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KENDALL RANDOM WALKS

BY

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Abstract. The paper deals with a new class of random walks strictly connected with the Pareto distribution. We consider stochastic processes in the sense of generalized convolution or weak generalized convolution. The processes are Markov processes in the usual sense. Their structure is similar to perpetuity or autoregressive model. We prove the theorem which describes the magnitude of the fluctuations of random walks generated by generalized convolutions.

We give a construction and basic properties of random walks with respect to the Kendall convolution. We show that they are not classical Lévy processes. The paper proposes a new technique to cumulate the Pareto-type distributions using a modification of the Williamson transform and contains many new properties of weakly stable probability measure connected with the Kendall convolution. It seems that the Kendall convolution produces a new class of heavy tailed distributions of Pareto-type.

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1. INTRODUCTION

In 2009 Nguyen Van Thu in [19], considering only the Kingman convolution, showed that each generalized convolution, together with an infinitely divisible distribution with respect to this convolution, defined a Markov process, which could be treated as a Lévy process in the sense of this convolution. The most important example is the Bessel process defined by the Kingman convolution (sometimes called also the Bessel convolution) – widely applied and intensively studied in many different areas of mathematics.

The Lévy processes with respect to generalized convolutions and weak generalized convolutions were introduced in [1]. The paper deals with the Lévy processes constructed as the Markov processes in the usual sense.

We consider here the Markov chain $\{X_n : n \in \mathbb{N}_0\}$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, given by the construction proposed in [1].

The possibility of many different interpretations of the cumulation of independent steps $\Delta X_1, \ldots, \Delta X_k$ into the state X_k-cumulation, which is not just a simple addition, promises wide applicability of such processes. Similarly to the classical theory, the discrete time processes can be built on the basis of any step distribution (in the sense that distribution of the unit step does not have to be infinitely divisible in any sense). The main generalization is that instead of classical addition we consider here two binary operations. The first one is the generalized convolution, which is an associative and commutative operation on the set of probability measures on the Borel subsets of the positive half line. We also consider generalized convolution extended to the Borel subsets of the real line in the sense given in [6], in particular, weak generalized convolution. Since weakly stable probability measure, which is strictly connected with the weak generalized convolution, is a natural generalization of symmetric α-stable distribution, it is worth investigating and it can be used in applications.

The next section deals with the definition and the existence of the Markov chains based on generalized convolutions and weak generalized convolutions. We recall construction given in [1]. A theorem describing the magnitude of the fluctuations of constructed random walks will be proved.

In the third section we study properties of random walks under the Kendall generalized and weak Kendall generalized convolutions. The main tool which we use is a transform called homomorphism:

$$h(\delta_t) = (1 - t^\alpha)_+$$

for $\alpha > 0$. For weakly stable probability measure with the characteristic function,

$$\widehat{\mu}_\alpha(t) = (1 - |t|^\alpha)_+$$

for $0 < \alpha \leqslant 1$, we arrive at random walks on the real line.

In the Kendall generalized convolution case we obtain the following random walk:

$$X_{n+1} = (X_n \vee \Delta X_{n+1})\, \theta_n^{Q_n} \text{ a.e.},$$

where $\vee$ denotes maximum, (θ_n) is a proper i.i.d. sequence with the Pareto distribution with density

$$\pi_{2\alpha}(dx) = 2\alpha / x^{2\alpha+1} \mathbf{1}_{(1,\infty)} dx$$

such that θ_n is independent of $(X_n \vee \Delta X_{n+1})$. The random variables (Q_n) take values zero and one provided that we know the position of $(X_n \vee \Delta X_{n+1})$.

We show that the obtained random walks are not the Lévy processes in the usual sense. Their structure is similar to the first order random coefficients autoregessive model (see, e.g., [2]) but with a different dependence structure.

We also present many new properties of the weakly stable probability measure μ_α, in particular, we obtain a characterization of the Pareto distribution $\pi_{2\alpha}$ in terms of μ_α for $0 < \alpha \leqslant 1$.

In order to start our constructions we give some basic facts on generalized convolution and weak generalized convolution.

The main mathematical tool used here is the generalized convolution defined by Urbanik (see [18]) on the set $\mathcal{P}_+$ of probability measures on the Borel subsets of the positive half line. For simplicity, we will use the notation T_a for the rescaling operator defined by $(T_a\lambda)(A) = \lambda(A/a)$ for every Borel set A when $a \neq 0$, and $T_0\lambda = \delta_0$ is the probability measure concentrated at zero.

DEFINITION 1.1. A commutative and associative $\mathcal{P}_+$-valued binary operation $\diamond$ defined on $\mathcal{P}_+^2$ is called a *generalized convolution* if for all $\lambda, \lambda_1, \lambda_2 \in \mathcal{P}_+$ and $a \geqslant 0$ we have:

(i) $\delta_0 \diamond \lambda = \lambda$;

(ii) $\big(p\lambda_1 + (1-p)\lambda_2\big) \diamond \lambda = p(\lambda_1 \diamond \lambda) + (1-p)(\lambda_2 \diamond \lambda)$ whenever $p \in [0,1]$;

(iii) $T_a(\lambda_1 \diamond \lambda_2) = (T_a\lambda_1) \diamond (T_a\lambda_2)$;

(iv) if $\{\lambda_n : n \geqslant 1\} \subset \mathcal{P}_+$ with $\lambda_n \to \lambda$, then $\lambda_n \diamond \eta \to \lambda \diamond \eta$ for all $\eta \in \mathcal{P}_+$ (here $\to$ denotes a weak convergence of probability measures);

(v) there exists a sequence $(c_n)_{n\in\mathbb{N}}$ of positive numbers such that the sequence $T_{c_n}\delta_1^{\diamond n}$ converges to a probability measure different from δ_0.

We call the set $(\mathcal{P}_+, \diamond)$ a *generalized convolution algebra*. A continuous mapping $h : \mathcal{P}_+ \to \mathbb{R}$ such that

$$h\big(p\lambda + (1-p)\nu\big) = ph(\lambda) + (1-p)h(\nu) \quad \text{and} \quad h(\lambda \diamond \nu) = h(\lambda)h(\nu)$$

for all $\lambda, \nu \in \mathcal{P}_+$ and $p \in (0,1)$ is called a *homomorphism* of $(\mathcal{P}_+, \diamond)$. For every probability measure $\lambda \in \mathcal{P}_+$ we have

$$h(\lambda) = \int_{\mathbb{R}_+} h(\delta_x)\lambda(dx).$$

The algebra $(\mathcal{P}_+, \diamond)$ is *regular* if it admits a non-trivial homomorphism, i.e. such an h that $h \not\equiv 0$ and $h \not\equiv 1$. Since for all $\lambda_1, \lambda_2 \in \mathcal{P}_+$ we have

$$\lambda_1 \diamond \lambda_2(A) = \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \rho_{x,y}(A)\lambda_1(dx)\lambda_2(dy),$$

every generalized convolution is uniquely determined by the *probability kernel*

$$\rho_{x,y} := \delta_x \diamond \delta_y.$$

Evidently, it follows that for all $x, y, c \geqslant 0$

- $\rho_{x,0} = \delta_x$,
- $\rho_{x,y} = \rho_{y,x}$,
- $T_c\rho_{x,y} = \rho_{cx,cy}$,
- $\rho_{x,y} = T_v\rho_{z,1}$, where $v = x \vee y,\ z = (x \wedge y)/(x \vee y)$.

The origin of the generalized convolution can be found in the Kingman paper [11], where the first example of random walk under generalized convolution (called the *Kingman* or *Bessel convolution*) is considered. The Kingman convolution has a natural interpretation at the interference phenomena (see [20]). In the series of papers (see [17] and [18]) Urbanik developed the theory of generalized convolutions. In [21] and [22] Vol'kovich was investigating this theory. Many open problems connected with generalized convolutions were given by Vol'kovich, Toledano-Kitai and Avros (see [23]), Hazod [4] or Van Thu [19]. In 2012 the paper [7] appeared about generalized convolutions in the non-commutative probability theory. Hence, it can be supposed that the results obtained here will also be useful in the non-commutative probability theory. On the other hand, the general theory of the Lévy processes in the generalized convolution sense was first established in [1]. In this paper we develop that theory giving an explicit recipe for random walks with respect to the Kendall convolution, presenting new results on this topic and proposing a new technique of the Williamson transform to investigate constructed stochastic processes.

EXAMPLE 1.1. The best known examples of generalized convolutions are the following:

- α-convolution $\star_\alpha$ for $\alpha > 0$ given by the formula $\delta_a \star_\alpha \delta_b = \delta_c$, where $c^\alpha = a^\alpha + b^\alpha$; the corresponding homomorphism $h(\delta_t) = \exp\{-t^\alpha\}$;
- max-convolution $\star_\infty$ given by $\delta_a \star_\infty \delta_b = \delta_c$, where $c = a \vee b$; the corresponding homomorphism $h(\delta_t) = \mathbf{1}_{[0,1]}(t)$;
- symmetric convolution $\star_{1,1}$ given by $\delta_a \star_{1,1} \delta_b = \frac{1}{2}\delta_{a+b} + \frac{1}{2}\delta_{|a-b|}$; the corresponding homomorphism $h(\delta_t) = \cos(t)$;
- the Kendall convolution $\triangle_\alpha$ for $\alpha > 0$ with the probability kernel $\delta_a \triangle_\alpha \delta_b = (1 - a^\alpha/b^\alpha)\,\delta_b + (a^\alpha/b^\alpha) T_b \pi_{2\alpha}$ for $0 \leqslant a \leqslant b$, where $\pi_{2\alpha}$ is the Pareto distribution with the density $\pi_{2\alpha}(dx) = 2\alpha/x^{2\alpha+1}\mathbf{1}_{(1,\infty)}(x)dx$ and $h(\delta_t) = (1 - t^\alpha)_+$;
- the Kingman convolution $\star_{1,\beta}$ $(\beta > 1)$ given by

$$\delta_a \star_{1,\beta} \delta_b(dx) =$$
$$= B\left(\frac{\beta-1}{2}, \frac{1}{2}\right) \frac{\left[\left(x^2 - (a-b)^2\right)\left((a+b)^2 - x^2\right)\right]^{(\beta-3)/2}}{(2ab)^{\beta-3}} \mathbf{1}_{[|a-b|, a+b]}(x)dx$$

and

$$h(\delta_t) = \Gamma(\beta/2)\left(\frac{2}{t}\right)^{\beta/2-1} J_{\beta/2-1}(t),$$

where $J_{\beta/2-1}(t)$ is the Bessel function of order $\beta/2 - 1$, and $B(a,b)$ is the beta function with parameters a and b;

- the Kucharczak convolution $\circ_\alpha$ $(0 < \alpha < 1)$ given by

$$\delta_a \circ_\alpha \delta_b(dx) = \frac{\sin(\pi\alpha)\cdot(ab)^\alpha(2x - a - b)}{\pi\cdot\big(x(x-a-b)\big)^\alpha\cdot(x-a)\cdot(x-b)} \mathbf{1}_{[(a^\alpha+b^\alpha)^{1/\alpha},\infty)}(x)dx$$

and $h(\delta_t)(t) = \Gamma(\alpha)^{-1}\Gamma(\alpha, t)$, where $\Gamma(\alpha, t)$ is the incomplete Γ-function; for more details see [12].

In [6] one can find the definition and basic properties of generalized convolution on the set $\mathcal{P}$ of probability measures on the Borel subsets of the real line. An example of such an object is a weak generalized convolution connected with weakly stable distribution. We recall the definition of weak stability of probability measures.

DEFINITION 1.2. A probability measure $\mu \in \mathcal{P}$ is *weakly stable* if

$$\forall\, a, b \in \mathbb{R}\ \exists \lambda \in \mathcal{P} \quad T_a\mu * T_b\mu = \mu \circ \lambda$$

or, equivalently,

$$\forall\, \lambda_1, \lambda_2 \in \mathcal{P}\ \exists \lambda \in \mathcal{P} \quad \mu \circ \lambda_1 * \mu \circ \lambda_2 = \mu \circ \lambda,$$

where $(\mu \circ \lambda)(A) = \int \mu(A/s)\lambda(ds)$ for every Borel set A.

From Theorem 6 in [16] we know that if μ is a weakly stable probability measure on a separable Banach space $\mathbb{E}$, then either there exists $a \in \mathbb{E}$ such that $\mu = \delta_a$ or there exists $a \in \mathbb{E} \setminus \{0\}$ such that $\mu = \frac{1}{2}(\delta_a + \delta_{-a})$ or $\mu(\{a\}) = 0$ for every $a \in \mathbb{E}$. In this paper we consider non-trivial weakly stable measures, i.e. weakly stable probability measures without any atoms. In [15] Misiewicz defined a weak generalized convolution:

DEFINITION 1.3. For a weakly stable measure μ, a *weak generalized convolution* of λ_1 and λ_2 (denoted by $\lambda_1 \otimes_\mu \lambda_2$) is defined by the formula

$$\lambda_1 \otimes_\mu \lambda_2 = \begin{cases} \lambda & \text{if } \mu \text{ is non-symmetric,} \\ |\lambda| & \text{if } \mu \text{ is symmetric,} \end{cases}$$

where $|\lambda| = \mathcal{L}(|\theta|)$ if $\lambda = \mathcal{L}(\theta)$. Instead of $|\lambda| \in \mathcal{P}_+$ we can take its symmetrization $\widetilde{\lambda} = \frac{1}{2}|\lambda| + \frac{1}{2}T_{-1}|\lambda|$ since in both cases we have uniqueness of $\lambda_1 \otimes_\mu \lambda_2$. We will consider only the case of weak generalized convolution defined by $\widetilde{\lambda}$, which is more convenient.

It can be shown that, for each weakly stable probability measure μ, weak generalized convolution $\otimes_\mu$ has properties (i)–(iv) on $\mathcal{P}$ but it does not have to satisfy condition (v). A wide discussion on condition (v) for generalized convolution on $\mathcal{P}$ can be found in [6]. In particular, if a weakly stable measure belongs to the domain of attraction of some strictly stable measure, then it generates weak generalized convolution having property (v). Similarly to the generalized convolution theory every weak generalized convolution $\otimes_\mu$ is uniquely determined by a *weak probability kernel*:

$$\rho_{z,1} = \delta_z \otimes_\mu \delta_1,$$

since $\rho_{x,y} = T_v \rho_{z,1}$ for $v = |x| \vee |y|,\ z = (|x| \wedge |y|)/(|x| \vee |y|)$ for all $x, y \in \mathbb{R}$, where $\vee$ and $\wedge$ denote maximum and minimum, respectively. Moreover,

$$\lambda_1 \otimes_\mu \lambda_2(A) = \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}} \rho_{x,y}(A) \lambda_1(dx) \lambda_2(dy)$$

for all $\lambda_1, \lambda_2 \in \mathcal{P}$. It follows that it is sufficient to define $\rho_{z,1}$ for $|z| \leqslant 1$.

EXAMPLE 1.2. The most popular examples of weakly stable distributions are the following:

• symmetric α-stable measure γ_α with the characteristic function $\widehat{\gamma}_\alpha(t) = \exp\{-A|t|^\alpha\}_+$, where A is a constant, for $0 < \alpha \leqslant 2$, defines weak generalized convolution $\otimes_{\gamma_\alpha}$ by

$$\delta_a \otimes_{\gamma_\alpha} \delta_b = \widetilde{\delta}_c,$$

where $|c|^\alpha = |a|^\alpha + |b|^\alpha$, since

$$T_a \gamma_\alpha * T_b \gamma_\alpha = T_c \gamma_\alpha;$$

• probability measure with the characteristic function $\widehat{\mu}_\alpha(t) = (1 - |t|^\alpha)_+$ generates the Kendall convolution $\otimes_{\mu_\alpha}$ $(0 < \alpha \leqslant 1)$:

$$\delta_a \otimes_{\mu_\alpha} \delta_b = \left(1 - \frac{|a|^\alpha}{|b|^\alpha}\right) \widetilde{\delta}_b + \frac{|a|^\alpha}{|b|^\alpha} T_b \widetilde{\pi}_{2\alpha}$$

for $|a| < |b|$, where $\widetilde{\pi}_{2\alpha}$ is the symmetrization of the Pareto distribution with the density $\widetilde{\pi}_{2\alpha}(dx) = \alpha/|x|^{2\alpha+1} \mathbf{1}_{(1,\infty)}(|x|) dx$, since

$$T_a \mu_\alpha * T_b \mu_\alpha = T_c \mu_\alpha \quad \text{for all } a, b \in \mathbb{R};$$

• uniform distribution on the unit sphere $S_{n-1} \subset \mathbb{R}^n$ corresponding to the Kingman convolution;

• every k-dimensional projection of a weakly stable random vector is weakly stable;

• distributions introduced by Cambanis, Keener and Simons in [3].

Notice that we can produce new classes of weak generalized convolutions generated by generalized convolution. For details see [7] or [12]. The weak generalized convolution theory contains many open problems. Jarczyk and Misiewicz [5] consider pseudoisotropic distributions connected with weak stability and solve many open problems from the point of view of functional equations. In [13] Mazurkiewicz describes weakly stable distributions connected with distributions introduced by Cambanis, Keener and Simons. The recipe for weak generalized convolution connected with these distributions is still an open problem.

2. THE MARKOV CHAIN UNDER GENERALIZED AND WEAK GENERALIZED CONVOLUTION

Following paper [1] we consider here a discrete time Lévy process (independent increments random walk) in the sense of generalized convolution. More precisely: we investigate $\{X_n : n \in \mathbb{N}_0\}$, where $X_0 = 0$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, based on the set of i.i.d. random variables $\{\Delta X_k : k \in \mathbb{N}\}$ with distribution $\nu \in \mathcal{P}_+$. The random element X_n is a kind of cumulation of variables $\Delta X_1, \Delta X_2, \ldots, \Delta X_n$ in the following sense:

$$\mathcal{L}(X_n) = \mathcal{L}(\Delta X_1) \diamond \ldots \diamond \mathcal{L}(\Delta X_n) \quad \text{for all } n \geqslant 1.$$

Moreover, we assume that the increment of the process from X_n to X_{n+k} in the sense of generalized convolution depends only on $\Delta X_{n+1}, \ldots, \Delta X_{n+k}$, i.e.

$$\mathcal{L}(X_n) \diamond \mathcal{L}(\Delta X_{n+1}) \diamond \ldots \diamond \mathcal{L}(\Delta X_{n+k}) = \mathcal{L}(X_{n+k}) \quad \text{for all } n, k \geqslant 1.$$

The existence theorem of the process $\{X_n : n \in \mathbb{N}_0\}$ was proved in [1].

THEOREM 2.1. *There exists a Markov process $\{X_n : n \in \mathbb{N}_0\}$ with the transition probabilities*

$$P_{k,n}(x, A) = P(X_n \in A | X_k = x) := \delta_x \diamond \nu^{\diamond n-k}(A),$$

where $x \geqslant 0$, $n, k \in \mathbb{N}$ and $A \in \mathcal{B}\big((0, \infty)\big)$.

On the other hand, we can consider stochastic processes generated by a weakly stable probability measure μ (in the sense of weak generalized convolution $\otimes_\mu$). Then we construct two associated processes $\{\widetilde{X}_n : n \in \mathbb{N}_0\}$ and $\{\widetilde{S}_n : n \in \mathbb{N}_0\}$ such that $\{\widetilde{X}_n : n \in \mathbb{N}_0\}$ is defined as the above process $\{X_n : n \in \mathbb{N}_0\}$ but on the real line. We assume, without loss of generality, that $\widetilde{X}_0 = 0$. This means that the families $\{\Delta\widetilde{X}_k\}$ and $\{\Delta\widetilde{X}_{k,n}\}$ are also specified so that $(\Delta\widetilde{X}_k)_{k \in \mathbb{N}}$ are i.i.d. with distribution $\nu \in \mathcal{P}$ and $\widetilde{X}_k \perp \Delta\widetilde{X}_{k,n}$ for every $k \in \mathbb{N}$ and $n > k$. Additionally, for a sequence $Y, (Y_i)_{i \in \mathbb{N}_0}$ of i.i.d. random variables with weakly stable distribution μ we define

$$\widetilde{S}_n := \sum_{k=1}^{n} (\Delta\widetilde{X}_k \cdot Y_k),$$

where $(Y_i)_{i \in \mathbb{N}_0}$ and $(\Delta\widetilde{X}_i)_{i \in \mathbb{N}}$ are independent. Since $\widetilde{X}_0 = 0$, we have $S_0 = 0$. Notice that $\{\widetilde{S}_n : n \in \mathbb{N}_0\}$ is a discrete time Lévy process in the classical sense such that $\widetilde{S}_n \stackrel{d}{=} \widetilde{X}_n Y$.

In much the same way as in the previous one, for all $k, n \in \mathbb{N},\ k \leqslant n$, and the Borel set $A \in \mathcal{B}(\mathbb{R})$, we see that the transition probabilities are given by

$$\widetilde{P}_{k,n}(x, A) = P(\widetilde{X}_n \in A | \widetilde{X}_k = x) := \delta_x \otimes_\mu \nu^{\otimes_\mu n-k}(A), \quad x \in \mathbb{R}.$$

The proof of the next corollary is basically the same as the proof of Theorem 2.1.

COROLLARY 2.1. *Let μ be a non-trivial weakly stable probability measure. Then $\{\widetilde{X}_n\colon n \in \mathbb{N}_0\}$ is a Markov process.*

Using the *characterizing exponent*

$$\varkappa := \varkappa(\mu) = \sup\left\{p \in [0,2]\colon \int_{\mathbb{R}} |x|^p \mu(dx) < \infty\right\}$$

for the weakly stable distribution μ we can prove the following theorem describing the magnitude of the fluctuations of random walk under weak generalized convolution. The parameter $\varkappa(\mu)$ plays a similar role to the parameter α for α-stable distribution. More information about characterizing exponent can be found in [9].

THEOREM 2.2. *Let μ be a non-trivial weakly stable probability measure with $\varkappa(\mu) > 0$ and let $\{\widetilde{X}_n : n \in \mathbb{N}_0\}$ be the random walk, under weak generalized convolution $\otimes_\mu$, defined above with i.i.d. increments $(\Delta\widetilde{X}_i)_{i\in\mathbb{N}}$ with distribution ν. Let $Y, Y_1, Y_2, \ldots$ be an i.i.d. sequence of random variables with distribution μ, which is independent of $(\Delta\widetilde{X}_i)_{i\in\mathbb{N}}$. If there exist sequences of positive numbers $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ such that $a_n \nearrow \varkappa,\ b_n \to \infty$ and*

$$\limsup_{n\to\infty} b_n^{-1}\mathbb{E}(|\widetilde{S}_n|^{a_n}) = c \in (0,\infty),$$

then for every sequence of positive numbers $(c_n)_{n\in\mathbb{N}}$ such that

$$\sum_{n=1}^{\infty} c_n^{-1} < \infty,$$

we have

$$\mathbf{P}\left(\bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty}\left\{|\widetilde{X}_k| \leqslant \left(\frac{c_k b_k}{d_k}\right)^{1/a_k}\right\}\right) = 1,$$

where $d_n = \mathbb{E}|Y|^{a_n}$.

Proof. Let $A_n = \{|\widetilde{X}_n|^{a_n} > c_n b_n/d_n\}$ and $a_n \nearrow \varkappa,\ b_n \to \infty$. Since $\widetilde{S}_n := \sum_{k=1}^n (\Delta\widetilde{X}_k \cdot Y_k) \stackrel{d}{=} \widetilde{X}_n Y$ for $\widetilde{X}_n,\ Y$ independent, we have

$$\mathbb{E}(|\widetilde{S}_n|^{a_n}) = \mathbb{E}(|\widetilde{X}_n|^{a_n})\mathbb{E}(|Y|^{a_n}).$$

By the Tchebyshev inequality there exists $n_0 \in \mathbb{N}_0$ such that

$$\mathbf{P}(A_n) \leqslant \frac{d_n\mathbb{E}(|\widetilde{X}_n|^{a_n})}{b_n c_n} = \frac{\mathbb{E}(|\widetilde{S}_n|^{a_n})}{b_n c_n} \leqslant \frac{c}{c_n}$$

for every $n \geqslant n_0$. Since $\sum_{n=1}^\infty c_n^{-1} < \infty$, we arrive at $\sum_{n=1}^\infty \mathbf{P}(A_n) < \infty$. Now it is sufficient to apply the Borel–Cantelli lemma. ∎

3. RANDOM WALK UNDER THE KENDALL GENERALIZED CONVOLUTION

In 2011 in [8] the authors showed that the Kendall convolution for δ_x and δ_1 is the unique generalized convolution which can be written as a convex linear combination of two fixed probability measures such that only coefficients of this combination depend on x. Moreover, it can be shown (see Theorem 1 in [8]) that if $0 < \alpha \leqslant 1$, then the Kendall convolution is a weak generalized convolution with respect to a symmetric weakly stable measure μ_α with the density function

$$\mu_\alpha(dy) = \frac{\alpha}{\pi y} \int_0^1 \sin(ty) t^{\alpha-1} dt dy$$

and the characteristic function $\widehat{\mu}_\alpha(t) = (1 - |t|^\alpha)_+$.

In [14] one can find connections between the Kendall convolution and the Archimedean copulas theory, i.e. a generator of Archimedean copula is the homomorphism of the Kendall convolution. It follows that we can also use the Williamson transform (see [24]) to get our results.

For a random variable $X \sim \nu$ with cumulative distribution function F, the classical Williamson transform is defined by

$$\mathcal{M}_d F(t) = \int_t^\infty \left(1 - \frac{t}{x}\right)^{d-1} dF(x) = \begin{cases} \mathbb{E}(1 - t/X)_+^{d-1} & \text{if } t > 0, \\ 1 - F(0) & \text{if } t = 0, \end{cases}$$

where $d \geqslant 2$ is an integer.

In this paper we investigate the random walks under the Kendall convolution using the technique of homomorphism (respectively, characteristic function) for given generalized convolution (respectively, weak generalized convolution), which is strictly connected with a modification of the Williamson transform. To see this notice that the homomorphism

$$h(\delta_t) = (1 - t)_+$$

is the Williamson transform for probability measure δ_1 and $d = 2$.

We use a modification of the Williamson transform given by

$$\begin{aligned} \Phi_\nu(t) = h(T_t \nu) &= \int_0^\infty \big(1 - (ts)^\alpha\big)_+ \nu(ds) \\ &= F\left(\frac{1}{t}\right) - t^\alpha \int_0^{1/t} s^\alpha \nu(ds) = \alpha t^\alpha \int_0^{1/t} s^{\alpha-1} F(s)(ds), \end{aligned}$$

which is easy to invert. Since

$$\alpha^{-1} t^\alpha \Phi_\nu\left(\frac{1}{t}\right) = \int_0^t s^{\alpha-1} F(s)(ds),$$

we have

$$F(t) = \Phi_\nu\left(\frac{1}{t}\right) + \alpha^{-1} t\,\frac{d}{dt}\left[\Phi_\nu\left(\frac{1}{t}\right)\right].$$

This section is devoted to the random walk under the Kendall convolution $\triangle_\alpha$, $\alpha > 0$, with unit step with distribution ν concentrated on the positive half line. Next, we extend our construction to obtain the random walk under the weak Kendall convolution with unit steps having distribution on the real line. In particular, we consider a random walk under the Kendall convolution in case $\nu = \delta_1$ or $\nu = \widetilde{\delta_1}$ under the weak Kendall convolution.

Let $(\Delta X_i)_{i\in\mathbb{N}}$ be a sequence of i.i.d. random variables with distribution $\nu \in \mathcal{P}_+$ and let $(\theta_i)_{i\in\mathbb{N}_0}$ be a sequence of i.i.d. random variables with the Pareto distribution $\pi_{2\alpha}$ such that θ_n is independent of X_n and ΔX_{n+1} for all $n \in \mathbb{N}$. The Markov chain $\{X_n : n \in \mathbb{N}_0\}$ with $\Delta X_1 \sim \nu$ is such that

$$\lambda_{0,n,\alpha}(\nu) := \mathcal{L}(X_n) = \nu^{\triangle_\alpha n}$$

for $n \in \mathbb{N}_0$ and $\nu^{\triangle_\alpha 0} = \delta_0$.

Our construction of random walk implies that X_n and $\Delta X_{n,n+k}$ are independent and $X_k \stackrel{d}{=} \Delta X_{n,n+k}$ for all $n, k \geqslant 0$.

Without loss of generality we can assume that we start from zero, i.e. $X_0 = 0$ a.e. It remains to find the measures $\lambda_{0,n,\alpha}(\nu)$ for $n \geqslant 2$.

PROPOSITION 3.1. *Let $\nu \in \mathcal{P}_+$. For each natural number $n \geqslant 2$ and $\alpha > 0$ we have*

$$\lambda_{0,n,\alpha}(\nu)(0,x) = \Phi_\nu^n\left(\frac{1}{x}\right) + \alpha^{-1} x \frac{d}{dx}\left(\Phi_\nu^n\left(\frac{1}{x}\right)\right)$$

or, *equivalently*,

$$\lambda_{0,n,\alpha}(\nu)(0,x) = \frac{d}{ds}\left(s\Phi_\nu^n(s^{-1/\alpha})\right)\Big|_{s=x^\alpha},$$

where

$$\Phi_\nu(t) = h(T_t\nu) = \int_0^\infty \left(1-(xt)^\alpha\right)_+ \nu(dx)$$

is the homomorphism of the unit step variable $\triangle X_1$.

P r o o f. Since μ_α is weakly stable, we have

$$(\mu_\alpha \circ \nu)^{*n} = \mu_\alpha \circ \nu^{\otimes_{\mu_\alpha} n}.$$

It follows that

$$\Phi_\nu^n(t) = \int_0^\infty \left(1-(xt)^\alpha\right)_+ \lambda_{0,n,\alpha}(\nu)(dx),$$

i.e.

$$\Phi_{\nu}^{n}(t) = \Phi_{\nu^{\triangle_{\alpha} n}}(t).$$

Consequently, if we convert the transform, we get

$$\lambda_{0,n,\alpha}(\nu)(0,x) = \Phi_{\nu}^{n}\left(\frac{1}{x}\right) + \alpha^{-1}x\frac{d}{dx}\left(\Phi_{\nu}^{n}\left(\frac{1}{x}\right)\right).$$

In order to obtain an equivalent equation it is sufficient to calculate the derivative. ■

COROLLARY 3.1. *Let $\beta_{a,b}$ be the beta distribution with parameters $a, b > 0$, i.e. the distribution with the density function*

$$\beta_{a,b}(dx) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}x^{a-1}(1-x)^{b-1}\mathbf{1}_{(0,1)}(x)dx.$$

Then for each natural number $n \geqslant 2$ and $\alpha > 0$ the distribution function of the measure $\lambda_{0,n,\alpha}(\beta_{a,b})$ is given by

$$\frac{d}{ds}\left[s\left(B(a,b,s^{1/\alpha}) - \frac{\Gamma(a+\alpha)\Gamma(a+b)}{s\Gamma(a)\Gamma(a+b+\alpha)}\,B(a+\alpha,b,s^{1/\alpha})\right)^{n}\right]\Bigg|_{s=x^{\alpha}},$$

where

$$B(a,b,s) = \int_{0}^{s}\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\,x^{a-1}(1-x)^{b-1}dx.$$

In particular, for the uniform distribution $\mathcal{U}(0,1)$ we have

$$\begin{aligned}\big(\lambda_{0,n,\alpha}(\beta_{1,1})\big)\big([0,x)\big) &= \left(\frac{\alpha}{\alpha+1}\right)^{n}\left(1+\frac{n}{\alpha}\right)x^{n}\mathbf{1}_{[0,1)}(x)\\ &\quad + \left(1-\frac{1}{(\alpha+1)x^{\alpha}}\right)^{n-1}\left(1+\frac{n-1}{(\alpha+1)x^{\alpha}}\right)\mathbf{1}_{[1,\infty)}(x).\end{aligned}$$

COROLLARY 3.2. *Let $\gamma_{a,b}$ be the gamma distribution with parameters $a, b > 0$ with the density*

$$\gamma_{a,b}(dx) = \frac{b^{a}}{\Gamma(a)}x^{a-1}e^{-bx}\mathbf{1}_{(0,\infty)}(x)dx.$$

Then for each natural number $n \geqslant 2$ and $\alpha > 0$ the measure $\lambda_{0,n,\alpha}(\gamma_{a,b})$ has the cumulative distribution function

$$\frac{d}{ds}\left[s\left(\Gamma(a,b,s^{1/\alpha}) - \frac{\Gamma(a+\alpha)}{s\Gamma(a)b^{\alpha}}\Gamma(a+\alpha,b,s^{1/\alpha})\right)^{n}\right]\Bigg|_{s=x^{\alpha}},$$

where

$$\Gamma(a,b,s) = \int_{0}^{s}\frac{b^{a}}{\Gamma(a)}x^{a-1}e^{-bx}dx.$$

For $\nu = \delta_1$ it is clear that $P(X_1 = 1) = 1$ and $X_2 \stackrel{d}{=} \theta_1$.

COROLLARY 3.3. *For each natural number $n \geqslant 2$ and real number $\alpha > 0$*

$$\big(\lambda_{0,n,\alpha}(\delta_1)\big)(dx) = \frac{\alpha n(n-1)}{x^{2\alpha+1}} \left(1 - \frac{1}{x^\alpha}\right)^{n-2} \mathbf{1}_{[1,\infty)}(x)dx.$$

Notice that

$$\lambda_{0,2,\alpha}(\delta_1) = \pi_{2\alpha}$$

is the Pareto distribution and by weak stability of μ_α we have

$$\mu_\alpha * \mu_\alpha = \mu_\alpha \circ \pi_{2\alpha}.$$

In the terms of transforms we arrive at

$$\Phi_{\pi_{2\alpha}}(t)^n = \Phi_{\lambda_{0,2n,\alpha}(\delta_1)}(t).$$

It means that we are able to cumulate the Pareto distributions in the Kendall convolution algebra.

Now we construct random walk under the Kendall convolution with the unit step $\triangle X_1$ with distribution δ_1.

THEOREM 3.1. *The Markov process $\{X_n : n \in \mathbb{N}_0\}$, with $X_0 = 0$ and $X_1 \sim \delta_1$, based on the Kendall convolution $\triangle_\alpha$, has the following properties*:

$$X_2 = \theta_1,\ X_{n+1} = X_n \cdot \theta_n^{Q_n} \ \textit{a.e.},$$

where θ_n is independent of X_n for $n \geqslant 1$ and

$$P(Q_n = k|X_n) = \begin{cases} 1/X_n^\alpha & \textit{for } k = 1, \\ 1 - 1/X_n^\alpha & \textit{for } k = 0. \end{cases}$$

Moreover,

$$\begin{aligned} P_{n-1,n}(x, A) &:= P(X_n \in A|X_{n-1} = x) \\ &= \frac{1}{x^\alpha} P(x\theta_{n-2} \in A) + \left(1 - \frac{1}{x^\alpha}\right) \mathbf{1}_A(x) \end{aligned}$$

for every Borel set $A \subseteq [0, \infty)$.

P r o o f. By construction the transition probabilities are given by

$$\begin{aligned} P(X_n \in A|X_{n-1} = x) &= \delta_x \diamond \delta_1^{\diamond n-k}(A) \\ &= \frac{1}{x^\alpha} P(x\theta_{n-1} \in A) + \left(1 - \frac{1}{x^\alpha}\right) \mathbf{1}_A(x). \end{aligned}$$

We see that

$$
\begin{aligned}
\big(\lambda_{0,n,\alpha}(\delta_1)\big)(A) &= \int_1^\infty (\delta_x \triangle_\alpha \delta_1)(A)\big(\lambda_{0,n-1,\alpha}(\delta_1)\big)(dx)\\
&= \int_1^\infty \left(T_x\left(\frac{1}{x^\alpha}\,\delta_1\triangle_\alpha\delta_1 + \left(1-\frac{1}{x^\alpha}\right)\delta_1\right)\right)(A)\big(\lambda_{0,n-1,\alpha}(\delta_1)\big)(dx)\\
&= \int_1^\infty \left(\frac{1}{x^\alpha}P(x\theta_{n-1}\in A) + \left(1-\frac{1}{x^\alpha}\right)\mathbf{1}_A(x)\right)\big(\lambda_{0,n-1,\alpha}(\delta_1)\big)(dx)
\end{aligned}
$$

for all $A \in \mathcal{B}\big((0,\infty)\big)$ and $n \geqslant 2$. By Corollary 5.11 in [10] we can find sequences $\{\theta_n\}$ and $\{Q_n\}$ such that

$$X_{n+1} = X_n \cdot \theta_n^{Q_n} \text{ a.e.}$$

Finally, X_n has distribution $\lambda_{0,n,\alpha}(\delta_1)$. ∎

In the next theorem we give generalization of Theorem 3.1 for ΔX_1 with any distribution ν concentrated on the positive half line.

THEOREM 3.2. *The Markov process* $\{X_n : n \in \mathbb{N}\}$ *with* $\Delta X_1 \sim \nu \in \mathcal{P}_+$ *based on the Kendall convolution* $\triangle_\alpha$, $\alpha > 0$, *has the following properties*:

$$X_0 = 0,\ X_1 = \Delta X_1,\ X_{n+1} = (X_n \vee \triangle X_{n+1})\cdot \theta_n^{Q_n} \textit{ a.e.},$$

where θ_n *is independent of* $(X_n \vee \triangle X_{n+1})$ *for* $n \geqslant 1$ *and*

$$P(Q_n = k | X_n, \triangle X_{n+1}) = \begin{cases} z(X_n,\triangle X_{n+1})^\alpha & \textit{for } k=1,\\ 1 - z(X_n,\triangle X_{n+1})^\alpha & \textit{for } k=0\end{cases}$$

for $z(x,y) = (x\wedge y)/(x\vee y)$. *Moreover*,

$$
\begin{aligned}
P_{n-1,n}(x,y,A) &:= P(X_n \in A | X_{n-1} = x, \triangle X_n = y)\\
&= z(x,y)^\alpha P\big(v(x,y)\cdot\theta_{n-2}\in A\big) + \big(1 - z(x,y)^\alpha\big)\mathbf{1}_A\big(v(x,y)\big)
\end{aligned}
$$

for $v(x,y) = x \vee y$ *and every Borel set* $A \subseteq [0,\infty)$.

P r o o f. As in the proof of Theorem 3.1 we can show that

$$
\begin{aligned}
\big(\lambda_{0,n,\alpha}(\nu)\big)(A) &= \Big(\big(\lambda_{0,n-1,\alpha}(\nu)\big)\triangle_\alpha \nu\Big)(A)\\
&= \int_0^\infty\int_0^\infty (\delta_x\triangle_\alpha\delta_y)(A)\big(\lambda_{0,n-1,\alpha}(\nu)\big)(dx)\nu(dy).
\end{aligned}
$$

Then, substituting $v(x,y)=v$ and $z(x,y)=z$, we obtain

$$\int_0^\infty\int_0^\infty \Big(T_v\big(z^\alpha\left(\delta_1 \triangle_\alpha \delta_z\right)+\left(1-z^\alpha\right)\delta_1\big)\Big)(A)\big(\lambda_{0,n-1,\alpha}(\nu)\big)(dx)\nu(dy)$$
$$=\int_0^\infty\int_0^\infty \Big(z^\alpha P(v\theta_{n-1}\in A)+\left(1-z^\alpha\right)\mathbf{1}_A\big(v(x,y)\big)\Big)\big(\lambda_{0,n-1,\alpha}(\nu)\big)(dx)\nu(dy)$$

for all $A\in\mathcal{B}\big((0,\infty)\big)$ and $n\geqslant 2$. It yields, by Corollary 5.11 in [10],

$$X_{n+1}=v\left(X_n,\triangle X_{n+1}\right)\cdot\theta_n^{Q_n}\ \text{a.e.}$$

for Q_n with desired distribution, the proper sequence $\{\theta_n\}$, and X_n has distribution $\lambda_{0,n,\alpha}(\nu)$. ■

One would think that the Markov chain $\{X_n : n\in\mathbb{N}_0\}$ with respect to a generalized convolution $\diamond$, with unit step distribution $\nu\in\mathcal{P}_+$, is a kind of the Lévy process in the classical sense. The following two propositions give a negative answer to such a hypothesis.

PROPOSITION 3.2. *Let $\triangle_1$ be the Kendall convolution and $\{X_n : n\in\mathbb{N}\}$ be a random walk under $\triangle_1$ with unit step distribution δ_1. Then*

$$P(X_{k+1}-X_k<w)=1-\frac{2}{k+1}E\left(\left(1+wY\right)^{-2}\right)$$

for every $k\in\mathbb{N}$, where Y has distribution $\beta_{3,k-1}$, which means that the increments of this chain are not stationary in time.

P r o o f. Notice that, by constructing the random walk given in Theorem 3.2, we have

$$P(X_{k+1}=X_k)=\int_1^\infty P(Q_k=0|X_k=s)\big(\lambda_{0,k,\alpha}(\delta_1)\big)(ds)=\frac{k-1}{k+1}.$$

The continuous part of the distribution of (X_k,X_{k+1}) has the weight $2/(k+1)$ and the density

$$f(u,v)=\frac{(k+1)k(k-1)}{u^2v^3}\left(1-\frac{1}{u}\right)^{k-2}\mathbf{1}_{\{1\leqslant u\leqslant v\}}.$$

Since $X_1\sim\delta_1$ and

$$P(X_{k+1}-X_k<w)=\frac{k-1}{k+1}+\int_1^\infty\int_u^{u+w} f(u,v)dvdu$$
$$=1-k(k-1)\int_1^\infty\frac{(u-1)^{k-2}}{u^k\,(u+w)^2}\,du,$$

the Markov process $\{X_n : n\in\mathbb{N}_0\}$ does not have stationary increments. ■

PROPOSITION 3.3. *The increments of the Markov chain $\{X_n : n \in \mathbb{N}\}$ with respect to the Kendall convolution $\triangle_1$, with unit step δ_1, are not independent. In particular, for each $k \in \mathbb{N}$ the random variables X_k and $X_{k+1} - X_k$ are not independent.*

P r o o f. By simple computation we obtain

$$\begin{aligned} &P(X_{k+1} - X_k < w, X_k < z) = \\ &= P(X_{k+1} - X_k < w, X_k < z | X_{k+1} = X_k) P(X_{k+1} = X_k) \\ &\quad + P(X_{k+1} - X_k < w, X_k < z, X_{k+1} > X_k) \\ &= \frac{k-1}{k+1} \big(\lambda_{0,k,\alpha}(\delta_1)\big)\big([0, z)\big) + \frac{2}{k+1}[1 - B(3, k-1, z^{-1})] \\ &\quad - \frac{2}{k+1} \int_{1/z}^{1} \frac{k(k-1)}{(1+wy)^2}(1-y)^{k-2} dy = \frac{k-1}{k+1}\big(\lambda_{0,k,\alpha}(\delta_1)\big)\big([0, z)\big) \\ &\quad + \frac{2}{k+1}[1 - B(3, k-1, z^{-1})] - \frac{2}{k+1} E\left(\frac{1}{(1+wY)^2}\mathbf{1}_{\{\omega : Y(\omega) > 1/w\}}\right), \end{aligned}$$

where $Y \sim \beta_{3,k-1}$, and $B(3, k-1, z^{-1})$ is the function given in Corollary 3.1. In particular,

$$\begin{aligned} &P(X_3 - X_2 < w, X_2 < z) = 1 - \frac{1}{3z^2}\left(1 + \frac{2}{z}\right) \\ &\quad - \frac{2}{w^3}\left[\ln\left(\frac{w+z}{z(1+w)}\right)^2 + \frac{w(z-1)}{z(w+z)(1+w)}\big(z - (w+z)(1+w)\big)\right] \end{aligned}$$

for $w > 0, z > 1$ and $P(X_2 < z) = 1 - 1/z^2$ for $z > 1$. Moreover,

$$P(X_3 - X_2 < w) = \frac{2}{3} - 2\left[\frac{1}{w^2} - \frac{w}{1+w} + \frac{3}{w^2(1+w)^2} - \frac{1}{w^3}\ln(1+w)^2\right]$$

for $w > 0$, which implies that X_2 and $X_3 - X_2$ are not independent. ■

Notice that in the same manner we can construct a family of random walks $\{Z_n^{(k)} : n \in \mathbb{N}\}$ under the Kendall convolution such that

$$Z_n^{(k)} := X_{kn}$$

for every $k \in \mathbb{N}$. For every fixed k the unit step of this random walk has distribution $\lambda_{0,k,\alpha}(\nu)$ given in Proposition 3.1. In particular, one can prove that probability measures which belong to the family $\{\lambda_{0,k,\alpha}(\delta_1) : k \in \mathbb{N}\}$ are heavy tailed and $\lambda_{0,2,\alpha}(\delta_1) = \pi_{2\alpha}$.

In a similar way we construct random walk under the Kendall weak generalized convolution $\otimes_{\mu_\alpha}$, where $\alpha \in (0, 1]$. Let $Y, (Y_i)_{i\in\mathbb{N}_0}$ be a sequence of i.i.d.

random variables with weakly stable distribution μ_α, $(\Delta\widetilde{X}_i)_{i\in\mathbb{N}_0}$ be a sequence of i.i.d. random variables with distribution ν which are concentrated on the real line. Let $(\widetilde{\theta}_i)_{i\in\mathbb{N}_0}$ be a sequence of i.i.d. random variables with distribution $\widetilde{\pi}_{2\alpha}$ such that $\widetilde{\theta}_n$ is independent of $\widetilde{X}_n$ and $\Delta\widetilde{X}_{n+1}$ for all $n\in\mathbb{N}$. Moreover, let the sequences $Y,(Y_i)_{i\in\mathbb{N}_0}$ and $(\widetilde{\theta}_i)_{i\in\mathbb{N}_0}$ be independent. Then $\widetilde{X}_n$ means the position of moving particle at the n-th step such that the unit step has distribution ν and $\widetilde{X}_n$ has distribution being symmetrization of $\lambda_{0,n,\alpha}(\nu)$, i.e.

$$\widetilde{\lambda}_{0,n,\alpha}(\nu) := \nu^{\otimes_{\mu_\alpha} n},$$

where $\widetilde{\lambda}_{0,n,\alpha}(\nu)$ is the probability measure on the real line and $\widetilde{\lambda}_{0,0,\alpha}(\nu)=\delta_0$. Just as in the case of the random walk under the Kendall generalized convolution we can get the series of dual results for random walk with respect to $\otimes_{\mu_\alpha}$:

LEMMA 3.1. *For each natural number $n\geqslant 2$ and $\alpha\in(0,1]$ the probability measure $\widetilde{\lambda}_{0,n,\alpha}(\widetilde{\delta}_1)$ has the density*

$$\big(\widetilde{\lambda}_{0,n,\alpha}(\widetilde{\delta}_1)\big)(dx) = \frac{\alpha n(n-1)}{2|x|^{2\alpha+1}}\left(1-\frac{1}{|x|^\alpha}\right)^{n-2}\mathbf{1}_{[1,\infty)}(|x|)dx.$$

The above lemma is a modification of Corollary 3.3 except that here we take the characteristic function of μ_α as the kernel of the corresponding homomorphism. In the next theorem we construct the Markov process with distribution $\widetilde{\lambda}_{0,n,\alpha}(\nu)$ at the n-th step.

THEOREM 3.3. *The Markov process $\{\widetilde{X}_n : n\in\mathbb{N}_0\}$ with $\triangle\widetilde{X}_1\sim\nu$ has the following properties*:

$$\widetilde{X}_0=0,\ \widetilde{X}_1=\Delta\widetilde{X}_1,\ \widetilde{X}_{n+1}=v(|\widetilde{X}_n|,|\triangle\widetilde{X}_{n+1}|)\cdot u(\widetilde{X}_n,\triangle\widetilde{X}_{n+1})\cdot\widetilde{\theta}_n^{\widetilde{Q}_n}\ \textit{a.e.}$$

for $n\geqslant 1$, where θ_n is independent of $v(|\widetilde{X}_n|,|\triangle\widetilde{X}_{n+1}|)\cdot u(\widetilde{X}_n,\triangle\widetilde{X}_{n+1})$, $v(x,y)=x\vee y,\ z(x,y)=(x\wedge y)/(x\vee y)$,

$$u(x,y)=\begin{cases}\operatorname{sgn}(x) & \textit{for } |x|\geqslant|y|,\\ \operatorname{sgn}(y) & \textit{for } |y|\geqslant|x|,\end{cases}$$

and

$$P(\widetilde{Q}_n=k|\widetilde{X}_n,\triangle\widetilde{X}_{n+1})=\begin{cases}\big(z(|\widetilde{X}_n|,|\triangle\widetilde{X}_{n+1}|)\big)^\alpha & \textit{for } k=1,\\ 1-\big(z(|\widetilde{X}_n|,|\triangle\widetilde{X}_{n+1}|)\big)^\alpha & \textit{for } k=0.\end{cases}$$

Moreover, $\widetilde{X}_n$ has distribution $\widetilde{\lambda}_{0,n,\alpha}(\nu)$ for every $n\in\mathbb{N}$ and

$$\begin{aligned}P_{n-1,n}(x,y,A) &:= P(\widetilde{X}_n\in A|\widetilde{X}_{n-1}=x,\triangle\widetilde{X}_n=y)\\ &= \big(z(|x|,|y|)\big)^\alpha P\big(u(x,y)\cdot v(|x|,|y|)\cdot\widetilde{\theta}_{n-2}\in A\big)\\ &\quad+\frac{1}{2}\Big(1-\big(z(|x|,|y|)\big)^\alpha\Big)\mathbf{1}_{\widetilde{A}}\big(u(x,y)\cdot v(|x|,|y|)\big)\end{aligned}$$

for every Borel set $A\subseteq\mathbb{R}$, where $\widetilde{A}$ is symmetrization of A.

P r o o f. In order to prove this theorem it is sufficient to follow the proof of Theorem 3.2, substituting $\otimes_{\mu_\alpha}$ instead of $\triangle_\alpha$ and integrating over $\mathbb{R}$ but using the probability kernel of the weak Kendall convolution. It is easy to see that for every Borel set $A \in \mathcal{B}(\mathbb{R})$ we have

$$\big(\widetilde{\lambda}_{0,n,\alpha}(\nu)\big)(A) = \int_{\mathbb{R}}\int_{\mathbb{R}} \Big(\big(z(|x|,|y|)\big)^{\alpha} P\big(u(x,y)\cdot v(|x|,|y|)\cdot \widetilde{\theta}_{n-2} \in A\big)$$
$$+ \Big(1 - \big(z(|x|,|y|)\big)^{\alpha}\Big)\widetilde{\delta}_A\big(u(x,y)\cdot v(|x|,|y|)\big)\Big)\nu(dy)\big(\widetilde{\lambda}_{0,n-1,\alpha}(\nu)\big)(dx). \quad \blacksquare$$

In particular, for $\nu = \widetilde{\delta}_1$ we have the following construction.

COROLLARY 3.4. *The Markov process* $\{\widetilde{X}_n : n \in \mathbb{N}\}$ *with* $\triangle\widetilde{X}_1 \sim \widetilde{\delta}_1$ *has the following properties*:

$$\widetilde{X}_0 = 0,\ \widetilde{X}_1 = \Delta\widetilde{X}_1,\ \widetilde{X}_2 = \widetilde{\theta}_1,\ \widetilde{X}_{n+1} = \widetilde{X}_n \cdot \widetilde{\theta}_n^{\widetilde{Q}_n} \text{ a.e.}$$

for $n \geqslant 2$, *where* θ_n *is independent of* $\widetilde{X}_n$ *and*

$$P(\widetilde{Q}_n = k|\widetilde{X}_n) = \begin{cases} 1/|\widetilde{X}_n|^{\alpha} & \text{for } k = 1,\\ 1 - 1/|\widetilde{X}_n|^{\alpha} & \text{for } k = 0.\end{cases}$$

Moreover, $\widetilde{X}_n$ *has distribution* $\widetilde{\lambda}_{0,n,\alpha}(\widetilde{\delta}_1)$ *for every* $n \in \mathbb{N}$ *and*

$$\begin{aligned}\widetilde{P}_{n-1,n}(x,A) &:= P(\widetilde{X}_n \in A|\widetilde{X}_{n-1} = x)\\ &= \frac{1}{|x|^{\alpha}}P(x\widetilde{\theta}_{n-2} \in A) + \frac{1}{2}\left(1 - \frac{1}{|x|^{\alpha}}\right)\mathbf{1}_{\widetilde{A}}(x)\end{aligned}$$

for every Borel set $A \in \mathcal{B}(\mathbb{R})$, *where* $\widetilde{A}$ *is symmetrization of* A.

Notice that using the sequence $(Y_i)_{i\in\mathbb{N}}$ we have constructed also a random walk in the usual sense $\{\widetilde{S}_n : n \in \mathbb{N}_0\}$ associated with the random walk $\{\widetilde{X}_n : n \in \mathbb{N}_0\}$ under the weak Kendall convolution $\otimes\mu_\alpha$ with unit step $\triangle\widetilde{X}_1 \sim \widetilde{\delta}_1$. The unit step $\widetilde{S}_1$ has distribution μ_α and

$$\widetilde{S}_0 = 0,\ \widetilde{S}_n = Y_1 + Y_2 + \ldots + Y_n \text{ a.e.}$$

The relation between $\{\widetilde{S}_n : n \in \mathbb{N}_0\}$ and $\{\widetilde{X}_n : n \in \mathbb{N}_0\}$ is given by the following distribution equation:

$$\widetilde{S}_n \stackrel{d}{=} Y\widetilde{X}_n, \quad \text{where } Y \sim \mu_\alpha,$$

for every $n \in \mathbb{N}_0$.

In the next few lemmas we give some properties for the random walk $\{\widetilde{S}_n : n \in \mathbb{N}_0\}$ associated with $\{\widetilde{X}_n : n \in \mathbb{N}_0\}$, where $\triangle\widetilde{X}_1 \sim \widetilde{\delta}_1$. In particular, we have the following relation between μ_α and μ_1.

LEMMA 3.2. *Let $0 < \alpha \leqslant 1$ and let μ_α be a weakly stable probability measure which induces the Kendall convolution. Then*

$$\mu_\alpha = \mu_1 \circ \big(\alpha\widetilde{\delta}_1 + (1-\alpha)\widetilde{\pi}_\alpha\big).$$

P r o o f. In order to find a probability measure $\nu \in \mathcal{P}$ such that $\mu_\alpha = \mu_1 \circ \nu$ we have to solve the integral equation

$$(1 - |t|^\alpha)_+ = \int_{\mathbb{R}} (1 - |ts|)_+ \, \nu(ds).$$

This equation can be solved in the same manner as in the proof of Proposition 3.1, however, it would be much simpler to check that for the measure $\nu = \alpha\widetilde{\delta}_1 + (1-\alpha)\widetilde{\pi}_\alpha$ the desired equality holds. ∎

PROPOSITION 3.4. *Let $0 < \alpha \leqslant 1$ and let $(Y_i)_{i\in\mathbb{N}_0}$ be the sequence of i.i.d. random variables with distribution μ_α. Then $\widetilde{S}_n = Y_1 + Y_2 + \ldots + Y_n$ has the distribution*

$$\mu_\alpha^{*n} = \Big(\mu_1 \circ \big(\alpha\widetilde{\delta}_1 + (1-\alpha)\widetilde{\pi}_\alpha\big)\Big)^{*n} = \mu_1 \circ \big(\alpha\widetilde{\delta}_1 + (1-\alpha)\widetilde{\pi}_\alpha\big)^{\otimes_{\mu_1} n},$$

where

$$\begin{aligned}\big(\alpha\widetilde{\delta}_1 &+ (1-\alpha)\widetilde{\pi}_\alpha\big)^{\otimes_{\mu_1} n}(dx) \\ &= \frac{\alpha n}{2|x|^{\alpha+1}} \left(1 - \frac{1}{|x|^\alpha}\right)^{n-2} \left[1 - \alpha + \frac{\alpha n - 1}{|x|^\alpha}\right] \mathbf{1}_{(1,\infty)}(|x|)\, dx.\end{aligned}$$

P r o o f. By Lemma 3.2 it follows that the characteristic function of the measure $\mu_1 \circ \big(\alpha\widetilde{\delta}_1 + (1-\alpha)\widetilde{\pi}_\alpha\big)$ is given by the formula

$$G(1/t) := \alpha\widehat{\mu}_1(t) + (1-\alpha)\widehat{\mu_1 \circ \pi_\alpha}(t) = \widehat{\mu}_\alpha(t).$$

Since

$$\big(G(1/t)\big)^n = \int_0^\infty (1 - tx)_+ F_n(dx) = t\int_0^{1/t} F_n(x)dx,$$

where F_n denotes the distribution function of $\big(\alpha\widetilde{\delta}_1 + (1-\alpha)\widetilde{\pi}_\alpha\big)^{\otimes_{\mu_1} n}$, so substituting $x := 1/t$, we get

$$F_n(x) = \frac{d}{dx}\left[xG(x)^n\right],$$

which leads to the explicit formula for F_n. ∎

The above proposition says that, by weak stability of μ_α, $\alpha \in (0,1]$, we can consider the random walk under the Kendall convolution $\otimes_{\mu_1}$ with unit step $\triangle X_1 \sim \alpha\widetilde{\delta}_1 + (1-\alpha)\widetilde{\pi}_\alpha$ instead of the random walk under $\otimes_{\mu_\alpha}$ with unit step with distribution $\widetilde{\delta}_1$. By this property we see that random walk in the usual sense with unit step μ_α is also associated with the random walk under the Kendall convolution $\otimes_{\mu_1}$. We have a similar property also for random walk with unit step with distribution $\nu \in \mathcal{P}$.

REMARK 3.1. *Let $\nu \in \mathcal{P}$ and $\alpha \in (0,1]$. By Lemma 3.2 and weak stability of measures μ_α we have*

$$\begin{aligned}
&\mu_1 \circ \big(\alpha\widetilde{\delta}_1 + (1-\alpha)\widetilde{\pi}_\alpha\big) \circ \nu^{\otimes_{\mu_\alpha} n} \\
&= \Big(\mu_1 \circ \big(\alpha\widetilde{\delta}_1 + (1-\alpha)\widetilde{\pi}_\alpha\big) \circ \nu\Big)^{*n} = \mu_1 \circ \Big(\big(\alpha\widetilde{\delta}_1 + (1-\alpha)\widetilde{\pi}_\alpha\big) \circ \nu\Big)^{\otimes_{\mu_1} n},
\end{aligned}$$

which implies that

$$\big(\alpha\widetilde{\delta}_1 + (1-\alpha)\widetilde{\pi}_\alpha\big) \circ \nu^{\otimes_{\mu_\alpha} n} = \Big(\big(\alpha\widetilde{\delta}_1 + (1-\alpha)\widetilde{\pi}_\alpha\big) \circ \nu\Big)^{\otimes_{\mu_1} n}.$$

It is worth noticing that for the random walk with unit step μ_1 we have the following recurrence relation.

LEMMA 3.3. *For every natural number $n \geqslant 3$ we have*

$$\begin{aligned}
\mu_1(dx) &= \frac{1}{\pi x^2}\,(1-\cos x)\,dx,\\
\mu_1^{*n}(dx) &= \frac{n}{\pi x^2}dx - \frac{(n-1)n}{\pi x^2}\mu_1^{*n-2}(dx).
\end{aligned}$$

P r o o f. Denoting by g_n the density of the measure μ_1^{*n} and applying the Fourier inverse transform, we obtain

$$g_1(x) = \frac{1}{2\pi}\int_0^1 \cos(tx)(1-t)dt \quad \text{and} \quad g_n(x) = \frac{1}{\pi}\int_0^1 \cos(tx)(1-t)^n dt.$$

Now it is sufficient to use integration by parts. ∎

LEMMA 3.4. *Let $0 < \alpha \leqslant 1$ and let $\otimes_{\mu_\alpha}$ be the weak Kendall convolution. Then the random walk $\{\widetilde{X}_n : n \in \mathbb{N}\}$ with unit step $\triangle\widetilde{X}_1 \sim \widetilde{\delta}_1$ is not recurrent.*

P r o o f. By Corollary 3.3 we have

$$\lambda_{0,n,\alpha}(\delta_1)\big((0,x]\big) = \left(1 + \frac{n-1}{x^\alpha}\right)\left(1 - \frac{1}{x^\alpha}\right)_+^{n-1}.$$

Since

$$|\widetilde{\lambda}_{0,n,\alpha}(\widetilde{\delta}_1)|\big((-\infty,x]\big) = \lambda_{0,n,\alpha}(\delta_1)\big((0,x]\big)$$

and $|\widetilde{X}_1| \sim \delta_1$, we arrive at

$$\sum_{n=1}^{\infty} P(|\widetilde{X}_n| < x) = x^{\alpha}(2 - x^{-\alpha})\mathbf{1}_{[1,\infty)}(x) < \infty.$$

By the Borel–Cantelli lemma we obtain $P(\limsup_{n\to\infty}\{|\widetilde{X}_n| < x\}) = 0$, which implies that $\{\widetilde{X}_n : n \in \mathbb{N}_0\}$ is not recurrent. ∎

Now we present the result describing the magnitude of the fluctuations for random walk under the weak Kendall convolution.

PROPOSITION 3.5. *For every $r > \frac{1}{2}$ and random walk under the Kendall convolution $\{\widetilde{X}_n : n \in \mathbb{N}_0\}$ with unit step $\triangle\widetilde{X}_1 \sim \widetilde{\delta}_1$ we have*

$$\mathbf{P}\Bigg(\bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty}\left\{|\widetilde{X}_n|^{\alpha} \leqslant \frac{n^{r+1}}{\ln n}\right\}\Bigg) = 1.$$

Proof. To see this it is sufficient to notice that

$$\mathbf{P}\left(|\widetilde{X}_n|^{\alpha} > \frac{n^{r+1}}{\ln n}\right) = 1 - \left(1 + \frac{n-1}{n^{r+1}}\ln n\right)\left(1 - n^{-r-1}\ln n\right)_{+}^{n-1}.$$

Let $A_n = \{|\widetilde{X}_n|^{\alpha} > n^{r+1}/\ln n\}$. It is a matter of laborious but straightforward calculations to show that

$$\lim_{n\to\infty}\frac{\mathbf{P}(A_n)}{n^{-2r}(\ln n)^2} = 1.$$

Moreover, for $2r - 1 > 0$ we have

$$\int_1^{\infty} x^{-2r}(\ln x)^2 dx = \int_0^{\infty} u^2\, e^{(1-2r)u}\, du = \frac{2}{(2r-1)^3} < \infty$$

and we obtain the assertion. ∎

REMARK 3.2. *Since $\mathbb{E}(|\widetilde{X}_n|^{\alpha}) = n$, by the Tchebyshev inequality we also have*

$$\mathbf{P}\left\{|\widetilde{X}_n|^{\alpha} \geqslant \frac{n^2}{\ln n}\right\} \leqslant \frac{\mathbb{E}(|\widetilde{X}_n|^{\alpha})}{n^2}\ln n = \frac{\ln n}{n} \to 0 \quad \textit{as } n \to \infty.$$

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