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CONTRACTIONS AND CENTRAL EXTENSIONS OF QUANTUM WHITE NOISE LIE ALGEBRAS

BY

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Abstract. We show that the Renormalized Powers of Quantum White Noise Lie algebra $RPQWN_*$, with the convolution type renormalization $\delta^n(t-s) = \delta(s)\,\delta(t-s)$ of the $n \geqslant 2$ powers of the Dirac delta function, can be obtained through a contraction of the Renormalized Powers of Quantum White Noise Lie algebra $RPQWN_c$ with the scalar renormalization $\delta^n(t) = c^{n-1}\,\delta(t)$, $c > 0$. Using this renormalization, we also obtain a Lie algebra $W_\infty(c)$ which contains the w_∞ Lie algebra of Bakas and the Witt algebra as contractions. Motivated by the W_∞ algebra of Pope, Romans and Shen, we show that $W_\infty(c)$ can also be centrally extended in a non-trivial fashion. In the case of the Witt subalgebra of W_∞, the central extension coincides with that of the Virasoro algebra.

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1. INTRODUCTION

1.1. Origins of the problem. Classical white noise was introduced in statistical mechanics and was used for several decades both in physics and engineering. The attempts to give a rigorous meaning to the Langevin equation can be considered to be the common root of Itô stochastic calculus and of Hida's program – stated in his 1975 Carleton lectures – of a mathematical approach to the theory of classical white noise (see [35] for history, the main ideas of this development and bibliography).

In classical probability the problem to give a meaning to Langevin equations driven by higher powers of WN was also considered in physics and engineering but no significant steps in this direction seem to be present in the literature.

On the other hand, **quantum white noise** has played a central role in quantum physics since its early origins because it coincides with the basic object of study in quantum field theory: the **free non-relativistic boson Fock field**. In this context the local field operators are identified with the operators of multiplication by white noise and the so-called **smeared fields** with the operators of multiplication by stochastic integrals. In the following we will mainly use the probabilistic terminology.

The basic physical principle of locality requires the fundamental forces of nature to be expressed as (usually non-linear) functions of the local fields rather than of their smeared versions. This poses a mathematical problem because, contrary to multiplication by stochastic integrals which are *bona fide* operators, white noises are **operator-valued distributions**, therefore even the definition of their simplest non-linear functions, such as their powers, poses formidable mathematical problems. This explains why, in quantum theory, the problem to give a meaning to the powers of white noise has been at the core of a large number of investigations.

Unfortunately, even in the quadratic case, no satisfactory solution was available to this problem as can be seen, for example, in Segal's paper [39] where a negative result is proved and in Berezin's monograph [30] where an attempt is made to extend Friedrichs's results on quadratic Hamiltonians from finitely many degrees of freedom to a field theoretical framework. This attempt however cannot be considered successful because, in order to realize quadratic fields in standard Fock space, the author was obliged to introduce Hilbert–Schmidt type conditions which, among other things, imply the loss of translation invariance.

The attack on the problem of defining the third and fourth power of white noise has been the object of a huge number of papers within the program called **constructive quantum field theory**, which in the period between the late 1960's and the late 1990's involved a large number of brilliant mathematicians and mathematical physicists. This program had an interesting technical fall out in mathematics in the so-called hypercontractive estimates, but it is generally recognized that it has not produced a solution that can become a new tool in applications to physics. Thus several people became convinced that the approach generally used during those decades to achieve this goal, i.e., to introduce a regularization (called cut-off in physics and corresponding to the replacement of white noise by a stochastic integral) and then trying to remove it by various kinds of limiting procedures, was not the right one and that new ideas and new mathematical tools should be introduced in order to produce substantial steps forward.

1.2. Emergence of white noise Hamiltonian equations from stochastic limit. These new ideas and tools began to emerge in the early 1990's when the developments of the stochastic limit of quantum theory led to the discovery that all **(classical and quantum) stochastic differential equations are equivalent to white noise Hamiltonian equations**, involving the first powers and the normally ordered second power of white noise. This equivalence, new even in the classical case, is highly non-trivial, e.g., the coefficients of the WN Hamiltonian equation turn out to be related to those of the associated stochastic differential equation by a Cayley transform. The precise meaning of the term *white noise Hamiltonian equations* and the development of the estimates necessary to establish on a solid mathematical basis the above-mentioned equivalence theorem as well as the main result of the theory, i.e., an existence, uniqueness and unitarity theorem for the Heisenberg type white noise equations, began in the paper [25] and was completed in the papers [2], [3] in the case of equations with **bounded coefficients**. Its extension to the unbounded case is an important open problem because its solution will contribute to reopen a communication between physicists, who hardly ever use stochastic equations, and mathematicians, who hardly ever use white noise equations.

Since the first and normally ordered second order WN Hamiltonian equations exhaust, up to equivalence, all stochastic differential equations, it was natural to ask oneself: **what kind of equations could arise from powers of WN higher than these ones?** Such equations, if existent, should provide a natural generalization of stochastic calculus as well as an answer to the problems mentioned above and studied since several decades in the engineering and physics literature. The solution of this problem is the main objective of **non-linear white noise calculus**.

1.3. Quadratic second quantization: the relation between $\mathfrak{sl}(2,\mathbb{R})$ and the Meixner distributions. The first attempts towards the development of a non-linear white noise calculus, based on higher powers of WN (see [27], [24], [5], [4]) were not satisfactory for several reasons, which would be too long to explain here.

The situation changed in 1999 when, in the series of lectures [25], a new idea was proposed which can be formulated as follows: **instead of renormalizing directly the equations of motion first renormalize the Lie-algebra structure, i.e., the commutation relations, thus obtaining a new *-Lie-algebra structure, then construct (non-trivial) Hilbert space representations of the new Lie algebra** whose self-adjoint generators are the candidates for the non-linear powers of the field operators (WN).

The paper [25] was the first step towards the realization of this program for the simplest non-linear power of WN, the square, and the simplest representation, the Fock one, and led to the *-Lie algebra of **Renormalized Second Powers of WN** (**$RSPWN$**).

The next step followed almost immediately: Śniady, in the paper [40], extended to the free case the construction of the renormalized quadratic boson Fock representation and noticed that, both in the free and the boson case, the first and second order representations cannot be combined into a single one in a non-trivial way. In the language of Lie algebras this result can be rephrased saying that, while the Fock representation of the current algebra over $\mathbb{R}^d$ of the Heisenberg algebra (i.e. the usual free boson Fock field) and of $\mathfrak{sl}(2,\mathbb{R})$ (i.e. the quadratic field) separately exist, the Fock representation of the Schrödinger Lie algebra (the smallest Lie algebra containing all first and second powers of WN) cannot exist: **this remark contains the core difficulties of the no-go theorems** to be discussed further in this introduction (see Section 1.4 below).

An important further step in this development was the paper [23] in which the *-Lie algebra $RSPWN$ of [25] was identified with the current algebra of $\mathfrak{sl}(2,\mathbb{R})$

over $\mathbb{R}$, more precisely – of its central extension: such a central extension is necessarily trivial and unique up to a scalar multiple of the central element; moreover, this scalar is precisely the renormalization constant which cannot be taken equal to zero because otherwise the Fock representation becomes the trivial one (identically zero). This identification was used to frame the construction of the quadratic Fock representation within the general context of the Araki–Woods–Parthasarathy–Schmidt theory of factorizable representations of Lie algebras and Lie groups. This theory, together with the fact that the unitary representations of $\mathfrak{sl}(2,\mathbb{R})$ are completely classified, was used to realize the quadratic Fock field as a usual boson Fock field with an infinite-dimensional multiplicity space. This new representation is conceptually important but, like all abstract representation theorems, has some drawbacks when one tries to use it for concrete calculations, for example: the Itô table of the fields turns out to be infinite dimensional and so complicated that it was not possible to find a solution for the quadratic unitary conditions (this problem was solved some years later in the paper [6] by using Hilbert module techniques).

Another result of [23] was the identification of the vacuum distributions of the quadratic field operators with the three non-standard classes of Meixner distributions which therefore play for quadratic quantization the role played by the first two (standard) classes (Gaussian and Poisson) for usual quantization. In other terms, while the classical stochastic processes which appear as vacuum distributions of the field operators in the usual quantization (i.e., the usual quantum fields) are the Brownian motion and the Poisson process, the vacuum distributions of the second powers of WN are the Meixner processes which have been widely studied in probability theory and in mathematical finance (cf. [34], [33]; the survey paper [7] contains a brief description of the multiplicity of contexts in which Meixner distributions have appeared in various branches of classical probability and statistics). A corollary to the above-mentioned result was the **quantum decomposition of the Meixner stochastic processes** which, contrarily to the quantum decomposition of the Brownian motion and the Poisson processes, was not known before, neither in mathematical nor in physical literature.

This result opened the way to a quantum probabilistic approach to the Meixner distributions and was followed by a vast multiplicity of papers dealing with and generalizing in various contexts different aspects of Meixner distributions (references on these developments, in particular concerning the classical and the free case, can be found in the paper [31]).

In the specific direction of boson quadratic quantization, the construction of the Fock representation was only the first step towards the construction of the Fock functor, a topic to which several papers were dedicated (see [21] and references therein).

The explicit form of the quadratic scalar product was deduced in [29] by using the Faà di Bruno formula, well known in combinatorics and widely used in probability theory. The **quadratic Weyl operators** and the corresponding **quadratic Heisenberg group** were constructed in [26].

The first examples, and at the moment the only known ones, of non-Fock (equilibrium) representations were constructed in [1]. One of the most interesting open problems concerning the quadratic case is to **enlarge the class of non-Fock representations** of the Renormalized Square of White Noise (RSWN) *-Lie algebra: there are several indications that the class constructed in [1] constitutes a tiny fraction of the representations that can be of interest for physics.

1.4. Higher order second quantizations. The above description of the results obtained in the quadratic case is motivated by the fact that they constitute the main and best understood model for what one would like to do for powers higher than 2.

The program to associate a *-Lie algebra to general renormalized powers of (quantum) white noise and to construct their representations was initiated by Accardi and Boukas [7] about 10 years ago. The first idea was to use, for higher powers, the same renormalization used for the square. This leads to the algebra that in the present paper is denoted by $RPQWN_c$. One could say, following [23], that this approach corresponds to finding a representation for the current algebra over $\mathbb{R}^d$ of the universal enveloping algebra of the Heisenberg algebra. Since this algebra contains the current algebra of the Schrödinger algebra, we know from Śniady's result that its Fock representation cannot exist. This was also clearly shown in Accardi and Boukas [42] by using the Schrödinger Fock kernel. However, there was a hope that, as happens for $\mathfrak{sl}(2,\mathbb{R})$, for some subalgebra (e.g., the *-Lie algebra generated by the cube of creation and annihilation operators) it might exist.

This hope was frustrated by the generalized no-go theorem proved in [19]. This motivated our search for new renormalization prescriptions.

A careful analysis of the structure of the no-go theorems suggested the use of the convolution renormalization (see Section 2.2 below). This new renormalization gives rise to commutators which do not correspond to current algebras of known Lie algebras: this fact prevents the possibility to follow the strategy used in [23], i.e., to apply the results of the Araki–Woods–Parthasarathy–Schmidt theory to this problem. Moreover, in this case the very *-Lie algebra structure has to be verified by direct calculation. This approach gave rise to the algebra that in the present paper is denoted by $RPQWN_*$.

1.5. The $RPQWN_*$ algebra, the W_∞-algebra, arising from the Virasoro–Zamolodchikov hierarchy, and their identification. The analysis of the new infinite-dimensional Lie algebra $RPQWN_*$ brought to light some striking similarities to other Lie algebras widely studied in string theory and conformal field theory (see [9], [10] and Section 1 below).

Even though some strong dissimilarities were present (namely in the involution, see formula (2.14) below), we were convinced that the fact that the structure constants for the two algebras are almost equal (compare (2.12) and (2.13) below) could not be attributed to chance.

After a few years (and a lot of work) this conjecture turned out to be true in the sense that we were able to express the generators of each algebra as series in the generators of the other one (cf. [13] and [12]). This implies that the closures – in a suitably defined topology – of the Lie algebras which emerge from the non-linear WN calculus program coincide with those emerged in string theory and conformal field theory following a completely disjoint program.

Since these structures are highly non-trivial, we interpret this fact as an indication that they are **new canonical objects in mathematics**, destined to play a role both in mathematics and in physics.

As explained below, **the algebras which arise in non-linear WN calculus are a second quantization** (more specifically, current algebras over $\mathbb{R}^d$) **of those that have emerged in physics**: the latter are obtained from the former by restricting the test function space to the multiples of a single indicator function of a bounded Borel set and by suitable rescaling. This is probably the reason why the no-go theorem appears in the two fields with completely different characteristics, being related, in physics, to group invariance and specific models, while in WN calculus to the problem of infinite divisibility of certain probability distributions (see Section 1.6.1 below).

1.6. The role of central extensions of Lie algebras. The enthusiasm for the discovery described above was balanced by the discovery – in [16] and the previous result in [8] – that, unfortunately, the no-go theorem applies also to the Lie algebra obtained with the new renormalization, i.e., $RPQWN_*$. The analysis of the structure of the Virasoro algebra led us to realize that this algebra is obtained by *gluing together* in a non-commutative way countably many copies of central extensions of the $\mathfrak{sl}(2,\mathbb{R})$ algebra. Moreover, in each of these copies the constant defining the Virasoro central extension coincides, up to a positive rational multiple, with the renormalization constant in $RSPWN$. This fact suggests a deep connection between renormalization and central extension and motivated our investigation in the direction of **central extensions** of Lie algebras.

In physics central extensions were of help in overcoming some no-go theorems (this was the case with the Virasoro central extension of the Witt algebra). Our hope is that a similar situation can take place in the non-linear white noise program. This motivated our search for non-trivial central extensions of the algebras we were dealing with. This new branch of our program was formulated in [18].

1.6.1. *Non-triviality of the second cohomology group of the Heisenberg algebra.* The first non-trivial extension we found, in [17], was the one of the Heisenberg algebra ($[a,a^\dagger]=1$). Since all low dimensional Lie algebras are classified, this extension was known as a Lie algebra; in fact, as we discovered later, this algebra plays a relevant role also in quantum physics where it is known under the name of **Galilei algebra**. To our knowledge neither this identification nor the fact that this four-dimensional Lie algebra is the unique non-trivial central extension of the Heisenberg algebra seems to have previously appeared in the literature.

In the Schrödinger representation the Galilei algebra can be realized by the generators $\{1,q,q^2,p\}$, so it interpolates between the Heisenberg algebra and the

Schrödinger one. Therefore, *a priori* the no-go theorem does not apply to this case and our conjecture was that the Fock representation of its current algebra over $\mathbb{R}$ exists.

Proving the validity of this conjecture is equivalent to proving the infinite divisibility of the vacuum distribution of all the real linear combinations of the generators (field operators).

The characteristic function of these distributions was explicitly calculated [20] and infinite divisibility has been established for a large set of parameters, but at the moment not for all. We have looked for help asking this question to some of the best known experts in classical infinite divisibility, but at the moment the problem is still open.

Thus even for a simple algebra like the Galilei one the possibility to exorcize the no-go theorems relies on a difficult problem of classical probability.

At the moment the status of this conjecture is not clear because a recent result, based on C^*-algebra techniques [22], seems to point out, contrary to our expectations, towards a negative answer.

1.6.2. *Calculation of the second cohomology group of infinite-dimensional Lie algebras.* The second powers of WN are the highest ones for which the Lie algebra generated by them is finite dimensional.

Starting from $n = 3$, even after renormalization, the n-th powers of quantum WN generate an infinite-dimensional subalgebra of the algebra of all renormalized powers of WN.

They correspond to higher order extension of the centerless Virasoro (or Witt) algebra. In order to obtain higher order extensions of the proper Virasoro algebra the problem of determining the second cohomology group of these algebras, i.e., the classification of the two-cocycles, had to be solved.

This program was realized in several steps in the papers [14], [15], [18] and in Section 7 of the present one. The results of this section constitute a non-trivial generalization of Virasoro's original construction of the central extension of the Witt algebra.

The main open problem in this direction is the construction of unitary representations of these algebras and the identification of the spaces of these representations as L^2-spaces of appropriate functional measures. This problem is at the moment open even at the level of the **one mode** realization of these algebras and the second problem, the identification of the underlying measures, is open even for the Virasoro algebra notwithstanding the huge literature available on it.

2. SOME INFINITE-DIMENSIONAL *-LIE ALGEBRAS

2.1. The $*$-Lie algebra $RPQWN_c$ of the renormalized higher powers of white noise with scalar renormalization. For $t, s \geqslant 0$, the Quantum White Noise (QWN) creation and annihilation densities $a_t^\dagger$ and a_s satisfy the commutation relations

$$[a_t, a_s^\dagger] = \delta(t-s), \quad [a_t^\dagger, a_s^\dagger] = [a_t, a_s] = 0, \quad (a_s)^* = a_s^\dagger, \tag{2.1}$$

whose formal generalization, i.e., involving formal powers of Dirac delta function, is (see [19])

$$\begin{aligned}[][{a_t^\dagger}^n a_t^k, {a_s^\dagger}^N a_s^K] &= \epsilon_{k,0}\,\epsilon_{N,0} \sum_{L\geqslant 1} \binom{k}{L} N^{(L)}\, {a_t^\dagger}^n\, {a_s^\dagger}^{N-L}\, a_t^{k-L}\, a_s^K\, \delta^L(t-s)\\ &\quad - \epsilon_{K,0}\,\epsilon_{n,0} \sum_{L\geqslant 1} \binom{K}{L} n^{(L)}\, {a_s^\dagger}^N\, {a_t^\dagger}^{n-L}\, a_s^{K-L}\, a_t^k\, \delta^L(t-s),\end{aligned}$$

where $n, k \geqslant 0$, $\delta_{n,k}$ is Kronecker's delta,

$$\epsilon_{n,k} := 1 - \delta_{n,k},$$

and

$$x^{(y)} = x(x-1)\ldots(x-y+1), \quad x^{(0)} := 1, \quad x^{(1)} := x, \tag{2.2}$$

$$(x)_y = x(x+1)\ldots(x+y-1), \quad (x)_0 := 1, \quad (x)_1 := x, \tag{2.3}$$

are the falling and rising factorials, respectively.

In order to give a meaning to the formal expression (2.1), the renormalization prescription (see [25], [19])

$$\delta^l(t) := c^{l-1}\,\delta(t), \quad l = 2, 3, \ldots;\ c > 0 \text{ arbitrary constant}, \tag{2.4}$$

was used in [19] and it was shown that, after this renormalization, the smeared operators, heuristically defined by

$$B_k^n(f; c) := \int_{\mathbb{R}} f(t)\, {a_t^\dagger}^n\, a_t^k\, dt, \tag{2.5}$$

satisfy the commutation and duality relations

$$[B_k^n(f; c), B_K^N(g; c)] = \sum_{L=1}^{(k\wedge N)\vee(K\wedge n)} \theta_L(n, k; N, K)\, c^{L-1}\, B_{k+K-L}^{n+N-L}(fg; c), \tag{2.6}$$

$$\big(B_k^n(f; c)\big)^* = B_n^k(\bar{f}; c), \tag{2.7}$$

where

$$\theta_L(n, k; N, K) := \epsilon_{k,0}\,\epsilon_{N,0} \binom{k}{L} N^{(L)} - \epsilon_{K,0}\,\epsilon_{n,0} \binom{K}{L} n^{(L)},$$

and, here and in the following, we use the convention that, whenever $a > b$,

$$\sum_{L=a}^{b} = 0.$$

By using the standard procedure of distribution theory, the result of these formal manipulations was taken as the definition of a new mathematical object:

DEFINITION 2.1. Let $\mathcal{S}(\mathbb{R})$ denote the Schwartz space of rapidly decreasing smooth functions on $\mathbb{R}$ and let $c > 0$ be a real number. The Lie algebra $RPQWN_c$ (Renormalized Powers of Quantum White Noise with renormalization constant c) is the *-Lie algebra with generators

$$\{B_k^n(f) := B_k^n(f;c) : f \in \mathcal{S}(\mathbb{R});\ k,n \in \mathbb{N}\} \tag{2.8}$$

such that the maps $f \in \mathcal{S}(\mathbb{R}) \mapsto B_k^n(f)$ are complex linear, and the commutation relations and involution are defined respectively by (2.6) and (2.7).

In [19] it was proved that Definition 2.1 is coherent, i.e., that $RPQWN_c$ is effectively a *-Lie algebra.

2.2. The $*$-Lie algebra $RPQWN_*$ of renormalized higher powers of white noise with convolution type renormalization. Motivated by a detailed analysis of the no-go theorems (see [19]), the following, convolution type, renormalization was introduced in [9] and [10]:

$$\delta^l(t-s) = \delta(s)\,\delta(t-s), \qquad l = 2,3,\dots, \tag{2.9}$$

where the distribution on the right-hand side is defined on the space of rapidly decreasing smooth functions that vanish at zero. The new renormalization leads to the commutation relations

$$[B_k^n(g), B_K^N(f)] = (k\,N - K\,n)\,B_{k+K-1}^{n+N-1}(gf) \tag{2.10}$$

which, with the same involution (2.7), also define a *-Lie algebra, called in the following the $RPQWN_*$ **-Lie algebra*.

Fixing an open set $I \subset \mathbb{R} \setminus \{0\}$ with

$$|I| := \text{Lebesgue measure of } I < \infty \tag{2.11}$$

and restricting the test function space to the single function

$$f(x) = g(x) = \chi_I(x) := \begin{cases} 0 & \text{if } x \notin I, \\ 1 & \text{if } x \in I, \end{cases}$$

one obtains a *-sub-Lie algebra of $RPQWN_*$. When I varies among all subsets of $\mathbb{R} \setminus \{0\}$ satisfying (2.11) the corresponding *-Lie algebras are isomorphic (possibly up to a multiplication of the generators by a positive scalar depending only on $|I|$). The additional condition

$$|I| = 1$$

defines the *one mode $RPQWN_*$-*-Lie algebra*

$$[B_k^n, B_K^N] = (k\,N - K\,n)\,B_{k+K-1}^{n+N-1}. \tag{2.12}$$

2.3. The $*$-Lie algebra w_∞. The *-Lie algebra w_∞, introduced by Bakas [28], is defined by generators $(\hat{B}^n_k)$ with commutation and involution relations

$$[\hat{B}^n_k, \hat{B}^N_K] = \big(k\,(N-1) - K\,(n-1)\big)\hat{B}^{n+N-2}_{k+K}, \tag{2.13}$$

$$(\hat{B}^n_k)^* = \hat{B}^n_{-k}, \tag{2.14}$$

where $n, k \in \mathbb{Z}$ with $n \geqslant 2$. For $n = 2$, the relations (2.13) and (2.14) reduce to the Witt (or centerless Virasoro) Lie algebra commutation relations

$$[\hat{B}^2_k, \hat{B}^2_K] = (k-K)\,\hat{B}^2_{k+K}, \quad (\hat{B}^2_k)^* = \hat{B}^2_{-k}.$$

Traditionally, the notation $L_k := \hat{B}^2_k$ is used and the Witt algebra commutation relations are written in the form

$$[L_k, L_K] = (k-K)\,L_{k+K}, \quad (L_k)^* = L_{-k}.$$

3. CONTRACTION OF $RPQWN_c$ TO $RPQWN_*$ AS $c \to 0$

The main result of this section implies that in the definition of $RPQWN_*$ one can eliminate the restriction that the test functions considered vanish at zero.

DEFINITION 3.1. A family $(C^\gamma_{\alpha,\beta})_{\alpha,\beta,\gamma\in T}$ of structure constants of some Lie algebra (or *-Lie algebra) $\mathcal{L}$ is called *locally finite* if for each pair $\alpha, \beta \in T$ one has

$$C^\gamma_{\alpha,\beta} \neq 0$$

only for a finite number of $\gamma \in T$. A set $(\ell_\alpha)_{\alpha\in T}$ of generators of a *-Lie algebra $\mathcal{L}$ is called *locally finite* if the associated family of structure constants is locally finite.

DEFINITION 3.2. Let I be a topological space and T be a set. A family of structure constants

$$\{C^\gamma_{\alpha,\beta}(c) \ : \ \alpha,\beta,\gamma \in T\}, \quad \forall c \in I,$$

of some Lie algebra (or *-Lie algebra) $\mathcal{L}_c$ is said to be *convergent* as $c \to c_0$ if

$$\lim_{c\to c_0} C^\gamma_{\alpha,\beta}(c) =: C^\gamma_{\alpha,\beta}(c_0), \quad \forall \alpha,\beta,\gamma \in T, \tag{3.1}$$

in the sense that the limit exists and defines the right-hand side.

If this is the case, it is not difficult to verify that

$$\{C^\gamma_{\alpha,\beta} = C^\gamma_{\alpha,\beta}(c_0) : \alpha,\beta,\gamma \in T\}$$

is a family of structure constants of some Lie algebra (or *-Lie algebra) $\mathcal{L}$. Moreover, condition (3.1) implies that if the family $\big(C^\gamma_{\alpha,\beta}(c)\big)$ is locally finite, then the same is true for the limit family $(C^\gamma_{\alpha,\beta})$ because in the limit the family of non-zero structure constants can only decrease.

DEFINITION 3.3. In the above notation the Lie algebra (or *-Lie algebra) $\mathcal{L}$ is called a *contraction* of the family of Lie algebras (or *-Lie algebras) $(\mathcal{L}_c)_{c\in I}$ as $c \to c_0$.

THEOREM 3.1. *$RPQWN_*$ is a contraction of the family $(RPQWN_c)_{c>0}$ as $c \to 0$.*

P r o o f. From (2.6) we see that the non-zero structure constants of $RPQWN_c$ with respect to the set of generators (2.8) are

$$\theta_L(n,k;N,K)\,c^{L-1} = \left(\epsilon_{k,0}\,\epsilon_{N,0}\binom{k}{L}N^{(L)} - \epsilon_{K,0}\,\epsilon_{n,0}\binom{K}{L}n^{(L)}\right)c^{L-1}$$

with $L \in \{1,\dots,(k\wedge N)\vee(K\wedge n)\}$. For $L=1$ this gives

$$\left(\epsilon_{k,0}\,\epsilon_{N,0}\binom{k}{1}N^{(1)} - \epsilon_{K,0}\,\epsilon_{n,0}\binom{K}{1}n^{(1)}\right) = (\epsilon_{k,0}\,\epsilon_{N,0}\,kN - \epsilon_{K,0}\,\epsilon_{n,0}\,Kn)$$

and for $L>1$ we have

$$\left(\epsilon_{k,0}\,\epsilon_{N,0}\binom{k}{L}N^{(L)} - \epsilon_{K,0}\,\epsilon_{n,0}\binom{K}{L}n^{(L)}\right)c^{L-1} \to 0 \quad \text{as } c \to 0.$$

Consequently,

$$\lim_{c\to 0}\theta_L(n,k;N,K)\,c^{L-1} = \begin{cases} kN - Kn & \text{if } L=1,\\ 0 & \text{if } L>1.\end{cases}$$

But (2.10) implies that these are the structure constants of $RPQWN_*$ in the basis $\big(B_k^n(f)\big)$ and this proves the statement. ■

4. THE $W_\infty(c)$ LIE ALGEBRA

In [13] we proved that the closures – in appropriate topologies – of the *-Lie algebras w_∞ and $RPQWN_*$ coincide up to isomorphism and we have constructed explicit representations of the generators of each of the two *-Lie algebras in terms of infinite series of generators of the other, converging in the above-mentioned topology.

In the previous section we have proved that the *-Lie algebra $RPQWN_*$ is a contraction of $RPQWN_c$ as $c \to 0$. Therefore, it is natural to conjecture that the *-Lie algebra w_∞ is a contraction of a natural closure of $RPQWN_c$ as $c \to 0$. The present section is devoted to the proof of this conjecture.

THEOREM 4.1. *If the higher powers of the delta function are renormalized with the generalized scalar renormalization prescription* (2.4), *then the QWN operators*

$$\hat{B}_k^n(f) := \int_{\mathbb{R}} f(t)\,e^{\frac{k}{2}(a_t - a_t^\dagger)}(a_t + a_t^\dagger)^{n-1}\,e^{\frac{k}{2}(a_t - a_t^\dagger)}\,dt, \tag{4.1}$$

where $n, k \in \mathbb{Z}$ *with* $n \geqslant 2$, *satisfy the involution condition* (2.14) *and the commutation relations*

$$(4.2) \qquad [\hat{B}_k^n(f), \hat{B}_K^N(g)] = \sum_{m=0}^{n-1} \sum_{l=0}^{N-1} \beta_{m,l}(n,k;N,K;c)\, \hat{B}_{k+K}^{m+l+1}(fg),$$

where, *by definition*, $0^0 := 1$ *and the remaining structure constants are given by*

$$(4.3) \quad \beta_{m,l}(n,k;N,K;c) = (1-\delta_{(n-1-m)+(N-1-l),0}) \binom{n-1}{m}\binom{N-1}{l} \\ \times \big((-1)^{n-m-1} - (-1)^{N-l-1}\big) k^{N-l-1} K^{n-m-1}\, c^{n+N-(m+l)-3}.$$

P r o o f. Introducing the position and momentum densities,

$$ip_t := a_t - a_t^\dagger, \qquad q_t := a_t + a_t^\dagger,$$

and using the representation (4.1), one finds

$$\begin{aligned}
&[\hat{B}_k^n(f), \hat{B}_K^N(g)] \\
&\quad = \int_{\mathbb{R}}\int_{\mathbb{R}} f(t)g(s)[e^{(kip_t)/2}\, q_t^{n-1} e^{(kip_t)/2}, e^{(Kip_s)/2}\, q_s^{N-1} e^{(Kip_s)/2}]\, dt\, ds \\
&\quad = \int_{\mathbb{R}}\int_{\mathbb{R}} f(t)g(s)\, e^{(kip_t)/2}\, q_t^{n-1} e^{(kip_t)/2} e^{(Kip_s)/2}\, q_s^{N-1} e^{(Kip_s)/2}\, dt\, ds \\
&\quad\quad - \int_{\mathbb{R}}\int_{\mathbb{R}} f(t)g(s)\, e^{(Kip_s)/2}\, q_s^{N-1} e^{(Kip_s)/2} e^{(kip_t)/2}\, q_t^{n-1} e^{(kip_t)/2}\, dt\, ds.
\end{aligned}$$

Since $[p_t, p_s] = 0$, this is equal to

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}} f(t)g(s)\, e^{(kip_t)/2}\, q_t^{n-1} e^{(Kip_s)/2} e^{(kip_t)/2}\, q_s^{N-1} e^{(Kip_s)/2}\, dt\, ds \\
&\quad - \int_{\mathbb{R}}\int_{\mathbb{R}} f(t)g(s)\, e^{(Kip_s)/2}\, q_s^{N-1} e^{(kip_t)/2} e^{(Kip_s)/2}\, q_t^{n-1} e^{(kip_t)/2}\, dt\, ds.
\end{aligned}$$

Starting from the well-known relation

$$(4.4) \qquad e^{i\tau p_s} q_t e^{-\tau t p_s} = q_t + 2\delta(t-s)\tau 1, \qquad \tau \in \mathbb{R},$$

one deduces the formal identities

$$e^{(Kip_s)/2} q_t^{n-1} = \sum_{m=0}^{n-1} \binom{n-1}{m} q_t^m K^{n-1-m} \delta^{n-1-m}(t-s)\, e^{(Kip_s)/2},$$

$$e^{(kip_t)/2} q_s^{N-1} = \sum_{m=0}^{N-1} \binom{N-1}{m} q_s^m k^{N-1-m} \delta^{N-1-m}(t-s)\, e^{(kip_t)/2},$$

$$q_t^{n-1} e^{(Kip_s)/2} = e^{(Kip_s)/2} \sum_{m=0}^{n-1} \binom{n-1}{m} q_t^m K^{n-1-m} (-1)^{n-1-m} \delta^{n-1-m}(t-s),$$

$$q_s^{N-1} e^{(kip_t)/2} = e^{(kip_t)/2} \sum_{m=0}^{N-1} \binom{N-1}{m} q_s^m k^{N-1-m} (-1)^{N-1-m} \delta^{N-1-m}(t-s),$$

involving powers of the δ-function. Using the renormalization prescription (2.4) to give a meaning to these powers, we find that

$$\begin{aligned}
&[\hat{B}_k^n(f), \hat{B}_K^N(g)] \\
&\quad = \sum_{m_1=0}^{n-1} \sum_{m_2=0}^{N-1} \binom{n-1}{m_1} \binom{N-1}{m_2} (-1)^{n-1-m_1} K^{n-1-m_1} k^{N-1-m_2} \\
&\qquad \times \int_{\mathbb{R}} \int_{\mathbb{R}} f(t) g(s)\, e^{(kip_t)/2} e^{(Kip_s)/2} q_t^{m_1} q_s^{m_2} e^{(kip_t)/2} e^{(Kip_s)/2} \\
&\qquad\qquad \times \delta^{n-1-m_1}(t-s) \delta^{N-1-m_2}(t-s)\, dt\, ds \\
&\qquad - \sum_{m_3=0}^{N-1} \sum_{m_4=0}^{n-1} \binom{N-1}{m_3} \binom{n-1}{m_4} (-1)^{N-1-m_3} k^{N-1-m_3} K^{n-1-m_4} \\
&\qquad \times \int_{\mathbb{R}} \int_{\mathbb{R}} f(t) g(s)\, e^{(Kip_s)/2} e^{(kip_t)/2} q_s^{m_3} q_t^{m_4} e^{(Kip_s)/2} e^{(kip_t)/2} \\
&\qquad\qquad \times \delta^{N-1-m_3}(t-s) \delta^{n-1-m_4}(t-s)\, dt\, ds \\
&\quad = \sum_{m_1=0}^{n-1} \sum_{m_2=0}^{N-1} \binom{n-1}{m_1} \binom{N-1}{m_2} (-1)^{n-1-m_1} K^{n-1-m_1} k^{N-1-m_2} \\
&\qquad \times \int_{\mathbb{R}} \int_{\mathbb{R}} f(t) g(s)\, e^{(kip_t)/2} e^{(Kip_s)/2} q_t^{m_1} q_s^{m_2} e^{(kip_t)/2} e^{(Kip_s)/2} \\
&\qquad\qquad \times \delta^{n+N-2-m_1-m_2}(t-s)\, dt\, ds \\
&\qquad - \sum_{m_3=0}^{N-1} \sum_{m_4=0}^{n-1} \binom{N-1}{m_3} \binom{n-1}{m_4} (-1)^{N-1-m_3} k^{N-1-m_3} K^{n-1-m_4} \\
&\qquad \times \int_{\mathbb{R}} \int_{\mathbb{R}} f(t) g(s)\, e^{(Kip_s)/2} e^{(kip_t)/2} q_s^{m_3} q_t^{m_4} e^{(Kip_s)/2} e^{(kip_t)/2} \\
&\qquad\qquad \times \delta^{n+N-2-(m_3+m_4)}(t-s)\, dt\, ds \\
&\quad = \sum_{m_1=0}^{n-1} \sum_{m_2=0}^{N-1} \binom{n-1}{m_1} \binom{N-1}{m_2} (-1)^{n-1-m_1} K^{n-1-m_1} k^{N-1-m_2} \\
&\qquad \times c^{n+N-3-(m_1+m_2)} \int_{\mathbb{R}} \int_{\mathbb{R}} f(t) g(s)\, e^{(kip_t)/2} e^{(Kip_s)/2} \\
&\qquad\qquad \times q_t^{m_1} q_s^{m_2} e^{(kip_t)/2} e^{(Kip_s)/2} \delta(t-s)\, dt\, ds \\
&\qquad - \sum_{m_3=0}^{N-1} \sum_{m_4=0}^{n-1} \binom{N-1}{m_3} \binom{n-1}{m_4} (-1)^{N-1-m_3} k^{N-1-m_3} K^{n-1-m_4}
\end{aligned}$$

$$
\begin{aligned}
&\times\ c^{n+N-3-(m_3+m_4)} \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}} f(t)g(s)\, e^{(Kip_s)/2} e^{(kip_t)/2} \\
&\qquad\qquad\qquad\qquad \times q_s^{m_3} q_t^{m_4} e^{(Kip_s)/2} e^{(kip_t)/2} \delta(t-s)\, dt\, ds \\
&= \sum_{m_1=0}^{n-1} \sum_{m_2=0}^{N-1} \binom{n-1}{m_1} \binom{N-1}{m_2} (-1)^{n-1-m_1} K^{n-1-m_1} k^{N-1-m_2} \\
&\quad \times c^{n+N-3-(m_1+m_2)} \int\limits_{\mathbb{R}} f(s)g(s)\, e^{(kip_s)/2} e^{(Kip_s)/2} \\
&\qquad\qquad\qquad\qquad \times q_s^{m_1} q_s^{m_2} e^{(kip_s)/2} e^{(Kip_s)/2}\, ds \\
&\quad - \sum_{m_3=0}^{N-1} \sum_{m_4=0}^{n-1} \binom{N-1}{m_3} \binom{n-1}{m_4} (-1)^{N-1-m_3} k^{N-1-m_3} K^{n-1-m_4} \\
&\quad \times c^{n+N-3-(m_3+m_4)} \int\limits_{\mathbb{R}} f(t)g(s)\, e^{(Kip_s)/2} e^{(kip_s)/2} \\
&\qquad\qquad\qquad\qquad \times q_s^{m_3} q_s^{m_4} e^{(Kip_s)/2} e^{(kip_s)/2}\, ds \\
&= \sum_{m_1=0}^{n-1} \sum_{m_2=0}^{N-1} \binom{n-1}{m_1} \binom{N-1}{m_2} (-1)^{n-1-m_1} K^{n-1-m_1} k^{N-1-m_2} \\
&\quad \times c^{n+N-3-(m_1+m_2)} \hat{B}_{k+K}^{m_1+m_2+1}(gf) \\
&\quad - \sum_{m_3=0}^{N-1} \sum_{m_4=0}^{n-1} \binom{N-1}{m_3} \binom{n-1}{m_4} (-1)^{N-1-m_3} k^{N-1-m_3} K^{n-1-m_4} \\
&\quad \times\ c^{n+N-3-(m_3+m_4)} \hat{B}_{k+K}^{m_3+m_4+1}(fg).
\end{aligned}
$$

Letting $m_1 = m_4 = m$, $m_2 = m_3 = l$, we obtain the above equality in the form

$$
\begin{aligned}
&\sum_{m=0}^{n-1} \sum_{l=0}^{N-1} \binom{n-1}{m} \binom{N-1}{l} (-1)^{n-1-m} K^{n-1-m} k^{N-1-l} \\
&\times c^{n+N-3-(m+l)} \hat{B}_{k+K}^{m+l+1}(fg) \\
&- \sum_{l=0}^{N-1} \sum_{m=0}^{n-1} \binom{N-1}{l} \binom{n-1}{m} (-1)^{N-1-l} k^{N-1-l} K^{n-1-m} \\
&\times c^{n+N-3-(m+l)} \hat{B}_{k+K}^{m+l+1}(fg) \\
&= \sum_{m=0}^{n-1} \sum_{l=0}^{N-1} \binom{n-1}{m} \binom{N-1}{l} \\
&\times \big((-1)^{n-m-1} - (-1)^{N-l-1}\big) k^{N-l-1} K^{n-m-1} c^{n+N-(m+l)-3} \hat{B}_{k+K}^{m+l+1}(fg).
\end{aligned}
$$

Thus, defining the coefficients

$$
\begin{aligned}
&\beta_{m,l}(n,k;N,K;c) := \\
&\binom{n-1}{m} \binom{N-1}{l} \big((-1)^{n-m-1} - (-1)^{N-l-1}\big) k^{N-l-1} K^{n-m-1} c^{n+N-(m+l)-3},
\end{aligned}
$$

we obtain

$$(4.5)\qquad [\hat{B}^n_k(f), \hat{B}^N_K(g)] = \sum_{m=0}^{n-1}\sum_{l=0}^{N-1} \beta_{m,l}(n,k;N,K;c)\, \hat{B}^{m+l+1}_{k+K}(fg).$$

Now notice that if $m = n-1$ and $l = N-1$, then

$$\big((-1)^{n-m-1} - (-1)^{N-l-1}\big) = 0 \Leftrightarrow \beta_{m,l}(n,k;N,K;c) = 0.$$

Therefore,

$$\begin{aligned}\beta_{m,l}(n,k;N,K;c) &= (1-\delta_{(n-1-m)+(N-1-l),0}) \binom{n-1}{m}\binom{N-1}{l}\\ &\times \big((-1)^{n-m-1} - (-1)^{N-l-1}\big) k^{N-l-1} K^{n-m-1} c^{n+N-(m+l)-3},\end{aligned}$$

which is (4.3). ∎

As usual we have to verify that the renormalization prescription does not break the *-Lie algebra structure. This is guaranteed by the following results.

LEMMA 4.1. *Let x, D and h satisfy the Heisenberg commutation relations $[D,x]=h$ and $[D,h]=[x,h]=0$. Then, for all $s,a,c\in\mathbb{C}$,*

$$e^{s(x+aD+ch)} = e^{sx}e^{saD}e^{(sc+(s^2a)/2)h} \quad \textit{and} \quad e^{sD}e^{ax} = e^{ax}e^{sD}e^{ash}.$$

P r o o f. The proof is well known. ∎

THEOREM 4.2. *Let $n \geqslant 2$ and $k\in\mathbb{Z}$. Then, in the sense of formal series, for all test functions f,*

$$\begin{aligned}(4.6)\qquad \hat{B}^n_k(f) = \sum_{m=0}^{n-1}\sum_{m'=0}^{n-1-m}\sum_{p,q=0}^{\infty} &\binom{n-1}{m}\binom{n-1-m}{m'}\\ &\times (-1)^p \frac{k^{p+q}}{p!\,q!}\,\phi_m(c,k)\, B^{m'+p}_{n-1-m-m'+q}(f),\end{aligned}$$

where

$$\phi_m(c,k) := \begin{cases} 0 & \textit{if } m \textit{ is odd},\\ \big(\delta_{m,0} + \epsilon_{m,0}\prod_{i=0}^{i=m/2-1}(m-2i-1)\,c^{m/2}\big)e^{-ck^2/2} & \textit{if } m \textit{ is even or zero},\end{cases}$$

and the case $k=0$ (only $p=q=0$ survives and we use $0^0=1$) is interpreted as

$$\hat{B}^n_0(f) = \sum_{m=0}^{n-1}\sum_{m'=0}^{n-1-m}\binom{n-1}{m}\binom{n-1-m}{m'}\phi_m(c,0)\, B^{m'}_{n-1-m-m'}(f).$$

P r o o f. For fixed $t, s \in \mathbb{R}$, we will make a repeated use of Lemma 4.1 with $D = a_t$, $x = a_s^\dagger$ and $h = \delta(t-s)$. We have

$$
\begin{aligned}
\hat{B}_k^n(f) &= \int\limits_{\mathbb{R}^d} f(s)\, e^{\frac{k}{2}(a_s - a_s^\dagger)}(a_s + a_s^\dagger)^{n-1}\, e^{\frac{k}{2}(a_s - a_s^\dagger)}\, ds \\
&= \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} f(t)\, e^{\frac{k}{2}(a_t - a_s^\dagger)}(a_t + a_s^\dagger)^{n-1}\, e^{\frac{k}{2}(a_t - a_s^\dagger)}\, \delta(t-s)\, dt\, ds \\
&= \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} f(t)\, e^{-\frac{k}{2}(a_s^\dagger - a_t)} \left(\frac{\partial^{n-1}}{\partial w^{n-1}}\Big|_{w=0} e^{w(a_t + a_s^\dagger)} \right) e^{-\frac{k}{2}(a_s^\dagger - a_t)}\, \delta(t-s) dt\, ds \\
&= \frac{\partial^{n-1}}{\partial w^{n-1}}\Big|_{w=0} \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} f(t)\, e^{-\frac{k}{2}(a_s^\dagger - a_t)} e^{w(a_t + a_s^\dagger)}\, e^{-\frac{k}{2}(a_s^\dagger - a_t)}\, \delta(t-s)\, dt\, ds \\
&= \frac{\partial^{n-1}}{\partial w^{n-1}}\Big|_{w=0} \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} f(t)\, e^{-\frac{k}{2}\, a_s^\dagger}\, e^{\frac{k}{2} a_t}\, e^{w\, a_s^\dagger}\, e^{w\, a_t}\, e^{-\frac{k}{2}\, a_s^\dagger}\, e^{\frac{k}{2}\, a_t} \\
&\quad \times e^{(w^2/2 - k^2/4)\, \delta(t-s)}\; \delta(t-s)\, dt\, ds \\
&= \frac{\partial^{n-1}}{\partial w^{n-1}}\Big|_{w=0} \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} f(t)\, e^{-\frac{k}{2}\, a_s^\dagger}\, e^{w\, a_s^\dagger}\, e^{\frac{k}{2} a_t}\, e^{-\frac{k}{2}\, a_s^\dagger}\, e^{w\, a_t}\, e^{\frac{k}{2}\, a_t} \\
&\quad \times e^{(w^2/2 - k^2/4)\, \delta(t-s)}\; \delta(t-s)\, dt\, ds \\
&= \frac{\partial^{n-1}}{\partial w^{n-1}}\Big|_{w=0} \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} f(t)\, e^{(w-k/2)\, a_s^\dagger}\, e^{\frac{k}{2} a_t}\, e^{-\frac{k}{2}\, a_s^\dagger}\, e^{(w+k/2)\, a_t} \\
&\quad \times e^{(w^2/2 - k^2/4)\, \delta(t-s)}\; \delta(t-s)\, dt\, ds \\
&= \frac{\partial^{n-1}}{\partial w^{n-1}}\Big|_{w=0} \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} f(t)\, e^{(w-k/2)\, a_s^\dagger}\, e^{-\frac{k}{2}\, a_s^\dagger}\, e^{\frac{k}{2} a_t}\, e^{(w+k/2)\, a_t} \\
&\quad \times e^{(w^2/2 - k^2/4)\, \delta(t-s)}\; \delta(t-s)\, dt\, ds \\
&= \frac{\partial^{n-1}}{\partial w^{n-1}}\Big|_{w=0} \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} f(t) e^{(w-k)\, a_s^\dagger}\, e^{(w+k)\, a_t} \\
&\quad \times e^{\left(w^2/2 - k^2/2\right)\delta(t-s)} \delta(t-s)\, dt\, ds \\
&= \frac{\partial^{n-1}}{\partial w^{n-1}}\Big|_{w=0} \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} f(t)\, e^{(w-k)\, a_s^\dagger}\, e^{(w+k)\, a_t} \\
&\quad \times \sum_{m=0}^{\infty} \frac{(w^2/2 - k^2/2)^m}{m!}\, \delta^m(t-s)\, \delta(t-s)\, dt\, ds \\
&= \frac{\partial^{n-1}}{\partial w^{n-1}}\Big|_{w=0} \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} f(t)\, e^{(w-k)\, a_s^\dagger}\, e^{(w+k)\, a_t} \\
&\quad \times \sum_{m=0}^{\infty} \frac{(w^2/2 - k^2/2)^m}{m!}\, \delta^{m+1}(t-s)\, dt\, ds
\end{aligned}
$$

$$= \frac{\partial^{n-1}}{\partial w^{n-1}}\Big|_{w=0} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(t)\, e^{(w-k)\,a_s^\dagger}\, e^{(w+k)\,a_t}$$

$$\times \sum_{m=0}^{\infty} \frac{c^m (w^2/2 - k^2/2)^m}{m!}\, \delta(t-s)\, dt\, ds$$

$$= \frac{\partial^{n-1}}{\partial w^{n-1}}\Big|_{w=0} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(t)\, e^{(w-k)\,a_s^\dagger}\, e^{(w+k)\,a_t}\, e^{c(w^2/2-k^2/2)}\, \delta(t-s)\, dt\, ds$$

$$= \frac{\partial^{n-1}}{\partial w^{n-1}}\Big|_{w=0} \big(e^{c(w^2/2-k^2/2)} \int_{\mathbb{R}^d} f(s)\, e^{(w-k)\,a_s^\dagger}\, e^{(w+k)\,a_s}\, ds\big)$$

$$= \sum_{m=0}^{n-1} \binom{n-1}{m} \frac{\partial^m}{\partial w^m}\Big|_{w=0} (e^{c(w^2/2-k^2/2)})$$

$$\times \frac{\partial^{n-1-m}}{\partial w^{n-1-m}}\Big|_{w=0} \big(\int_{\mathbb{R}^d} f(s)\, e^{(w-k)\,a_s^\dagger}\, e^{(w+k)\,a_s}\, ds\big)$$

by Leibniz's rule. By the same rule,

$$\frac{\partial^{n-1-m}}{\partial w^{n-1-m}}\Big|_{w=0} \big(\int_{\mathbb{R}^d} f(s)\, e^{(w-k)\,a_s^\dagger}\, e^{(w+k)\,a_s}\, ds\big)$$

$$= \sum_{m'=0}^{n-1-m} \binom{n-1-m}{m'} \int_{\mathbb{R}^d} f(s)\, \frac{\partial^{m'}}{\partial w^{m'}}\Big|_{w=0} (e^{(w-k)\,a_s^\dagger})$$

$$\times \frac{\partial^{n-1-m-m'}}{\partial w^{n-1-m-m'}}\Big|_{w=0} (e^{(w+k)\,a_s})\, ds$$

$$= \sum_{m'=0}^{n-1-m} \binom{n-1-m}{m'} \int_{\mathbb{R}^d} f(s)\, {a_s^\dagger}^{m'}\, e^{-k\,a_s^\dagger}\, a_s^{n-1-m-m'}\, e^{k\,a_s}\, ds$$

$$= \sum_{m'=0}^{n-1-m} \binom{n-1-m}{m'} \int_{\mathbb{R}^d} f(s)\, {a_s^\dagger}^{m'} \sum_{p=0}^{\infty} \frac{(-k)^p}{p!}\, {a_s^\dagger}^{p}\, a_s^{n-1-m-m'} \sum_{q=0}^{\infty} \frac{k^q}{q!}\, a_s^q\, ds$$

$$= \sum_{m'=0}^{n-1-m} \binom{n-1-m}{m'} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} (-1)^p \frac{k^{p+q}}{p!\,q!} \int_{\mathbb{R}^d} f(s)\, {a_s^\dagger}^{m'+p}\, a_s^{n-1-m-m'+q}\, ds$$

$$= \sum_{m'=0}^{n-1-m} \binom{n-1-m}{m'} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} (-1)^p \frac{k^{p+q}}{p!\,q!}\, B_{n-1-m-m'+q}^{m'+p}(f)$$

and

$$\frac{\partial^m}{\partial w^m}\Big|_{w=0} (e^{c(w^2/2-k^2/2)})$$

$$= \begin{cases} 0 & \text{if } m \text{ is odd,} \\ \big(\delta_{m,0} + \epsilon_{m,0} \prod_{i=0}^{i=m/2-1} (m-2i-1)\, c^{m/2}\big) e^{-ck^2/2} & \text{if } m \text{ is even or zero.} \end{cases}$$

Thus,

$$\hat{B}_k^n(f) = \sum_{m=0}^{n-1} \binom{n-1}{m} \phi_m(c,k) \sum_{m'=0}^{n-1-m} \binom{n-1-m}{m'} \times \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} (-1)^p \frac{k^{p+q}}{p!\,q!} B_{n-1-m-m'+q}^{m'+p}(f)$$

and the proof is completed. ■

THEOREM 4.3. *For each* $c > 0$ *there exists a unique *-Lie algebra, hereafter denoted by* $W_\infty(c)$, *with generators*

$$\{\hat{B}_k^n(f) = \hat{B}_k^n(f;c) : n,k \in \mathbb{Z};\ n \geqslant 2\ ;\ f \in \mathcal{S}(\mathbb{R})\},$$

the involution (2.14), *commutation relations* (4.2) *and structure constants given by* (4.3).

Proof. By Theorem 4.2, the Jacobi identity for $W_\infty(c)$ is reduced to the limit – in the appropriate topology – of the Jacobi identity on polynomials of elements of $RPQWN_c$, which is satisfied since $RPQWN_c$ is a Lie algebra. Thus $W_\infty(c)$ is also a Lie algebra.

A combinatorial proof could also be given as follows. For $i = 1,2,3$, let n_i, k_i, f_i be fixed, with $S := n_1 + n_2 + n_3$ and $M := k_1 + k_2 + k_3$, and let $l_i = \hat{B}_{k_i}^{n_i}(f_i) \in W_\infty(c)$. Then

$$\begin{aligned}
&\big[[l_1,l_2],l_3\big] + \big[[l_2,l_3],l_1\big] + \big[[l_3,l_1],l_2\big] \\
&= \sum_{m,l,m',l' \geqslant 0} \big(\beta_{m,l}(n_1,k_1;n_2,k_2;c)\beta_{m',l'}(m+l+1,k_1+k_2;n_3,k_3;c) \\
&\quad + \beta_{m,l}(n_2,k_2;n_3,k_3;c)\beta_{m',l'}(m+l+1,k_2+k_3;n_1,k_1;c) \\
&\quad + \beta_{m,l}(n_3,k_3;n_1,k_1;c)\beta_{m',l'}(m+l+1,k_3+k_1;n_2,k_2;c)\big)\hat{B}_{k_1+k_2+k_3}^{m'+l'+1}(f_1f_2f_3) \\
&= \sum_{m,l,m',l' \geqslant 0} \big(a_{m,l}(n_1,k_1;n_2,k_2)a_{m',l'}(m+l+1,k_1+k_2;n_3,k_3) \\
&\quad + a_{m,l}(n_2,k_2;n_3,k_3)a_{m',l'}(m+l+1,k_2+k_3;n_1,k_1) \\
&\quad + a_{m,l}(n_3,k_3;n_1,k_1)a_{m',l'}(m+l+1,k_3+k_1;n_2,k_2)\big) \\
&\quad \times c^{n_1+n_2+n_3-(m'+l')-5}\hat{B}_{k_1+k_2+k_3}^{m'+l'+1}(f_1f_2f_3),
\end{aligned}$$

where

$$a_{m,l}(n,k;N,K)c^{n+N-(m+l)-3} = \beta_{m,l}(n,k;N,K;c).$$

Letting

$$\begin{aligned}\hat{b}_{m,l,m',l'} &= a_{m,l}(n_1,k_1;n_2,k_2)a_{m',l'}(m+l+1,M-k_3;n_3,k_3)\\ &\quad + a_{m,l}(n_2,k_2;n_3,k_3)a_{m',l'}(m+l+1,M-k_1;n_1,k_1)\\ &\quad + a_{m,l}(n_3,k_3;n_1,k_1)a_{m',l'}(m+l+1,M-k_2;n_2,k_2)\end{aligned}$$

we find that

$$\begin{aligned}&\big[[l_1,l_2],l_3\big]+\big[[l_2,l_3],l_1\big]+\big[[l_3,l_1],l_2\big]\\ &\qquad = \sum_{m,l,m',l'\geqslant 0} \hat{b}_{m,l,m',l'}\cdot c^{S-(m'+l')-5}\hat{B}_M^{m'+l'+1}(f_1f_2f_3),\end{aligned}$$

which is a polynomial in c. Therefore, in order to prove the Jacobi identity, it suffices to show that for fixed $\alpha \in \mathbb{N}\cup\{0\}$

$$\sum_{\substack{m,l,m',l'\geqslant 0\\ m'+l'=\alpha}} \hat{b}_{m,l,m',l'} = 0,$$

which is seen to be true, after cancelation of terms. ■

5. CONTRACTION OF $(W_\infty(c))_{c>0}$ TO w_∞ AS $c\to 0$

THEOREM 5.1. *w_∞ is the contraction of the family $\big(W_\infty(c)\big)_{c>0}$ as $c\to 0$.*

P r o o f. In the expression (4.3) for the structure constants one has

$$m\leqslant n-1 \quad\text{and}\quad l\leqslant N-1 \tag{5.1}$$

and

$$\beta_{n-1,N-1}(n,k;N,K;c)=0.$$

Thus, in non-zero structure constants, at least one of the two inequalities in (5.1) must be strict, and this implies that

$$n+N-(m+l)-3\geqslant 0.$$

Therefore, in non-zero structure constants, the exponent in $c^{n+N-(m+l)-3}$ is always nonnegative and equal to zero if and only if

$$m=n-1 \text{ and } l=N-2 \quad\text{or}\quad m=n-2 \text{ and } l=N-1. \tag{5.2}$$

In conclusion: the limit of the structure constants,

$$\lim_{c\to 0}\beta_{m,l}(n,k;N,K;c),$$

always exists and the only possible pairs of indices (m,l) for which this limit is not equal to zero are those given by (5.2). From (4.3) it follows that the corresponding values of the structure constants are

$$\beta_{n-1,N-2}(n,k;N,K;c) = 2(N-1)k, \tag{5.3}$$

$$\beta_{n-2,N-1}(n,k;N,K;c) = -2(n-1)K. \tag{5.4}$$

Thus, denoting by

$$\{(1/\sqrt{2})\hat{B}^n_k(f;0) : n,k \in \mathbb{Z};\ n \geqslant 2;\ f \in \mathcal{S}(\mathbb{R})\}$$

the generators of the *-Lie algebra obtained in the limit $c \to 0$, one has

$$[(1/\sqrt{2})\hat{B}^n_k(f;0), (1/\sqrt{2})\hat{B}^N_K(f;0)] = (N-1)k - (n-1)K,$$

which are the w_∞ commutation relations (2.13). This proves the statement. ■

REMARK 5.1. *The white noise representation of the w_∞ generators, introduced in* [9] *and* [10] *and based not on the scalar renormalization, as here, but on the convolution type renormalization* (2.9) *of the powers of the delta function, is*

$$\hat{B}^n_k(f) := \int_{\mathbb{R}} f(t)\, e^{\frac{k}{2}(a_t - a_t^\dagger)} \left(\frac{a_t + a_t^\dagger}{2}\right)^{n-1} e^{\frac{k}{2}(a_t - a_t^\dagger)}\, dt. \tag{5.5}$$

With this notation the structure constants become

$$\begin{aligned}\beta_{m,l}(n,k;N,K;c) = \frac{1}{2^{n+N-2}}\,(1-\delta_{(n-1-m)+(N-1-l),0}) \binom{n-1}{m}\binom{N-1}{l}\\ \times \big((-1)^{n-m-1} - (-1)^{N-l-1}\big) k^{N-l-1} K^{n-m-1}\, c^{n+N-(m+l)-3}.\end{aligned}$$

REMARK 5.2. *The Witt algebra, the subalgebra of $W_\infty(c)$ generated by*

$$\hat{B}^2_k(f) := \int_{\mathbb{R}} f(t)\, e^{\frac{k}{2}(a_t - a_t^\dagger)}(a_t + a_t^\dagger)\, e^{\frac{k}{2}(a_t - a_t^\dagger)}\, dt,$$

remains fixed during the expansion of w_∞ to $W_\infty(c)$.

6. CENTRAL EXTENSIONS: BASIC CONCEPTS

If L and $\widetilde{L}$ are two complex Lie algebras, we say that $\widetilde{L}$ is a one-dimensional *central extension of L with central element E* if and only if $\widetilde{L}$ is the direct sum of L and $\mathbb{C}\,E$ as vector spaces with

$$[l_1, l_2]_{\widetilde{L}} = [l_1, l_2]_L + \phi(l_1, l_2)\, E \tag{6.1}$$

and

$$[l_1, E]_{\widetilde{L}} = 0 \tag{6.2}$$

for all $l_1, l_2 \in L$, where $[\cdot,\cdot]_{\widetilde{L}}$ and $[\cdot,\cdot]_L$ are the Lie brackets in $\widetilde{L}$ and L, respectively, and $\phi : L \times L \mapsto \mathbb{C}$ is a bilinear form (two-cocycle) on L satisfying the skew-symmetry condition

$$\phi(l_1, l_2) = -\phi(l_2, l_1) \tag{6.3}$$

and the Jacobi identity

$$\phi([l_1, l_2]_L, l_3) + \phi([l_2, l_3]_L, l_1) + \phi([l_3, l_1]_L, l_2) = 0. \tag{6.4}$$

In particular, (6.3) implies that $\phi(l, l) = 0$ for all $l \in L$. A two-cocycle ϕ corresponding to a trivial central extension is given by a linear function $f : L \mapsto \mathbb{C}$ satisfying

$$\phi(l_1, l_2) = f([l_1, l_2]_L). \tag{6.5}$$

It is known (see [15] and [18]) that, with the exception of its Heisenberg algebra sector, $RQPWN_*$ admits no non-trivial central extension. Precisely, the non-trivial central extensions of $RQPWN_*$ are given by

$$[B^n_k(f), B^N_K(g)] = (k\,N - K\,n)\, B^{n+N-1}_{k+K-1}(fg) + \rho_z(n, k; N, K)\, E,$$

where E is the (self-adjoint) central element and

$$\rho_z(n, k; N, K) = \delta_{n+k,0}\delta_{N,0}\delta_{K,1}\, z + \delta_{N+K,0}\delta_{n,1}\delta_{k,0}\, \bar{z}$$

with $z \in \mathbb{C} \setminus \{0\}$ arbitrary.

The same is true, with the exception of its Virasoro algebra sector, for w_∞ whose non-trivial central extensions are given by

$$\begin{aligned}&[\hat{B}^n_k(f), \hat{B}^N_K(g)]\\ &= \big(k\,(N-1) - K\,(n-1)\big)\hat{B}^{n+N-2}_{k+K}(fg) + \delta_{n,2}\,\delta_{N,2}\,\delta_{k+K,0}\, k\,(k^2-1)\, E,\end{aligned}$$

where traditionally $E = c/12$, and $c > 0$ is the "central charge".

7. THE W_∞ ALGEBRA

In [38] (see also [37] and [36]), Pope, Romans and Shen introduced the W_∞ Lie algebra as the inductive limit of the family of algebras (W_N) appearing in conformal field theory (W_3 is Zamolodchikov's algebra, see [41]). W_∞ is a Lie algebra with generators

$$\{V^j_n : n, j \in \mathbb{Z}, j \geqslant 0\} \tag{7.1}$$

and commutation relations

$$[V_m^i, V_n^j] = \sum_{l \geqslant 0} g_{2l}^{ij}(m,n)\, V_{m+n}^{i+j-2l} + c_i(m)\, \delta_{i,j}\, \delta_{m+n,0}, \tag{7.2}$$

where c_i are central charges determined by

$$c_i(m) = m(m^2-1)(m^2-4)\ldots\bigl(m^2-(i+1)^2\bigr)\, c_i, \tag{7.3}$$

$$c_i = \frac{2^{2i-3} i!\,(i+2)!}{(2i+1)!!\,(2i+3)!!}\, c \quad (c \in \mathbb{R} \text{ arbitrary})$$

(here, for an odd positive integer n, the *double factorial* sign !! denotes the product of all odd values up to n), and

$$g_l^{ij}(m,n) = \frac{1}{2(l+1)!}\, \phi_l^{i,j}\, N_l^{i,j}(m,n),$$

$$N_l^{i,j}(m,n) = \sum_{k=0}^{l+1} (-1)^k \binom{l+1}{k} \times (2i+2-l)_k (2j+2-k)^{(l+1-k)} (i+1+m)^{(l+1-k)} (j+1+n)^{(k)},$$

$$\phi_l^{i,j} = {}_4F_3 \left(\begin{matrix} -1/2, & 3/2, & -l/2-1/2, & -l/2 \\ -i-1/2, & -j-1/2, & i+j-l+5/2, & -l/2 \end{matrix} \; ; \; 1 \right).$$

REMARK 7.1. *The factor $\delta_{i,j}\, \delta_{m+n,0}$ is non-zero only if*

$$n = -m \quad \text{and} \quad i = j,$$

which corresponds to the subalgebras

$$[V_m^j, V_{-m}^j] = \sum_{l \geqslant 0} g_{2l}^{jj}(m,-m)\, V_0^{2(j-l)} + c_j(m), \quad j \in \{0,1,2,\ldots\}, \tag{7.4}$$

of which the case $j=0$ should correspond to the Virasoro central extension.

This suggests that we look for central extensions before taking the contraction $c \to 0$.

REMARK 7.2. *w_∞ can be obtained as a contraction of W_∞ by defining*

$$V_{m;q}^i = q^{-1}\, V_m^i$$

and letting $q \to 0$. Only the Virasoro central extension survives and we obtain the w_∞ Lie algebra commutation relations (2.13) *and their Virasoro central extension in the form*

$$[V_{m;0}^i\,, V_{n;0}^j] = \bigl((j+1)m - (i+1)n\bigr)\, V_{m+n;0}^{i+j} + \frac{c}{12} m(m^2-1) \delta_{i,0}\delta_{j,0}\delta_{m+n,0},$$

which can be put in the form of (2.13) *by defining*

$$\hat{B}^i_m = V^{i-2}_{m;0}.$$

Notice that the Witt–Virasoro algebra generators are

$$\hat{B}^2_m = V^0_{m;0}.$$

REMARK 7.3. *Letting*

$$\hat{B}^n_k = V^{n-2}_k, \quad n = 2, 3, \dots,$$

we see that the W_∞ *commutation relations take the form*

$$[\hat{B}^n_k, \hat{B}^N_K] = \sum_{l\geqslant 0} g^{(n-2)(N-2)}_{2l}(k, K)\, \hat{B}^{n+N-2(l+1)}_{k+K} + c_{n-2}(k)\, \delta_{n,N}\, \delta_{k+K,0}, \tag{7.5}$$

i.e.,

$$[\hat{B}^n_k, \hat{B}^n_{-n}] = \sum_{l\geqslant 0} g^{(n-2)(n-2)}_{2l}(k, -k)\, \hat{B}^{2(n-l-1)}_0 + c_{n-2}(k), \tag{7.6}$$

while, for $c_{n-2} = 0$, *we have the non-centrally extended commutation relations*

$$[\hat{B}^n_k, \hat{B}^N_K] = \sum_{l\geqslant 0} g^{(n-2)(N-2)}_{2l}(k, K)\, \hat{B}^{n+N-2(l+1)}_{k+K}. \tag{7.7}$$

REMARK 7.4. *Letting* $M = n - 1 - m$ *and* $L = N - 1 - l$ *we see that the* $W_\infty(c)$ *commutation relations of Theorem* 4.1 *can be put in the form*

$$[\hat{B}^n_k(f), \hat{B}^N_K(g)] = \sum_{M=0}^{n-1} \sum_{L=0}^{N-1} \hat{\beta}_{M,L}(n, k; N, K; c)\, \hat{B}^{n+N-(M+L+1)}_{k+K}(fg), \tag{7.8}$$

where

$$\begin{aligned} \hat{\beta}_{m,l}(n, k; N, K; c) \, &= (1 - \delta_{(n-1-m)+(N-1-l),0}) \binom{n-1}{m} \binom{N-1}{l} \\ &\times \big((-1)^{n-m-1} - (-1)^{N-l-1}\big) k^{N-l-1} K^{n-m-1} c^{n+N-(m+l)-3}. \end{aligned} \tag{7.9}$$

We notice that, due to the presence of the factor

$$\big((-1)^{n-m-1} - (-1)^{N-l-1}\big) = \big((-1)^M - (-1)^L\big),$$

the only non-zero contribution to the commutator (7.8) comes from terms with M, L of different even/odd parity, which, in turn, implies that $M + L + 1$ is always even. Therefore, just like in Pope's W_∞ algebra, the commutator contains only terms of the form $\hat{B}^{n+N-2(l+1)}_{k+K}$, where we have set $n + N - (M + L + 1) =$

$n+N-2(l+1)$, with l ranging from 0 to $n+N-2$. We may therefore write the $W_\infty(c)$ commutation relations (7.8) as

$$[\hat{B}^n_k(f),\hat{B}^N_K(g)]=\sum_{l\geqslant 0}\hat{b}_l(n,k;N,K;c)\,\hat{B}^{n+N-2(l+1)}_{k+K}(fg), \tag{7.10}$$

where

$$\hat{b}_l(n,k;N,K;c)=\sum_{\substack{M,L\in\{0,1,\dots,n-1\}\\ M+L=2l+1}}\hat{\beta}_{M,L}(n,k;N,K;c). \tag{7.11}$$

In the one-mode case, i.e., over a fixed interval, the commutation relations (7.10) become

$$[\hat{B}^n_k,\hat{B}^N_K]=\sum_{l\geqslant 0}\hat{b}_l(n,k;N,K;c)\,\hat{B}^{n+N-2(l+1)}_{k+K}. \tag{7.12}$$

We notice the similarity between the one-mode $W_\infty(c)$ commutation relations (7.12) and the non-centrally extended W_∞ commutation relations (7.7).

8. CENTRAL EXTENSIONS OF $W_\infty(c)$

THEOREM 8.1. *The non-trivial central extensions of the* $W_\infty(c)$ *commutation relations* (7.8) *are given by*

$$[\hat{B}^n_k(f),\hat{B}^N_K(g)]=\sum_{l\geqslant 0}\hat{b}_l(n,k;N,K;c)\,\hat{B}^{n+N-2(l+1)}_{k+K}(fg)$$
$$+\,\delta_{n,N}\,\delta_{k+K,0}\,k(k^2-1)\,\sigma(n,k)\,E,$$

i.e.,

$$[\hat{B}^n_k(f),\hat{B}^n_{-k}(g)]=\sum_{l\geqslant 0}\hat{b}_l(n,k;n,-k;c)\,\hat{B}^{2(n-l-1)}_0(fg)+k(k^2-1)\sigma(n,k)\,E, \tag{8.1}$$

where, *in the notation of* (2.2) *and* (2.3),

$$\sigma(n,k):=\begin{cases}1 & \text{if } n=2,\\ \displaystyle\prod_{i=2}^{n-1}\frac{(k-r_i-1)^{(k-2)}}{(k+r_i+1)^{(k-2)}} & \text{if } k\geqslant 0, n>2,\\ \displaystyle\prod_{i=2}^{n-1}\frac{(k+r_i+1)_{-k-2}}{(k-r_i-1)_{-k-2}} & \text{if } k\leqslant 0, n>2,\end{cases} \tag{8.2}$$

$\sigma(n,k)\in\mathbb{R}$, *where* $-2,r_2,r_3,\dots,r_{n-1}$ *are the* $n-1$ *roots of the Jacobi polynomial*

$$P^{(0,-2n+1)}_{n-1}(-2r-1)={}_2F_1(1-n,1-n;1,r+1)=\sum_{L=0}^{n-1}\binom{n-1}{L}^2(r+1)^L.$$

We may take $E = cI$, where $c \in \mathbb{R}$ and I is the identity operator. Traditionally, the range of values of the charge c is determined in the study of representations of the Lie algebra commutation relations.

P r o o f. The presence of the Virasoro factor $k(k^2-1)$ in (8.2) implies the non-triviality of the central extensions. For if $f : W_\infty(c) \mapsto \mathbb{C}$ was a linear function satisfying, for all $n, k, N, K \in \mathbb{Z}$ with $n, N \geqslant 2$,

$$f([B^n_k, B^N_K]) = \delta_{n,N}\, \delta_{k+K,0}\, \rho(n,k),$$

then, letting $n = N = 2$, we would conclude that the Virasoro central extension is trivial.

To find a formula for $\rho(n,k)$, suppressing test functions, by the cocycle Jacobi identity (6.4), for $l_i = B^{n_i}_{k_i}$, $i = 1,2,3$, we obtain

$$\begin{aligned}&\sum_{L\geqslant 0} \hat{b}_L(n_1,k_1;n_2,k_2;c)\,\phi(B^{n_1+n_2-2(L+1)}_{k_1+k_2}, B^{n_3}_{k_3})\\ &\qquad + \sum_{L'\geqslant 0} \hat{b}_{L'}(n_2,k_2;n_3,k_3;c)\,\phi(B^{n_2+n_3-2(L'+1)}_{k_2+k_3}, B^{n_1}_{k_1})\\ &\qquad + \sum_{L''\geqslant 0} \hat{b}_{L''}(n_3,k_3;n_1,k_1;c)\,\phi(B^{n_3+n_1-2(L''+1)}_{k_3+k_1}, B^{n_2}_{k_2}) = 0,\end{aligned}$$

which, after combining the three summations into one, yields

$$\begin{aligned}&\sum_{L\geqslant 0} \Big(\hat{b}_L(n_1,k_1;n_2,k_2;c)\,\phi\big(n_1+n_2-2(L+1), k_1+k_2;\, n_3,k_3\big)\\ &\quad + \hat{b}_L(n_2,k_2;n_3,k_3;c)\,\phi\big(n_2+n_3-2(L+1), k_2+k_3;\, n_1,k_1\big)\\ &\quad + \hat{b}_L(n_3,k_3;n_1,k_1;c)\,\phi\big(n_3+n_1-2(L+1), k_3+k_1;\, n_2,k_2\big)\Big) = 0,\end{aligned}$$

which, by using $\phi(n,k;N,K) := \delta_{n,N}\,\delta_{k+K,0}\,\rho(n,k)$, becomes

$$\begin{aligned}&\sum_{L\geqslant 0} \Big(\hat{b}_L(n_1,k_1;n_2,k_2;c)\,\delta_{n_1+n_2-2(L+1),n_3}\,\delta_{k_1+k_2+k_3,0}\\ &\times \rho\big(n_1+n_2-2(L+1), k_1+k_2\big)\\ &+ \hat{b}_L(n_2,k_2;n_3,k_3;c)\delta_{n_2+n_3-2(L+1),n_1}\delta_{k_1+k_2+k_3,0}\rho\big(n_2+n_3-2(L+1),k_2+k_3\big)\\ &+ \hat{b}_L(n_3,k_3;n_1,k_1;c)\delta_{n_3+n_1-2(L+1),n_2}\delta_{k_1+k_2+k_3,0}\rho\big(n_3+n_1-2(L+1),k_3+k_1\big)\Big)\\ &= 0,\end{aligned}$$

i.e.,

$$\sum_{L\geqslant 0}\Big(\hat{b}_L(n_1,k_1;n_2,k_2;c)\,\delta_{n_1+n_2-n_3,2(L+1)}\,\rho(n_3,k_1+k_2)$$
$$+\,\hat{b}_L\big(n_2,k_2;n_3,-(k_1+k_2);c\big)\,\delta_{n_2+n_3-n_1,2(L+1)}\,\rho(n_1,-k_1)$$
$$+\,\hat{b}_L\big(n_3,-(k_1+k_2);n_1,k_1;c\big)\,\delta_{n_3+n_1-n_2,2(L+1)}\,\rho(n_2,-k_2)\Big)=0.$$

We notice that if $n_1+n_2+n_3$ is odd (resp. even), then so also are $n_1+n_2-n_3$, $n_1-n_2+n_3$ and $-n_1+n_2+n_3$. If $n_1+n_2+n_3$ is odd, then

$$\delta_{n_1+n_2-n_3,\,2(L+1)}=\delta_{n_2+n_3-n_1,\,2(L+1)}=\delta_{n_3+n_1-n_2,\,2(L+1)}=0,$$

and the above cocycle identity is trivially satisfied for all functions ρ. This is true, in particular, for $n_1=n_2=n_3=n$, where n is odd. From now on we assume that $n_1+n_2+n_3$ is even, which implies that there exist indices L_1,L_2,L_3 for which

$$\delta_{n_1+n_2-n_3,\,2(L_1+1)}=\delta_{n_2+n_3-n_1,\,2(L_2+1)}=\delta_{n_3+n_1-n_2,\,2(L_3+1)}=1.$$

In fact,

$$L_1=\frac{n_1+n_2-n_3}{2}-1,\qquad L_2=\frac{n_2+n_3-n_1}{2}-1,\qquad L_3=\frac{n_3+n_1-n_2}{2}-1,$$

and the cocycle Jacobi identity becomes

(8.3)
$$\hat{b}_{L_1}(n_1,k_1;n_2,k_2;c)\,\rho(n_3,k_1+k_2)+\hat{b}_{L_2}\big(n_2,k_2;n_3,-(k_1+k_2);c\big)\rho(n_1,-k_1)$$
$$+\,\hat{b}_{L_3}\big(n_3,-(k_1+k_2);n_1,k_1;c\big)\,\rho(n_2,-k_2)=0.$$

Using (8.3) we can conclude the arbitrariness of $\rho(n,k)$ for odd n as follows. Suppose that n_1 is odd. Since $n_1+n_2+n_3$ is even, it follows that n_2+n_3 is odd. Without loss of generality, suppose that n_2 is even and n_3 is odd. We will show that the coefficient $\hat{b}_{L_2}\big(n_2,k_2;n_3,-(k_1+k_2);c\big)$ of $\rho(n_1,-k_1)$ in (8.3) is equal to zero by showing that, in view of (7.9) and (7.11), the factors of the form $\big((-1)^{n_2-M-1}-(-1)^{n_3-L-1}\big)$ appearing in the definition of the coefficient $\hat{b}_{L_2}\big(n_2,k_2;n_3,-(k_1+k_2);c\big)$ are all equal to zero. In the notation of (7.11), we notice that $M+L=2L_2+1$ implies that M and L have different parity. That implies that n_2-M-1 and n_3-L-1 have the same parity. Therefore,

$$\big((-1)^{n_2-M-1}-(-1)^{n_3-L-1}\big)=0.$$

For $n_1=n_2=n_3=2$ (implying $L_1=L_2=L_3=0$) (8.3) reduces to the cocycle Jacobi identity for the Witt–Virasoro algebra (see Lemma 3 of [14]) and yields the well-known non-trivial Witt–Virasoro cocycle (see [14] for a proof of the non-triviality) with

$$\rho(2,k) = k\,(k^2-1).$$

For $n_1 = n_2 = n_3 = n$, which implies that $L_1 = L_2 = L_3 = n/2-1$ and (by the assumption that $n_1+n_2+n_3$ is even) that n is even, the cocycle Jacobi identity (8.3) reduces to

(8.4)

$$\begin{aligned}\hat{b}_{n/2-1}(n,k_1;n,k_2;c)\,\rho(n,k_1+k_2) + \hat{b}_{n/2-1}\big(n,k_2;n,-(k_1+k_2);c\big)\,\rho(n,-k_1)&\\ + \hat{b}_{n/2-1}\big(n,-(k_1+k_2);n,k_1;c\big)\,\rho(n,-k_2) &= 0.\end{aligned}$$

By (7.11) and (4.3), the formula for $\hat{b}_{n/2-1}$ involves a summation over $M, L \in \{0,1,\dots,n-1\}$ such that $M+L = n-1$. Since n is assumed to be even, $n-1$ is always odd, and so M and L are of different parity. Thus, in the definition of $\hat{b}_{n/2-1}$,

$$\begin{aligned}(-1)^{n-M-1} - (-1)^{n-L-1} &= (-1)^{n-1}\big((-1)^M - (-1)^L\big)\\ &= (-1)^L - (-1)^M = \pm 2.\end{aligned}$$

Moreover, since $M+L = n-1$, we cannot simultaneously have $M = L = n-1$. Thus

$$1 - \delta_{(n-1-L)+(n-1-M),0} = 1$$

and equation (8.4) takes the form

$$\begin{aligned}&\sum_{\substack{M,L\in\{0,1,\dots,n-1\}\\ M+L=n-1}} \big((-1)^L - (-1)^M\big)\binom{n-1}{M}\binom{n-1}{L}\\ &\times\{k_1^{n-L-1}k_2^{n-M-1}\rho(n,k_1+k_2) + (-1)^{n-M-1}k_2^{n-L-1}(k_1+k_2)^{n-M-1}\rho(n,-k_1)\\ &\qquad + (-1)^{n-L-1}(k_1+k_2)^{n-L-1}k_1^{n-M-1}\,\rho(n,-k_2)\} = 0,\end{aligned}$$

which, letting $M = n-1-L$, becomes

(8.5)

$$\begin{aligned}&\sum_{L=0}^{n-1}(-1)^L\binom{n-1}{L}^2\{k_1^{n-L-1}k_2^{L}\,\rho(n,k_1+k_2)\\ &+ (-1)^L k_2^{n-L-1}(k_1+k_2)^L\rho(n,-k_1) + (-1)^{n-L-1}(k_1+k_2)^{n-L-1}k_1^L\rho(n,-k_2)\}\\ &= 0.\end{aligned}$$

Replacing k by $-k$ in (8.1) and then multiplying the resulting identity by (-1) we find that

$$\rho(n,k) = -\rho(n,-k),$$

which allows us to extend the definition of $\rho(n,k)$ to negative k, provided that a formula for $\rho(n,k)$ for positive k has been obtained. In particular, $\rho(n,0)=0$. Thus, (8.5) becomes

$$\sum_{L=0}^{n-1}(-1)^L\binom{n-1}{L}^2\{k_1^{n-L-1}k_2^L\rho(n,k_1+k_2)-(-1)^Lk_2^{n-L-1}(k_1+k_2)^L\rho(n,k_1) - (-1)^{n-L-1}(k_1+k_2)^{n-L-1}k_1^L\,\rho(n,k_2)\}=0,$$

i.e.,

$$\sum_{L=0}^{n-1}\binom{n-1}{L}^2\{(-1)^Lk_1^{n-L-1}k_2^L\,\rho(n,k_1+k_2) - k_2^{n-L-1}(k_1+k_2)^L\,\rho(n,k_1)+(k_1+k_2)^{n-L-1}k_1^L\,\rho(n,k_2)\}=0. \tag{8.6}$$

We point out that the cocycle identity (8.6) was obtained under the assumption that n is even.

For $n=2$, equation (8.6) becomes

$$\sum_{L=0}^{1}\binom{1}{L}^2\{(-1)^Lk_1^{1-L}k_2^L\,\rho(2,k_1+k_2) - k_2^{1-L}(k_1+k_2)^L\,\rho(2,k_1)+(k_1+k_2)^{1-L}k_1^L\,\rho(2,k_2)\}=0,$$

which is easily checked to be satisfied by the Witt–Virasoro cocycle $\rho(2,k)=k\,(k^2-1)$.

For the general, even $n>2$ case, for $k_2=1$ and $k_1=k\geqslant 2$, assuming $\rho(n,1)=\rho(n,0)=0$, from equation (8.6) we obtain

$$\rho(n,k+1)=\frac{p(n,k)}{q(n,k)}\rho(n,k), \tag{8.7}$$

where

$$p(n,k)=\sum_{L=0}^{n-1}\binom{n-1}{L}^2(k+1)^L,$$

$$q(n,k)=\sum_{L=0}^{n-1}\binom{n-1}{L}^2(-1)^Lk^{n-L-1}.$$

We notice that:

(i) $p(n,k)$ and $q(n,k)$ are monic polynomials in k, of odd degree $n-1$.

(ii) $p(n,-2)=0$ and $q(n,1)=0$ (for even n).

(iii) $q(n,k-1)$ is the alternating version of $p(n,k)$, i.e., the coefficient of k^L in $q(n,k-1)$ is $(-1)^{L+1}$ times the corresponding coefficient in $p(n,k)$. Thus the roots k of $q(n,k-1)$ are the negatives of the roots $k=r_1=-2, k=r_2, k=r_3,\ldots,k=r_{n-1}$ of $p(n,k)$.

(iv) For each $j \in \mathbb{Z}$, the roots k of $p(n,k-j)$ are given by $k-j=-2$, $k-j=r_2, k-j=r_3, \dots, k-j=r_{n-1}$, so $k=j-2, k=j+r_2, k=j+r_3, \dots, k=j+r_{n-1}$. By (iii), the roots $k-j$ of $q(n,k-j-1)$ are the negatives of $k-j=-2, k-j=r_2, k-j=r_3, \dots, k-j=r_{n-1}$. Therefore, $k-j=2, k-j=-r_2, k-j=-r_3, \dots, k-j=-r_{n-1}$, i.e., $k=j+2, k=j-r_2, k=j-r_3, \dots, k=j-r_{n-1}$.

Thus, for each $j \in \mathbb{Z}$, we may write

$$p(n,k-j) = (k-j+2)(k-r_2-j)\dots(k-r_{n-1}-j),$$

$$q(n,k-j-1) = (k-j-2)(k+r_2-j)\dots(k+r_{n-1}-j),$$

and

$$\frac{p(n,k-j)}{q(n,k-j-1)} = \frac{(k-j+2)(k-r_2-j)\dots(k-r_{n-1}-j)}{(k-j-2)(k+r_2-j)\dots(k+r_{n-1}-j)}. \tag{8.8}$$

Iterating (8.7) and using (8.8) we obtain

$$\begin{aligned}\rho(n,k+1) &= \frac{1}{q(n,k)}\frac{p(n,k)}{q(n,k-1)}\frac{p(n,k-1)}{q(n,k-2)}\dots\frac{p(n,3)}{q(n,2)}p(n,2)\rho(n,2)\\ &= \frac{1}{q(n,k)}\frac{(k+2)(k+1)k(k-1)}{24}\frac{\prod_{i=2}^{n-1}(k-r_i)^{(k-2)}}{\prod_{i=2}^{n-1}(k+r_i)^{(k-2)}}p(n,2)\rho(n,2).\end{aligned} \tag{8.9}$$

Since

$$p(n,k+1) = (k+3)\prod_{i=2}^{n-1}(k-r_i+1),$$

$$q(n,k) = (k-1)\prod_{i=2}^{n-1}(k+r_i+1),$$

and so

$$p(n,2) = 4\prod_{i=2}^{n-1}(2-r_i),$$

substituting in (8.9), after replacing k by $k-1$, defining

$$\rho(n,2) = 2(2^2-1) \tag{8.10}$$

and simplifying, we obtain for $k \geqslant 0$

$$\rho(n,k) = k(k^2-1)\prod_{i=2}^{n-1}\frac{(k-r_i-1)^{(k-2)}}{(k+r_i+1)^{(k-2)}}.$$

Now let $k \leqslant 0$. Then

$$\rho(n,k) = -\rho(n,-k) = -(-k)\big((-k)^2-1\big)\prod_{i=2}^{n-1}\frac{(-k-r_i-1)^{(-k-2)}}{(-k+r_i+1)^{(-k-2)}}$$

$$= k(k^2-1)\prod_{i=2}^{n-1}\frac{(-1)^{-k-2}(k+r_i+1)_{-k-2}}{(-1)^{-k-2}(k-r_i-1)_{-k-2}} = k(k^2-1)\prod_{i=2}^{n-1}\frac{(k+r_i+1)_{-k-2}}{(k-r_i-1)_{-k-2}},$$

where we have used

$$x^{(n)} = (-1)^n(-x)_n.$$

By (8.9) and (8.10), $\rho(n,k) \in \mathbb{R}$. ■

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