infty} \left((p+1)2^{-(p+1)\min_{1\leqslant i\leqslant n}\alpha_i + 1}\right)^{1/2}\right)^{-2}$, and if $\sigma^2 = 0$, then the second term on the right-hand side disappears.

3. INFINITE-TIME HORIZON

In this section we shall focus on the asymptotics of

$$\psi_\infty(u) = \mathbb{P}\big(\sup_{t\in[0,\infty)} \sum_{i=1}^{n} b_i^2 Y_i^2(t) - ct^\beta > u\big), \quad u \to \infty, \tag{3.1}$$

where Y_i's are the centered self-similar Gaussian processes as discussed in Section 1. Throughout the paper, for technical reasons we assume that $\beta > 2H$. As demonstrated in [17] and [18] it is useful to define, for $\beta > 2H$ and $c > 0$,

$$Z_i(t) = \frac{Y_i(t)}{\sqrt{1+ct^\beta}}, \quad t \geqslant 0, \quad 1 \leqslant i \leqslant n. \tag{3.2}$$

Indeed, by self-similarity of Y_i's, for any $u > 0$,

$$\psi_\infty(u) = \mathbb{P}\big(\sup_{t\geqslant 0} \sum_{i=1}^{n} b_i^2 Z_i^2(t) > u^{1-2H/\beta}\big). \tag{3.3}$$

Let $\sigma_Z(t) = \sqrt{\mathrm{Var}\big(Z_1(t)\big)}$. It is noted that $\sigma_Z(t)$ attains its maximum on $[0,\infty)$ at the unique point

$$t_0 = \left(\frac{2H}{c(\beta-2H)}\right)^{1/\beta}$$

and

$$\sigma_Z(t) = A^{-1/2}\left(1 - \frac{B}{4A}(t-t_0)^2\big(1+o(1)\big)\right), \quad t \to t_0, \tag{3.4}$$

with

$$A = \left(\frac{2H}{c(\beta-2H)}\right)^{-2H/\beta} \frac{\beta}{\beta-2H}, \quad B = 2\left(\frac{2H}{c(\beta-2H)}\right)^{-2(H+1)/\beta} H\beta.$$

In the rest of the paper we assume *local stationarity* for the standardized Gaussian process $\overline{Y}(t) := Y(t)/t^H, t > 0$, in a neighborhood of the point t_0, i.e.,

$$\lim_{s\to t_0, t\to t_0} \frac{\mathbb{E}\big(\big(\overline{Y}(s) - \overline{Y}(t)\big)^2\big)}{|s-t|^\alpha} = Q > 0 \tag{3.5}$$

holds for some $\alpha \in (0,2)$. Condition (3.5) is common in the literature; most of the known self-similar Gaussian processes (such as fBm, sub-fBm, and bi-fBm) satisfy (3.5), see, e.g., [16]. Note that the local stationarity at t_0 and the self-similarity of the process Y imply the local stationarity at any point $r \in (0,\infty)$.

Next we present our main result concerning the tail asymptotics of the supremum of the self-similar chi-square process with trend over an infinite-time horizon.

THEOREM 3.1. *Suppose that the generic process* $\{Y(t), t \geqslant 0\}$ *is a centered self-similar Gaussian process with index* $H \in (0,1)$ *and correlation function satisfying* (3.5). *If* $\beta > 2H$, *then*

$$\psi_\infty(u) = 2^{1-1/\alpha} Q^{1/\alpha} A^{1/\alpha} B^{-1/2} \pi^{1/2} \mathcal{H}_\alpha \prod_{i=k+1}^{n} (1-b_i^2)^{-1/2} \times u^{(1-2H/\beta)(1/\alpha-1/2)} \Upsilon_k(Au^{1-2H/\beta})\big(1+o(1)\big), \quad u \to \infty.$$

4. THRESHOLD-DEPENDENT-TIME HORIZON

In this section we are concerned about the asymptotics of

$$\psi_{T_u}(u) = \mathbb{P}\big(\sup_{t\in[0,T_u]} \sum_{i=1}^{n} b_i^2 Y_i^2(t) - ct^\beta > u\big), \quad u \to \infty.$$

Throughout this section we shall adopt the same notation as in Section 3. In addition, define

$$B(u) = 2^{1/2} B^{-1/2} u^{(H+1)/\beta - 1/2}, \quad u > 0.$$

In the sequel, the following two scenarios of $T_u > 0$ will be discussed:

(i) the short time horizon: $\lim_{u\to\infty} T_u/u^{1/\beta} = s_0 \in [0, t_0)$;

(ii) the long time horizon: $\lim_{u\to\infty}(T_u - t_0 u^{1/\beta})/B(u) = x \in (-\infty, \infty]$.

Clearly, $T = \infty$ is included in scenario (ii), and $T \in (0,\infty)$ is covered by scenario (i). We present below our main result of this section.

THEOREM 4.1. *Suppose that the generic process* $\{Y(t), t \geqslant 0\}$ *is a centered self-similar Gaussian process with index* $H \in (0,1)$ *and correlation function satisfying* (3.5). *Assume further that* $\beta > 2H$. *We have, as* $u \to \infty$:

(i) *If* $\lim_{u\to\infty} T_u/u^{1/\beta} = s_0 \in [0, t_0)$, *then*

$$\psi_{T_u}(u) = \prod_{i=k+1}^{n} (1-b_i^2)^{-1/2} \mathcal{M}_{\alpha,1,\frac{Q}{2}t_0^\alpha,D} \left(\frac{u+cT_u^\beta}{T_u^{2H}}\right)^{(1/\alpha-1)_+} \times \Upsilon_k\left(\frac{u+cT_u^\beta}{T_u^{2H}}\right)\big(1+o(1)\big),$$

where the constant $\mathcal{M}_{\alpha,1,\frac{Q}{2}t_0^\alpha,D}$ *is given as in* (2.2) *with*

$$D = \frac{2H - c(\beta-2H)s_0^\beta}{2(1+cs_0^\beta)}.$$

(ii) *If* $\lim_{u\to\infty}(T_u - t_0 u^{1/\beta})/B(u) = x \in (-\infty, \infty]$, *then*

$$\psi_{T_u}(u) = \psi_\infty(u)\Phi(x)\big(1+o(1)\big),$$

where the asymptotics of $\psi_\infty(u)$ *is given in Theorem* 3.1, *and* $\Phi(\cdot)$ *denotes the standard normal distribution function.*

As a corollary of Theorem 4.1 we derive an approximation of the first passage time of the chi-square process with trend, which goes in line with, e.g., [18], [7], [15]. Precisely, define

$$\tau_u = \inf\{t \geqslant 0 : R_u(t) \leqslant 0\} \;\; (\text{with} \inf\{\emptyset\} = \infty)$$

to be the first passage time to zero of the process $\{R_u(t), t \geqslant 0\}$ defined in (1.2). Denote by $\xrightarrow{d}$ convergence in distribution when the argument tends to infinity, let E be a unit mean exponential random variable, and $\mathcal{N}$ be a standard normal random variable. We have:

COROLLARY 4.1. *Under the conditions and notation of Theorem* 4.1 *the following holds*:

(i) *If* $\lim_{u\to\infty} T_u/u^{1/\beta} = s_0 \in [0, t_0)$, *then*

$$\frac{\big(2H - c\beta s_0^{\beta}/(1+cs_0^{\beta})\big)(u + cT_u^{\beta})}{2T_u^{2H+1}}(T_u - \tau_u)\big|(\tau_u \leqslant T_u) \xrightarrow{d} E, \quad u \to \infty.$$

(ii) *If* $\lim_{u\to\infty}(T_u - t_0u^{1/\beta})/B(u) = x \in (-\infty, \infty]$, *then*

$$\frac{\tau_u - t_0u^{1/\beta}}{B(u)}\big|(\tau_u \leqslant T_u) \xrightarrow{d} \mathcal{N}\big|(\mathcal{N} \leqslant x), \quad u \to \infty.$$

5. EXTENSIONS AND DISCUSSIONS

In Sections 3 and 4, we have derived asymptotical results for the case where the chi-square process is generated from a self-similar Gaussian process. In this section, we shall discuss two possible extensions: (a) instead of independent copies of a self-similar Gaussian process we shall consider independent but non-identical self-similar Gaussian processes; (b) instead of a polynomial function $|t-s|^{\alpha}$ in (3.5) we consider a regularly varying function $K^2(|t-s|)$ with index $\alpha \in (0, 2]$.

As we have seen, Theorems 2.1 and 6.1 are fundamental for the proofs of our results in the last two sections. Asymptotical results for the extended chi-square processes (as in the cases (a) and (b)) with trend will follow similarly if corresponding extended results for Theorems 2.1 and 6.1 are available. Therefore, it is sufficient at this point to present only an extension of Theorem 2.1; a corresponding extension for Theorem 6.1 can also be obtained.

5.1. Non-identical Gaussian processes X_i's. Let $\{X_i(t), t \geqslant 0\}, 1 \leqslant i \leqslant k$, be independent copies of the a.s. continuous Gaussian process X satisfying assumptions I–III with the parameters therein, and let $\{X_i(t), t \geqslant 0\}, k+1 \leqslant i \leqslant n$, be independent copies of another a.s. continuous Gaussian process $X^{(1)}$ satisfying assumption III with parameter γ_1 instead of γ. Moreover, we suppose that the standard deviation function $\sigma_{X^{(1)}}(\cdot)$ attains its maximum one over $[0, T]$ at t_0 as well. Besides, $\{X_i(t), t \geqslant 0\}, 1 \leqslant i \leqslant k$, and $\{X_i(t), t \geqslant 0\}, k+1 \leqslant i \leqslant n$, are assumed to be independent. Define also

$$\chi^2_{n,\boldsymbol{b}}(t) = \sum_{i=1}^n b_i^2 X_i^2(t), \quad t \geqslant 0,$$

with $1 = b_1 = \ldots = b_k \geqslant b_{k+1} \geqslant \ldots \geqslant b_n > 0$.

THEOREM 5.1. *Let $\{\chi^2_{n,\boldsymbol{b}}(t), t \geqslant 0\}$ be a chi-square process defined as above. If $\gamma \geqslant \nu$ and $\gamma_1 \geqslant \nu$, then we have, as $u \to \infty$,*

$$\mathbb{P}\big(\sup_{t\in[0,T]} \chi^2_{n,\boldsymbol{b}}(t) > u\big) = \prod_{i=k+1}^n (1-b_i^2)^{-1/2} \mathcal{M}_{\nu,\mu,d,a} u^{(1/\nu-1/\mu)_+} \Upsilon_k(u)\big(1+o(1)\big).$$

REMARKS 5.1. (a) *Suppose that the generic processes X and $X^{(1)}$ are both fBm with indexes $H \in (0,1)$ and $H_1 \in (0,1)$, respectively. If $H_1 \geqslant H$, then the conditions of the last theorem are fulfilled.*

(b) *From the proof of the last theorem we can see that the assumption that $\{X_i(t), t \geqslant 0\}, k+1 \leqslant i \leqslant n$, are identical (in distribution) is not really necessary; here to simplify the notation we chose to work under this assumption.*

5.2. General correlation structure. First, we formulate the general assumption about the correlation structure of the generic Gaussian process X.

ASSUMPTION II$'$. There exists some $K(\cdot)$, a regularly varying function at zero with index $\nu/2 \in (0,1]$, such that

$$r_X(s,t) = \mathrm{Corr}\big(X(s), X(t)\big) = 1 - K^2(|t-s|)\big(1+o(1)\big), \quad s,t \to t_0.$$

Next, we introduce some further notation. Let $q(u) = \overleftarrow{K}(u^{-1/2})$ be the inverse function of $K(\cdot)$ at point $u^{-1/2}$ (assumed to exist asymptotically). It follows that $q(u)$ is a regularly function at infinity with index $-1/\nu$ which can be further expressed as $q(u) = u^{-1/\nu} L(u^{-1/2})$, with $L(\cdot)$ a slowly varying function at zero. According to the values of $L(u^{-1/2})$ as $u \to \infty$, we consider the following three scenarios:

C1: $\mu > \nu$, or $\mu = \nu$ and $\lim_{u\to\infty} L(u^{-1/2}) = 0$;
C2: $\mu = \nu$ and $\lim_{u\to\infty} L(u^{-1/2}) = \mathcal{L} \in (0,\infty)$;
C3: $\mu < \nu$, or $\mu = \nu$ and $\lim_{u\to\infty} L(u^{-1/2}) = \infty$.
We present below our second extension of Theorem 2.1.

THEOREM 5.2. *Under the assumptions and conditions of Theorem 2.1 with assumption II replaced by assumption II$'$, we have, as $u \to \infty$,*

$$\mathbb{P}\big(\sup_{t\in[0,T]} \chi^2_{n,\boldsymbol{b}}(t) > u\big) = \prod_{i=k+1}^n (1-b_i^2)^{-1/2} \widetilde{\mathcal{M}}_{\nu,\mu,1,a}(u) u^{(1/\nu-1/\mu)_+} \Upsilon_k(u)\big(1+o(1)\big),$$

where

$$\widetilde{\mathcal{M}}_{\nu,\mu,1,a}(u) = \begin{cases} a^{-1/\mu}\Gamma(1/\mu+1)(1+I_{(t_0 \notin \{0,T\})})\mathcal{H}_\nu \overleftarrow{L}(u^{-1/2}) & \textit{for } \text{C1}, \\ \mathcal{P}_\nu^{a\mathcal{L}^\nu} & \textit{for } \text{C2}, \\ 1 & \textit{for } \text{C3}. \end{cases}$$

The proof of the last theorem follows by similar arguments to those in the proof of Theorem 2.1, and thus we give only some remarks. Note that the difference from the classical results in [31] is that for the case $\mu = \nu$ three subcases should be considered differently (depending on the property of $L(\cdot)$). This is not observed in the study of some other Gaussian random fields, e.g., [33] and [11], where it is shown that the substitution of a polynomial function $d|t-s|^\nu$ by a regularly varying function $K^2(|t-s|)$ in the correlation structure of the Gaussian random fields does not influence much the asymptotics. However, it seems not surprising to have these subcases if one examines the proof of Theorem 8.2 in [31].

6. FURTHER RESULTS AND PROOFS

This section is devoted to the proofs of Theorems 2.1, 3.1, 4.1 and 5.1 and Corollary 4.1. In the following let $\mathbb{Q}, \mathbb{Q}_i, i = 1, 2, \ldots,$ denote positive constants whose values may change from line to line.

First, we present a result concerning the tail asymptotics of the supremum of a chi-square process over a threshold-dependent time interval, which turns out to be crucial for the proofs of Theorems 2.1 and 4.1 and Corollary 4.1. The full technical proof of it is presented in arXiv version [26].

THEOREM 6.1. *Let* $\{\chi^2_{n,\boldsymbol{b}}(t), t \geqslant 0\}$ *be a chi-square process given as in* (2.4) *with generic* X *satisfying assumptions* I *and* II, *and* $1 = b_1 = \ldots = b_k > b_{k+1} \geqslant b_{k+2} \geqslant \ldots \geqslant b_n > 0$. *Let further* $\Delta_x(u) = [t_0 - x_1(u)u^{-2/\mu}, t_0 + x_2(u)u^{-2/\mu}]$ *with functions* $x_i(u), i = 1, 2,$ *such that*

$$\lim_{u\to\infty} x_i(u) = x_i \in [-\infty, \infty], \quad \lim_{u\to\infty} x_i(u)u^{-1/\mu} = 0, \quad i = 1, 2.$$

If $-x_1 < x_2$, *then*

$$\mathbb{P}\big(\sup_{t\in\Delta_x(u)} \chi^2_{n,\boldsymbol{b}}(t) > u^2\big) = \prod_{i=k+1}^{n} (1-b_i^2)^{-1/2}\widehat{\mathcal{M}}_{\nu,\mu,d,a}(x_1,x_2)u^{(2/\nu-2/\mu)_+}\Upsilon_k(u^2)\big(1+o(1)\big) \tag{6.1}$$

as $u \to \infty$, *where*

$$\widehat{\mathcal{M}}_{\nu,\mu,d,a}(x_1,x_2) = \begin{cases} d^{1/\nu}a^{-1/\mu}\mathcal{H}_\nu\left(G_\mu(a^{1/\mu}x_2) - G_\mu(-a^{1/\mu}x_1)\right) & \textit{if } \nu < \mu, \\ \mathcal{P}_\nu^{a/d}[-d^{1/\nu}x_1, d^{1/\nu}x_2] & \textit{if } \nu = \mu, \\ 1 & \textit{if } \nu > \mu, \end{cases}$$

with $G_\mu(x) = \int_{-\infty}^x e^{-|t|^\mu}dt, x > 0,$ *for any* $\mu > 0$.

P r o o f o f T h e o r e m 2.1. Without loss of generality we shall only consider the case where $t_0 \in (0,T)$. As in [30] we consider the Gaussian random field

$$Y_{\boldsymbol{b}}(t,\boldsymbol{v}) = \sum_{i=1}^{n} b_i X_i(t) v_i$$

defined on $\mathcal{G}_T = [0,T] \times \mathcal{S}_{n-1}$, where $\mathcal{S}_{n-1}$ stands for the $(n-1)$-dimensional unit sphere. Following the arguments as in [30] we conclude that $\sigma_{Y_{\boldsymbol{b}}}$ and the correlation function $r_{Y_{\boldsymbol{b}}}$ of $Y_{\boldsymbol{b}}$ have the following asymptotic expansions:

$$\sigma_{Y_{\boldsymbol{b}}}(t,\boldsymbol{v}) = 1 - a|t-t_0|^{\mu}\big(1+o(1)\big) - \sum_{i=k+1}^{n} \frac{1-b_i^2}{2} v_i^2 \big(1+o(1)\big) \tag{6.2}$$

as $t \to t_0$ and $v_{k+1}^2 + \ldots + v_n^2 \to 0$, and

$$r_{Y_{\boldsymbol{b}}}(t,\boldsymbol{v},t',\boldsymbol{v}') = 1 - d|t-t'|^{\nu}\big(1+o(1)\big) - \sum_{i=1}^{n} \frac{b_i^2}{2}(v_i - v_i')^2\big(1+o(1)\big) \tag{6.3}$$

as $t,t' \to t_0$, $v_{k+1}^2 + \ldots + v_n^2 \to 0$, and $v'^2_{k+1} + \ldots + v'^2_n \to 0$.

In addition, there exist $\delta > 0, \mathbb{Q} > 0$ such that

$$\mathbb{E}\left(\big(Y_{\boldsymbol{b}}(t,\boldsymbol{v}) - Y_{\boldsymbol{b}}(t',\boldsymbol{v}')\big)^2\right) \leqslant \mathbb{Q}\big(|t-t'|^{\gamma} + \sum_{i=1}^{n}(v_i - v_i')^2\big) \tag{6.4}$$

holds for all $(t,\boldsymbol{v}) \in ([t_0-\rho, t_0+\rho] \cap [0,T]) \times \mathcal{S}_{n-1}$. Since

$$\mathbb{P}\big(\sup_{t\in[0,T]} \chi^2_{n,\boldsymbol{b}}(t) > u^2\big) = \mathbb{P}\big(\sup_{(t,\boldsymbol{v})\in\mathcal{G}_T} Y_{\boldsymbol{b}}(t,\boldsymbol{v}) > u\big),$$

we shall focus on the tail asymptotics of

$$\mathbb{P}\big(\sup_{(t,\boldsymbol{v})\in\mathcal{G}_T} Y_{\boldsymbol{b}}(t,\boldsymbol{v}) > u\big), \quad u \to \infty.$$

Next define $\Delta_u = [t_0 - (\ln u/u)^{2/\mu}, t_0 + (\ln u/u)^{2/\mu}]$, $C_u = \{\boldsymbol{v} \in \mathcal{S}_{n-1} : v_i \in [-\ln u/u, \ln u/u], k+1 \leqslant i \leqslant n\}$ and let

$$\pi_1(u) := \mathbb{P}\big(\sup_{(t,\boldsymbol{v})\in\Delta_u\times C_u} Y_{\boldsymbol{b}}(t,\boldsymbol{v}) > u\big).$$

We have, for any $u > 0$ and any small $\rho > 0$,

$$\begin{aligned} \pi_1(u) &\leqslant \mathbb{P}\big(\sup_{(t,\boldsymbol{v})\in\mathcal{G}_T} Y_{\boldsymbol{b}}(t,\boldsymbol{v}) > u\big) \\ &\leqslant \pi_1(u) + \mathbb{P}\big(\sup_{(t,\boldsymbol{v})\in\mathcal{G}_T\setminus([t_0-\rho,t_0+\rho]\times C_u)} Y_{\boldsymbol{b}}(t,\boldsymbol{v}) > u\big) \\ &\quad + \mathbb{P}\big(\sup_{(t,\boldsymbol{v})\in([t_0-\rho,t_0+\rho]\setminus\Delta_u)\times C_u} Y_{\boldsymbol{b}}(t,\boldsymbol{v}) > u\big). \end{aligned} \tag{6.5}$$

Further, in view of Theorem 6.1,

(6.6)
$$\pi_1(u) = \prod_{i=k+1}^{n} (1-b_i^2)^{-1/2} \widehat{\mathcal{M}}_{\nu,\mu,d,a}(-\infty,\infty) u^{(2/\nu-2/\mu)_+} \Upsilon_k(u^2)\big(1+o(1)\big)$$

as $u \to \infty$. By (6.2) and the Borell–TIS inequality (see, e.g., [1]),

$$\mathbb{P}\big(\sup_{(t,\boldsymbol{v})\in \mathcal{G}_T \setminus ([t_0-\rho,t_0+\rho]\times C_u)} Y_{\boldsymbol{b}}(t,\boldsymbol{v}) > u\big) \leqslant \mathbb{Q} \exp\left(-\frac{(u-\mathbb{Q}_1)^2}{2(1-\delta_0)}\right) \tag{6.7}$$

holds for all u large, with some constants $\mathbb{Q}>0, \mathbb{Q}_1>0$ and $\delta_0 \in (0,1)$. Further, in the light of (6.2), (6.4) and the Piterbarg inequality given in Theorem 8.1 in [31]

$$\begin{aligned} \mathbb{P}\big(&\sup_{(t,\boldsymbol{v})\in ([t_0-\rho,t_0+\rho]\setminus\Delta_u)\times C_u} Y_{\boldsymbol{b}}(t,\boldsymbol{v}) > u\big) \\ &\leqslant \mathbb{Q}_2 u^{2(n+1)/(\gamma\wedge 2)-1} \exp\left(-\frac{u^2}{2\big(1-(\ln u/u)^2\mathbb{Q}_3\big)^2}\right) \end{aligned} \tag{6.8}$$

holds for all u large, with some positive constants $\mathbb{Q}_2, \mathbb{Q}_3$. Consequently, the claim for the case where $t_0 \in (0,T)$ follows from (6.5)–(6.8). This completes the proof. ■

Proof of Theorem 3.1. Let $T > t_0$ be some fixed large enough integer, and let

$$\pi(u) = \mathbb{P}\big(\sup_{t\in[0,T]} \sum_{i=1}^{n} b_i^2 Z_i^2(t) > u^{1-2H/\beta}\big), \tag{6.9}$$

$$\pi_1(u) = \mathbb{P}\big(\sup_{t\in[T,\infty)} \sum_{i=1}^{n} b_i^2 Z_i^2(t) > u^{1-2H/\beta}\big). \tag{6.10}$$

Clearly,

$$\pi(u) \leqslant \psi_\infty(u) \leqslant \pi(u) + \pi_1(u).$$

By the definition of Z_i's it follows that there exist some constants $\mathbb{Q}>0, \rho \geqslant 0$ such that

$$\mathbb{E}\Big(\big(Z_1(t)-Z_1(s)\big)^2\Big) \leqslant \mathbb{Q}|t-s|^\alpha$$

holds for any $t,s \in [t_0-\rho, t_0+\rho]$. Thus, in view of (3.4), (3.5) and Theorem 2.1 we conclude that, as $u \to \infty$,

$$\pi(u) = \prod_{i=k+1}^{n} (1-b_i^2)^{-1/2} \mathcal{M}_{\alpha,2,\frac{Q}{2},\frac{B}{4A}} \big(A(u)\big)^{2/\alpha-1} \Upsilon_k\Big(\big(A(u)\big)^2\Big)\big(1+o(1)\big),$$

where $A(u) = A^{1/2}u^{1/2-H/\beta}$. Therefore, to complete the proof it is sufficient to show that

$$\pi_1(u) = o\big(\pi(u)\big), \quad u \to \infty.$$

To this end, let $\widetilde{Y_{\boldsymbol{b}}}(t,\boldsymbol{v})=\sum_{i=1}^n b_i A^{1/2} Z_i(t) v_i, (t,\boldsymbol{v})\in[T,\infty)\times[-1,1]^n$. We have

$$\pi_1(u)=\mathbb{P}\big(\sup_{(t,\boldsymbol{v})\in[T,\infty)\times S_{n-1}}\widetilde{Y_{\boldsymbol{b}}}(t,\boldsymbol{v})>A(u)\big).$$

We split the interval $[T,\infty)$ into subintervals $[k,k+1), k\geqslant T$. For every $k\geqslant T$, we have (set $Y^*_{\boldsymbol{b}}(t,\boldsymbol{v})=\sqrt{1+ct^\beta}t^{-H}\widetilde{Y_{\boldsymbol{b}}}(t,\boldsymbol{v})$)

$$\begin{aligned}(6.11)\quad &\mathbb{P}\big(\sup_{(t,\boldsymbol{v})\in[k,k+1)\times\mathcal{S}_{n-1}}\widetilde{Y_{\boldsymbol{b}}}(t,\boldsymbol{v})>A(u)\big)\\ &\qquad\leqslant\mathbb{P}\left(\sup_{(t,\boldsymbol{v})\in[k,k+1)\times[-1,1]^n}Y^*_{\boldsymbol{b}}(t,\boldsymbol{v})>\frac{\sqrt{1+c(k-1)^\beta}}{(k-1)^H}A(u)\right).\end{aligned}$$

In addition, there exists some global constant $\mathbb{Q}$ such that for any $k\geqslant T$

$$\begin{aligned}\mathbb{E}\left(Y^*_{\boldsymbol{b}}(t,\boldsymbol{v})-Y^*_{\boldsymbol{b}}(t',\boldsymbol{v}')\right)^2&\leqslant 2An\mathbb{E}\left(\overline{Y}(t)-\overline{Y}(t')\right)^2+2A\sum_{i=1}^n(v_i-v_i^{'})^2\\ &\leqslant\mathbb{Q}\big(|t-t'|^\alpha+\sum_{i=1}^n(v_i-v_i^{'})^2\big)\end{aligned}$$

holds for all $t,t'\in[k,k+1),\boldsymbol{v},\boldsymbol{v}'\in[-1,1]^n$. Next we split $[-1,1]^n$ into 2^n subsets of the form $\prod_{i=1}^n\Delta_i^{j_i}, j_i=1,2$, where $\Delta_i^1=[-1,0]$ and $\Delta_i^2=[0,1]$. By using Lemma 2.1 we derive, for $k\geqslant T$,

$$\begin{aligned}&\mathbb{P}\left(\sup_{(t,\boldsymbol{v})\in[k,k+1)\times\prod_{i=1}^n\Delta_i^{j_i}}Y^*_{\boldsymbol{b}}(t,\boldsymbol{v})>\frac{\sqrt{1+c(k-1)^\beta}}{(k-1)^H}A(u)\right)\\ &\qquad\leqslant 2^{n+2}\exp\left(-\mathbb{Q}_1\frac{1+c(k-1)^\beta}{(k-1)^{2H}}\big(A(u)\big)^2\right)\end{aligned}$$

with $\mathbb{Q}_1=\min\left(c^*/\mathbb{Q},1/(8A)\right)$. This together with (6.11) yields

$$\begin{aligned}&\mathbb{P}\big(\sup_{(t,\boldsymbol{v})\in[k,k+1)\times S_{n-1}}\widetilde{Y_{\boldsymbol{b}}}(t,\boldsymbol{v})>A(u)\big)\\ &\qquad\leqslant 2^{2n+2}\exp\left(-\mathbb{Q}_1\frac{1+c(k-1)^\beta}{(k-1)^{2H}}\big(A(u)\big)^2\right).\end{aligned}$$

Consequently, since T was chosen large enough,

$$\begin{aligned}\pi_1(u)&\leqslant\sum_{k=T}^\infty 2^{2n+2}\exp\left(-\mathbb{Q}_1\frac{1+c(k-1)^\beta}{(k-1)^{2H}}\big(A(u)\big)^2\right)\\ &\leqslant 2^{2n+2}\int_{T-2}^\infty\exp\left(-\mathbb{Q}_2\big(A(u)\big)^2y^{\beta-2H}\right)dy\\ &\leqslant\mathbb{Q}_3\big(A(u)\big)^{-2}\exp\left(-\mathbb{Q}_2(T-2)^{\beta-2H}\big(A(u)\big)^2\right)=o\big(\pi(u)\big)\end{aligned}$$

as $u \to \infty$, where $\mathbb{Q}_3$ is a constant depending on T, and $\mathbb{Q}_2 = c\mathbb{Q}_1$. This completes the proof. ∎

Proof of Theorem 4.1. Case (i). We introduce a deterministic function $m(u) = (u + cT_u^\beta)/T_u^{2H}, u > 0$, and centered Gaussian processes

$$W_{u,i}(t) = \frac{Y_i(t)}{\sqrt{1 - c_u(1+c_u)^{-1}(1-t^\beta)}}, \quad t \geqslant 0, \quad 1 \leqslant i \leqslant n,$$

with $c_u = cT_u^\beta/u, u > 0$, such that $\lim_{u\to\infty} c_u = cs_0^\beta =: c_0$. By the self-similarity of Y we have

$$\psi_{T_u}(u) = \mathbb{P}\big(\sup_{t\in[0,1]} \sum_{i=1}^n b_i^2 W_{u,i}^2(t) > m(u)\big).$$

Let further $c_0^{\pm\epsilon} = \max(c_0 \pm \epsilon, 0)$ and define

$$W_i^{\pm\epsilon}(t) = \frac{Y_i(t)}{\sqrt{1 - c_0^{\pm\epsilon}(1+c_0^{\pm\epsilon})^{-1}(1-t^\beta)}}, \quad t \geqslant 0, \quad 1 \leqslant i \leqslant n,$$

for any sufficiently small $\epsilon > 0$. Thus we have, for u large enough,

$$\mathbb{P}\Big(\sup_{t\in[0,1]} \sum_{i=1}^n b_i^2 \left(W_i^{-\epsilon}(t)\right)^2 > m(u)\Big) \leqslant \psi_{T_u}(u) \leqslant \mathbb{P}\Big(\sup_{t\in[0,1]} \sum_{i=1}^n b_i^2 \left(W_i^{+\epsilon}(t)\right)^2 > m(u)\Big).$$

Next we consider the upper bound of $\psi_{T_u}(u)$. It follows that $\sigma_{W_1^{+\epsilon}}(t)$ attains its maximum over $[0,1]$ at the unique point $t_0 = 1$ and further

$$\sigma_{W_1^{+\epsilon}}(t) = 1 - \frac{2H - (\beta - 2H)c_0^{+\epsilon}}{2(1+c_0^{+\epsilon})}|t-1|\big(1+o(1)\big), \quad t \to 1,$$

$$\mathrm{Corr}\big(W_1^{+\epsilon}(t), W_1^{+\epsilon}(s)\big) = 1 - \frac{t_0^\alpha Q}{2}|t-s|^\alpha\big(1+o(1)\big), \quad s,t \to 1.$$

In addition, there exists some $\mathbb{Q} > 0$ such that

$$\mathbb{E}\Big(\big(W_1^{+\epsilon}(t) - W_1^{+\epsilon}(s)\big)^2\Big) \leqslant \mathbb{Q}|t-s|^\alpha$$

holds for all $s, t \in [1/2, 1]$. Therefore, in view of Theorem 2.1,

$$\begin{aligned}
&\mathbb{P}\Big(\sup_{t\in[0,1]} \sum_{i=1}^n b_i^2 \left(W_i^{+\epsilon}(t)\right)^2 > m(u)\Big) \\
&= \prod_{i=k+1}^n (1-b_i^2)^{-1/2} \mathcal{M}_{\alpha,1,\frac{Q}{2}t_0^\alpha,D_{+\epsilon}}\big(m(u)\big)^{(1/\alpha-1)_+} \Upsilon_k\big(m(u)\big)\big(1+o(1)\big)
\end{aligned}$$

as $u \to \infty$, with $D_{+\epsilon} = [2H - (\beta - 2H)c_0^{+\epsilon}]/2(1 + c_0^{+\epsilon})$. Similar arguments give the same lower bound as above (with $+\epsilon$ replaced by $-\epsilon$) for $\psi_{T_u}(u)$, and thus by letting $\epsilon \to 0$ the claim in (i) follows.

Case (ii). Again, using the self-similarity we derive

$$\psi_{T_u}(u) = \mathbb{P}\Big(\sup_{t\in[0,T_u u^{-1/\beta}]} \sum_{i=1}^n b_i^2 A Z_i^2(t) > \big(A(u)\big)^2\Big),$$

where $A(u) = A^{1/2}u^{1/2-H/\beta}$. Let $t_u = t_0 - u^{-1/2+H/\beta}\ln u$, and define

$$\pi_{t_u}(u) = \mathbb{P}\Big(\sup_{t\in[0,t_u]} \sum_{i=1}^n b_i^2 A Z_i^2(t) > \big(A(u)\big)^2\Big).$$

Clearly,

$$\begin{aligned}\mathbb{P}\Big(\sup_{t\in[t_u,T_u u^{-1/\beta}]} &\sum_{i=1}^n b_i^2 A Z_i^2(t) > \big(A(u)\big)^2\Big)\\ &\leqslant \psi_{T_u}(u)\\ &\leqslant \mathbb{P}\Big(\sup_{t\in[t_u,T_u u^{-1/\beta}]} \sum_{i=1}^n b_i^2 A Z_i^2(t) > \big(A(u)\big)^2\Big) + \pi_{t_u}(u).\end{aligned}$$

In the following, we shall first derive the asymptotics of the common term on both sides of the above formula, which will give the exact asymptotics of $\psi_{T_u}(u)$. Then we show that $\pi_{t_u}(u)$ is asymptotically negligible. In view of (3.4), (3.5) and Theorem 6.1, we have, for any $x \in (-\infty, \infty)$,

$$\mathbb{P}\Big(\sup_{t\in[t_u,T_u u^{-1/\beta}]} \sum_{i=1}^n b_i^2 A Z_i^2(t) > \big(A(u)\big)^2\Big) = \psi_\infty(u)\Phi(x)\big(1 + o(1)\big), \quad u \to \infty.$$

Next we show that the last formula is also valid for $x = \infty$. Since, for any fixed $y \geqslant 0$,

$$\begin{aligned}\mathbb{P}\Big(\sup_{t\in[t_u,t_0+yB(u)]} &\sum_{i=1}^n b_i^2 A Z_i^2(t) > \big(A(u)\big)^2\Big)\\ &\leqslant \mathbb{P}\Big(\sup_{t\in[t_u,T_u u^{-1/\beta}]} \sum_{i=1}^n b_i^2 A Z_i^2(t) > \big(A(u)\big)^2\Big) \leqslant \psi_\infty(u),\end{aligned}$$

we infer from Theorem 6.1 that

$$\begin{aligned}\Phi(y) &\leqslant \varliminf_{u\to\infty} \frac{\mathbb{P}\Big(\sup_{t\in[t_u,T_u u^{-1/\beta}]} \sum_{i=1}^n b_i^2 A Z_i^2(t) > \big(A(u)\big)^2\Big)}{\psi_\infty(u)}\\ &\leqslant \varlimsup_{u\to\infty} \frac{\mathbb{P}\Big(\sup_{t\in[t_u,T_u u^{-1/\beta}]} \sum_{i=1}^n b_i^2 A Z_i^2(t) > \big(A(u)\big)^2\Big)}{\psi_\infty(u)} \leqslant 1.\end{aligned}$$

Therefore, letting $y \to \infty$ we conclude that

$$\mathbb{P}\Big(\sup_{t\in[t_u, T_u u^{-1/\beta}]} \sum_{i=1}^n b_i^2 A Z_i^2(t) > \big(A(u)\big)^2 \Big) = \psi_\infty(u)\big(1+o(1)\big), \quad u \to \infty.$$

To complete the proof we prove that $\pi_{t_u}(u) = o\big(\psi_\infty(u)\big)$ as $u \to \infty$. We have

$$\mathbb{P}\Big(\sup_{t\in[0,t_u]} \sum_{i=1}^n b_i^2 A Z_i^2(t) > \big(A(u)\big)^2 \Big) = \mathbb{P}\big(\sup_{(t,\boldsymbol{v})\in[0,t_u]\times\mathcal{S}_{n-1}} \widetilde{Y_{\boldsymbol{b}}}(t,\boldsymbol{v}) > A(u)\big),$$

where $\widetilde{Y_{\boldsymbol{b}}}(t,\boldsymbol{v}) = \sum_{i=1}^n b_i A^{1/2} Z_i(t) v_i, (t,\boldsymbol{v}) \in [0, t_0+1]\times\mathcal{S}_{n-1}$. Further, there exist some constants $\delta\in(0,1), \mathbb{Q}>0$ such that

$$\mathbb{E}\Big(\big(\widetilde{Y_{\boldsymbol{b}}}(t,\boldsymbol{v})\big)^2 \Big) \leqslant 1-\delta < 1, \quad t \in [0, t_0-\rho], \boldsymbol{v}\in\mathcal{S}_{n-1},$$

$$\mathbb{E}\Big(\big(\widetilde{Y_{\boldsymbol{b}}}(t,\boldsymbol{v}) - \widetilde{Y_{\boldsymbol{b}}}(t,\boldsymbol{v})\big)^2 \Big) \leqslant \mathbb{Q}\big(|t-s|^\alpha + \sum_{i=1}^n (v_i - v_i^{'})^2\big),$$
$$t\in[t_0-\rho, t_0+\rho], \boldsymbol{v}\in\mathcal{S}_{n-1},$$

hold. Therefore, as in the proof of Theorem 2.1, by the Borell–TIS inequality we have

$$\mathbb{P}\Big(\sup_{t\in[0,t_0-\rho]} \sum_{i=1}^n b_i^2 A Z_i^2(t) > \big(A(u)\big)^2 \Big) \leqslant \exp\left(-\frac{\big(A(u)-\mathbb{Q}_0\big)^2}{2(1-\delta)^2} \right) = o\big(\psi_\infty(u)\big), \quad u\to\infty, \tag{6.12}$$

with $\mathbb{Q}_0 = \mathbb{E}\big(\sup_{(t,\boldsymbol{v})\in[0,t_0-\rho]\times\mathcal{S}_{n-1}} \widetilde{Y_{\boldsymbol{b}}}(t,\boldsymbol{v})\big) < \infty$, and by the Piterbarg inequality and (3.4) (or by a direct application of [35], Proposition 3.2) we have

$$\mathbb{P}\Big(\sup_{t\in[t_0-\rho,t_u]} \sum_{i=1}^n b_i^2 A Z_i^2(t) > \big(A(u)\big)^2 \Big) \leqslant \mathbb{Q}_1\big(A(u)\big)^{2(n+1)/\alpha} \Psi\left(\frac{A(u)}{1-\mathbb{Q}_2\big(A(u)^{-1}\ln A(u)\big)^2} \right) = o\big(\psi_\infty(u)\big) \tag{6.13}$$

as $u\to\infty$, where $\mathbb{Q}_1$ and $\mathbb{Q}_2$ are two positive constants. Consequently, we conclude from (6.12) and (6.13) that

$$\pi_{t_u}(u) = o\big(\psi_\infty(u)\big), \quad u\to\infty.$$

This completes the proof. ■

P r o o f o f C o r o l l a r y 4.1. Case (i). For notational simplicity, we let

$$f(u) = \frac{2T_u^{2H+1}}{\big(2H - c\beta s_0^\beta/(1+cs_0^\beta)\big)(u + cT_u^\beta)}, \quad u > 0.$$

By definition, for any $x > 0$

$$\mathbb{P}\left(\frac{T_u - \tau_u}{f(u)} > x \big| \tau_u \leqslant T_u\right) = \frac{\psi_{T_u - xf(u)}(u)}{\psi_{T_u}(u)}.$$

Further, it follows from Theorem 4.1 that

$$\lim_{u\to\infty} \frac{\psi_{T_u - xf(u)}(u)}{\psi_{T_u}(u)} = \lim_{u\to\infty} \exp\left(\frac{u + cT_u^\beta}{2T_u^{2H}} - \frac{u + c\big(T_u - xf(u)\big)^\beta}{2\big(T_u - xf(u)\big)^{2H}}\right) = e^{-x},$$

establishing the claim in (i).

Case (ii). By a similar reasoning to the above, in the light of Theorem 4.1 we have, for any $y \leqslant x$,

$$\lim_{u\to\infty} \mathbb{P}\left(\frac{\tau_u - t_0 u^{1/\beta}}{B(u)} < y \big| \tau_u \leqslant T_u\right) = \lim_{u\to\infty} \frac{\psi_{t_0 u^{1/\beta} + yB(u)}(u)}{\psi_{T_u}(u)} = \frac{\Phi(y)}{\Phi(x)}.$$

Thus, the proof is complete. ∎

P r o o f o f T h e o r e m 5.1. One approach is to follow a similar proof to that of Theorem 2.1 by using the double-sum method. Here, we give another proof based on the ideas and results in [16], [30] and [29]. We first show that

$$\begin{aligned}(6.14)\quad & \mathbb{P}\big(\sup_{t\in[0,T]} \chi^2_{n,\boldsymbol{b}}(t) > u\big) \\ &\qquad = \mathbb{P}\big(\sup_{t\in[0,T]} \chi^2_{k,\mathbf{1}}(t) + \sum_{i=k+1}^n b_i^2 X_i^2(t_0) > u\big)\big(1 + o(1)\big)\end{aligned}$$

holds as $u \to \infty$, which in view of Lemma 2.1 in [29] is sufficient. Indeed, letting $G(u) = \mathbb{P}\big(\sup_{t\in[0,T]} \chi^2_{k,\mathbf{1}}(t) \leqslant u\big)$ we infer from (2.5) that

$$\lim_{u\to\infty} \frac{1 - G(u+y)}{1 - G(u)} = \exp\left(-\frac{1}{2}y\right) \qquad \text{for all } y \in \mathbb{R}.$$

Further, let $H(u) = \mathbb{P}\left(\sum_{i=k+1}^n b_i^2 X_i^2(t_0) \leqslant u\right)$. It is known (cf. Example 2 in [22]) that, for some $r \in \mathbb{N}$,

$$1 - H(u) = O\left(u^r \exp\left(-\frac{u}{2b_{k+1}}\right)\right) = o\big(1 - G(u)\big).$$

Moreover, choosing some $\theta \in \big(1/2, 1/(2b_{k+1})\big)$, we have $\int_0^\infty e^{\theta x} dH(x) < \infty$. Therefore, by Lemma 2.1 in [29] the claim in (2.6) follows from (6.14).

It remains to show (6.14). To this end, we introduce the following two Gaussian random fields: for $t \geqslant 0, \boldsymbol{v} \in \mathbb{R}^n$,

$$Y_{\boldsymbol{b}}(t, \boldsymbol{v}) = \sum_{i=1}^n b_i v_i X_i(t), \qquad Z_{\boldsymbol{b}}(t, \boldsymbol{v}) = \sum_{i=1}^k v_i X_i(t) + \sum_{i=k+1}^n b_i v_i X_i(t_0).$$

As in the proof of Theorem 2.1 it is sufficient that

$$\mathbb{P}\Big(\sup_{(t,\boldsymbol{v}) \in \mathcal{G}_T} Y_{\boldsymbol{b}}(t, \boldsymbol{v}) > u \Big) = \mathbb{P}\Big(\sup_{(t,\boldsymbol{v}) \in \mathcal{G}_T} Z_{\boldsymbol{b}}(t, \boldsymbol{v}) > u \Big) \big(1 + o(1)\big) \tag{6.15}$$

holds as $u \to \infty$. Next we see that the standard deviations $\sigma_{Y_{\boldsymbol{b}}}(t, \boldsymbol{v})$ and $\sigma_{Z_{\boldsymbol{b}}}(t, \boldsymbol{v})$ attain their absolute maximum (equal to one) over $\mathcal{G}_T$ at all points of C_0 given as

$$C_0 = \{t_0\} \times \{\boldsymbol{v} \in \mathcal{S}_{n-1} : v_1^2 + \ldots + v_k^2 = 1\} \subset \mathcal{G}_T.$$

Further, we consider the expansions of the standard deviations and the correlations of the Gaussian random fields $Y_{\boldsymbol{b}}$ and $Z_{\boldsymbol{b}}$ around the sphere C_0. By direct calculations we have

$$\begin{aligned} \sigma_{Y_{\boldsymbol{b}}}(t, \boldsymbol{v}) &= 1 - a|t - t_0|^\mu \big(1 + o(1)\big) - \frac{1}{2} \sum_{i=k+1}^n (1 - b_i^2) v_i^2 \big(1 + o(1)\big), \\ \sigma_{Z_{\boldsymbol{b}}}(t, \boldsymbol{v}) &= 1 - a|t - t_0|^\mu \big(1 + o(1)\big) - \frac{1}{2} \sum_{i=k+1}^n (1 - b_i^2) v_i^2 \big(1 + o(1)\big) \end{aligned} \tag{6.16}$$

as $t \to t_0$ and $\sum_{i=k+1}^n v_i^2 \to 0$. Further, since $\gamma > \nu, \gamma_1 \geqslant \nu$,

$$\begin{aligned} r_{Y_{\boldsymbol{b}}}(t, \boldsymbol{v}, s, \boldsymbol{u}) &= \operatorname{Corr}\big(Y_{\boldsymbol{b}}(t, \boldsymbol{v}), Y_{\boldsymbol{b}}(s, \boldsymbol{u})\big) \\ &= 1 - d|t - s|^\nu \big(1 + o(1)\big) - \frac{1}{2} \sum_{i=1}^n b_i^2 (v_i - u_i)^2 \big(1 + o(1)\big), \\ r_{Z_{\boldsymbol{b}}}(t, \boldsymbol{v}, s, \boldsymbol{u}) &= \operatorname{Corr}\big(Z_{\boldsymbol{b}}(t, \boldsymbol{v}), Y_{\boldsymbol{b}}(s, \boldsymbol{u})\big) \\ &= 1 - d|t - s|^\nu \big(1 + o(1)\big) - \frac{1}{2} \sum_{i=1}^n b_i^2 (v_i - u_i)^2 \big(1 + o(1)\big) \end{aligned} \tag{6.17}$$

hold as $s, t \to t_0$, $\sum_{i=k+1}^n v_i^2 \to 0$ and $\sum_{i=k+1}^n u_i^2 \to 0$. The technical proof of (6.17) can be found in arXiv version [26].

Define a neighborhood C_u of C_0 as

$$C_u = \Big\{(t, \boldsymbol{v}) : a|t - t_0|^\mu + \frac{1}{2} \sum_{i=k+1}^n (1 - b_i^2) v_i^2 < \ln u / u\Big\} \cap \mathcal{G}_T.$$

By an application of the Borell–TIS inequality and the Piterbarg inequality as in the proof of Lemma 8.1 in [31] we can show that

$$\mathbb{P}\big(\sup_{(t,\boldsymbol{v})\in\mathcal{G}_T} Y_{\boldsymbol{b}}(t,\boldsymbol{v})>u\big)=\mathbb{P}\big(\sup_{(t,\boldsymbol{v})\in C_u} Y_{\boldsymbol{b}}(t,\boldsymbol{v})>u\big)\big(1+o(1)\big), \tag{6.18}$$

$$\mathbb{P}\big(\sup_{(t,\boldsymbol{v})\in\mathcal{G}_T} Z_{\boldsymbol{b}}(t,\boldsymbol{v})>u\big)=\mathbb{P}\big(\sup_{(t,\boldsymbol{v})\in C_u} Z_{\boldsymbol{b}}(t,\boldsymbol{v})>u\big)\big(1+o(1)\big) \tag{6.19}$$

hold as $u\to\infty$. Moreover, since we are concerned about the asymptotic results, it follows that the expansions of the standard deviations and the correlations of the Gaussian random field $Y_{\boldsymbol{b}}$ (or $Z_{\boldsymbol{b}}$) around the sphere C_0 are the only necessary properties influencing the asymptotics of (6.18) (or (6.19)); this is due to the fact that $Y_{\boldsymbol{b}}$ and $Z_{\boldsymbol{b}}$ are Gaussian and $C_u\to C_0$ as $u\to\infty$. Therefore, it follows from (6.16) and (6.17) that (6.15) is established. This completes the proof. ■

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