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SMALL DEVIATION PROBABILITIES OF WEIGHTED SUMS WITH FAST DECREASING WEIGHTS*

BY

LEONID V. ROZOVSKY (SAINT-PETERSBURG)

Abstract. We examine small deviation probabilities of weighted sums of i.i.d. positive random variables whose distribution function is regularly varying at zero provided that weights are decreasing fast enough.

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1. INTRODUCTION

Let $\{X_n\}_{n\geqslant 1}$ be independent copies of a positive random variable X with distribution function $F(x) = \mathbf{P}(X < x)$ and let $a(\cdot)$ be a continuous and *non-increasing* positive function on $[1, \infty]$ such that

$$\sum_{n\geqslant 1} \mathbf{E} \min\big(1, a(n)X\big) < \infty. \tag{1.1}$$

It is well known (see [9] or [2]) that (1.1) is the necessary and sufficient condition under which the series $S = \sum_{n\geqslant 1} a(n)\, X_n$ converges almost surely.

Our basic aim is to get asymptotics in an *explicit* form for $\log \mathbf{P}(S < r)$ as $r \to 0$, somewhat sharper than earlier known, assuming that

$$b(u) = a^{-1}(1/u) \in \mathbf{R}_0, \tag{1.2}$$

the class of slowly varying functions (here we assume that $u \geqslant u_0 \geqslant 1/a(1)$ and $a^{-1}(x) = \sup\{y : a(y) \geqslant x\}$ denotes the inverse function of a), and

$$F(1/\cdot) \in \mathbf{R}_{-\alpha}, \tag{1.3}$$

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the class of regularly varying functions with index $-\alpha < 0$ (or, in other words, $F(r) \sim r^{\alpha}\, h(1/r)$ as $r \to 0^{+}$ and $h(1/\cdot) \in \mathbf{R}_0$).

Note that if (1.2) holds, then the weights $a(n)$ have to decrease fast enough, faster than any power of n, at least, and that (see [11]) (1.1) is equivalent to

$$\mathbf{E}b(X)\, I[X > u_0] < \infty. \tag{1.4}$$

Let us recall a few earlier known results, the most close to the subject of the note (a complete bibliography on the theme can be found in [6]; see also [5]).

Set $f(u) = \mathbf{E}e^{-uX},\ u \geqslant 0$, and formulate one result following from Theorem 4 of [14].

THEOREM 1.1. *Let $a(\cdot)$ be a twice differentiable function on $[1, \infty]$ such that $\int_1^{\infty} \big|\big(\log a(t)\big)''\big|\, dt < \infty$ and*

$$\limsup_{n\to\infty} \sum_{l\geqslant 1} \mathbf{E} \min\left(1, \frac{a(l\,n)}{a(n)}\, X\right) < \infty. \tag{1.5}$$

Assume that the distribution F satisfies (1.3) *and*

(1.6) *the function $\big(s(\log f)'(s)\big)'$ is absolutely integrable at infinity.*

Then, as $r \to 0^{+}$,

$$\log \mathbf{P}(S < r) = I(u) - u\, I'(u) + \big(\log F(1/u) - \log a^{-1}(1/u)\big)/2 + O\,(1),$$

where $I(u) = \int_1^{\infty} \log f\big(ua(t)\big)\, dt$, and $u = u(r)$ is the unique solution of the equation $I'(u) + r = 0$.

Observe that (1.5) is appreciably milder than moment conditions in [4], where the *exact* asymptotics for $\mathbf{P}(S < r)$ was examined. For instance, (1.5) and (1.1) are equivalent if $\log\big(1/a(n)\big) = g(\log n) + O\,(1)$, where the function $g(y)/y$ does not decrease for all y large enough.

Let us note that several conditions under which (1.6) holds can be found in [4] and [12]. For instance, it is sufficient to assume that $u\,\big(\log F(u)\big)'$ tends monotonically to $-\alpha$ as $u \searrow 0$ (and therefore (1.3) holds).

The next result follows from Theorem 6 of [13] (see also [2], Theorem 4.1, and [7], Theorem 2, for the case $X = \xi^2$ with $\xi \sim \mathbf{N}(0, 1)$).

THEOREM 1.2. *Let a constant $\alpha > 0$ and*

$$\log F(r) \sim \alpha \log r \quad \textit{as } r \to 0. \tag{1.7}$$

If (1.2) *holds and*

$$\mathbf{E}g(X)\, I[X > 1] < \infty \tag{1.8}$$

for

$$g(t) = \sup_{u \geqslant u_0} \frac{b(tu)}{b(u)}, \tag{1.9}$$

then

$$-\log \mathbf{P}(S < r) \sim \alpha\, l(s) \quad \text{as } r \to 0^+,$$

where $l(s) = \int_{u_0}^{s} b(u)\, du/u$ *and* $s = s(r) > u_0$ *satisfies the condition* $l(s) \sim s\, r$.

In particular, *if*

$$a(n) = e^{-(n-1)/c}, \quad n \geqslant 1,\ c > 0, \tag{1.10}$$

then $u_0 = 1$, $b(u) = g(u) = 1 + c \log u$ *and*

$$-\log \mathbf{P}(S < r) \sim \frac{\alpha\, c}{2} \log^2 r \quad \text{as } r \to 0^+.$$

Observe that if $\{\lambda_n\}$ is a positive sequence such that $\log\big(\lambda_n/a(n)\big) = O\,(1)$ then, under the conditions of Theorem 1.2,

$$\log \mathbf{P}\Big(\sum_{n \geqslant 1} \lambda_n X_n < r\Big) \sim \log \mathbf{P}(S < r) \quad \text{as } r \to 0^+.$$

Note also that (1.7) is weaker than (1.3). Moreover (see [11], Remark 2, or [13], Lemma 1), if

$$\log\big(b(u)/\tilde{b}(u)\big) = O(1) \quad \text{and} \quad u\,\big(\log \tilde{b}(u)\big)' \searrow 0 \quad \text{as } u \to \infty \tag{1.11}$$

then

$$g(u) = O\big(b(u)\big) \quad \text{as } u \to \infty, \tag{1.12}$$

and therefore (1.8) is equivalent to the necessary condition (1.4). Let us note that if $-u\big(\log a(u)\big)' \nearrow \infty$ as $u \to \infty$, then (1.11) holds.

Remark that (1.5) follows from (1.8). To verify this fact one can take into account that (1.5) is equivalent to

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{l \geqslant n} \mathbf{E} \min\bigg(1, \frac{a(l)}{a(n)}\, X\bigg) < \infty,$$

and evaluate the sum above by using (1.2), (1.9) and the reasoning from [11], (18)–(20).

The following assertion takes an intermediate position between Theorems 1.1 and 1.2 (the case (1.10)).

THEOREM 1.3. *Let* $\mathbf{E}\log(1+X)<\infty$ *and, for some rational* $\alpha>0$,

$$F(r)\sim b\,r^{\alpha}\quad \textit{as } r\to 0^{+},\ b>0.$$

If (1.10) *holds*, *then*

$$-\log \mathbf{P}(S<r)=\frac{\alpha\,c}{2}\,s^{2}+\alpha\,c\,s\,\log s+\big(\kappa+o(1)\big)\,s\quad \textit{as } r\to 0^{+}, \tag{1.13}$$

where $s=|\log r|$ *and* $\kappa=\alpha/2-c\,\log b+\alpha\,c\,\log(\alpha\,c)-c\,\log\Gamma(1+\alpha)-\alpha\,c.$

Theorem 1.3 was proved in [3] by means of the reasoning using results on asymptotic analysis of the delayed differential equations. Such a rather subtle method led, in particular, to the redundant requirement of rationality of α.

Note also that (1.13) for *all* $\alpha>0$ under the additional assumption (1.6) follows from Theorem 1.1 (see the details in [14], Corollary 2).

The general aim of the present note is to obtain asymptotics for $\log\mathbf{P}(S<r)$, lying between ones of Theorems 1.1 and 1.2, more general and refined in comparison with Theorem1.3.

Our results are arranged in Section 2. Sections 3 and 4 contain some auxiliary results and the proofs of Theorems 2.1–2.3, respectively. In Section 5 we prove Corollaries 1.1–1.3.

2. RESULTS

In what follows, besides conditions (1.1)–(1.3) we assume that a positive non-increasing sequence $\{\lambda_n\}$ satisfies the condition

$$\lambda_n\sim a_n=a(n), \tag{2.1}$$

and

$$F(r)\sim r^{\alpha}\,F_0(r)\quad \text{as } r\to 0^{+}, \tag{2.2}$$

assuming without loss of generality that a positive function $F_0(\cdot)$, defined on the interval $(0,1]$, is continuous and slowly varying at zero (one can take, say, $F_0(r)=r^{-\alpha}\,f(1/r)/\Gamma(1+\alpha)$). For instance, if $X=|\xi|^{p}$ with $p>0$ and $\xi\sim\mathbf{N}(0,1)$, then $\alpha=1/p$ and $F_0(\cdot)=\sqrt{2/\pi}$.

Denote, for simplicity, $\mathbf{P}\big(\sum_{n\geqslant 1}\lambda_n\,X_n<r\big)$ by $V(r)$. Notice that the condition $V(\infty)=1$ is equivalent to (1.1) (or (1.4)).

Further we present some new asymptotics for $\log V(r)$ whose forms somewhat differ, depending on properties of $a(\cdot)$.

The first result is formulated under the assumption

$$|\log a(u)|=o(u)\ \text{(that is, } b(u)/\log u\to\infty)\quad \text{as } u\to\infty. \tag{2.3}$$

Thus, $a(\cdot)$ decreases faster than a power and slower than an exponent, as in the case

$$a(u) = e^{-c \log^\delta u} \text{ (or } b(t) = e^{(c^{-1} \log t)^{1/\delta}}\text{)} \tag{2.4}$$

with some $\delta > 1$ and $c > 0$.

Let, as in Theorem 1.2, $l(s) = \int_{u_0}^s b(u)\, du/u,\ s > u_0$.

THEOREM 2.1. *If* (2.3) *and* (1.8) *hold*, *then for any* $u_0 \geqslant 1/a(1)$

(2.5)

$$-\log V(r) = \alpha\, l(h) + \int_{u_0}^{h} -\log F_0(u/h)\, db(u) + \big(C_\alpha + o\,(1)\big)\, b(h) \quad \textit{as } r \to 0,$$

where $C_\alpha = \alpha \log \alpha - \alpha - \log \Gamma(1+\alpha)$ *and* $h = h(r) > u_0$ *is any function such that*

$$h/b(h) \sim 1/r \quad \textit{as } r \to 0. \tag{2.6}$$

Let us consider a consequence of Theorem 2.1 for the case (2.4), under which (1.8) is equivalent to the necessary condition (1.4) (see (1.11) and (1.12)).

We shall also assume that

$$F_0(e^{-u}) \in \mathbf{R}_\gamma \tag{2.7}$$

for some γ or, equivalently, $F_0(1/t) \sim (\log t)^\gamma\, H(t)$ as $t \to \infty$, where a positive function $H(t)$ is slowly varying at infinity.

COROLLARY 2.1. *Let* (2.4), (1.4) *and* (2.7) *hold. Then we have as* $r \to 0$:
in the case $\delta > 2$,

$$-\log V(r) = e^{\tilde s} \Big(\alpha\, c\, \delta\, \tilde s^{\delta-1} \big(e^{\tilde s^{2-\delta}/(c\,\delta)} + \sum_{l=1}^{[\delta-1]} \nu_l\, \tilde s^{-l}\big) + C(r) + o\,(1)\Big), \tag{2.8}$$

where

$$\tilde s = (s/c)^{1/\delta}, \qquad s = |\log r|, \qquad \nu_l = (-1)^l \prod_{k=1}^{l} (\delta - k),$$

$[x]$ *denotes the integer part of* x, $C(r) = C_\alpha - \gamma \log c\delta + \gamma\, \mathcal{E} - \log F_0(e^{-\tilde s^{\delta-1}})$ *and* $\mathcal{E} = -\int_0^\infty e^{-y} \log y\, dy$ *is the Euler constant*;

in the case $\delta = 2$,

$$-\log V(r) = e^{\tilde s + 1/(2c)} \big(2\,\alpha\, c\, \tilde s + C(r) - 2\,\alpha\, c + \alpha + \alpha/(4\,c) + o\,(1)\big); \tag{2.9}$$

in the case $1 < \delta < 2$,

$$-\log V(r) = e^{Y_M} \big(\alpha\, c\, \delta\, \tilde s^{\delta-1} + C(r) + \alpha\,(\delta-1) + o\,(1)\big) \tag{2.10}$$

provided that $Y_M = \tilde{s}\left(1 + \sum_{\nu=1}^{M} \alpha_{\nu+1}\,\tau^{\nu}\right)$, $M = [\delta/(\delta-1)]$, $\tau = \tilde{s}^{1-\delta}/c$ *and the coefficients* α_ν *are defined by the relation*

$$\alpha_1 = 1, \quad \alpha_{\nu+1} = \sum \prod_{l=0}^{s-1}(1/\delta - l) \prod_{m=1}^{\nu} \frac{{\alpha_m}^{k_m}}{k_m!}, \ \nu \geqslant 1, \tag{2.11}$$

where the summation is taken over all integers $k_m \geqslant 0$ *with* $1 \cdot k_1 + \ldots + \nu \cdot k_\nu = \nu$, *and* $s = k_1 + \ldots + k_\nu$ (*in particular*, $a_2 = 1/\delta$, $a_3 = (3-\delta)/(2\delta^2)$).

The next our result is valid if

$$-\left(\log a(u)\right)' \to 1/c > 0 \quad \text{as } u \to \infty, \tag{2.12}$$

which, in turn, is equivalent to $u\, b'(u) \to c$, and implies $\log\left(1/a(u)\right) \sim 1/c\, u$ and $b(u)/\log u \to c$.

THEOREM 2.2. *Let* (2.12) *hold and* $\mathbf{E}\log(1+X) < \infty$. *Then we have for any* $u_0 \geqslant 1/a(1)$ *as* $r \to 0$

$$(2.13) \quad -\log V(r)$$
$$= \alpha\, l(h) + \int_{u_0}^{h} -\log F_0(u/h)\, db(u) + \left(c\, C_\alpha + \alpha/2 + \alpha\, c \log c + o(1)\right)\log(1/r),$$

where $h = |\log r|/r$ (*see also the notation in Theorem* 2.1).

Note that the moment assumptions in Corollary 2.1 and in Theorem 2.2 are necessary and sufficient for $V(\infty) = 1$.

COROLLARY 2.2. *Let* $\mathbf{E}\log(1+X) < \infty$ *and* $\lambda_n \sim e^{d-n/c}$ *with some constants* d *and* $c > 0$. *Moreover*, *let* (2.7) *hold true. Then*, *as* $r \to 0$,

$$-\log V(r) = c\, s\left(\frac{\alpha}{2}\, s + \alpha\, \log s - \log F_0(r) + \kappa + o(1)\right), \tag{2.14}$$

where $\kappa = \alpha\log(\alpha c) + \alpha\,(d-1) + \alpha/(2\,c) - \log\Gamma(1+\alpha) + \gamma$ *and* $s = \log(1/r)$.

Putting $F_0(\cdot) = b > 0$ (or $\gamma = 0$) and $d = 1/c$ in (2.14), we get (1.13) for *any* α.

The relations (2.3) and (2.6) presuppose that λ_j (or $a(j)$) tends to zero not too fast, for instance, $-\log a(j) = j^\delta$, $0 < \delta \leqslant 1$. The following approach allows us to consider a more general (in comparison with Theorem 1.1) situation, including the case $1 < \delta < 2$.

Assume, in addition to (2.1) and (2.2), that the functions $a(t)$ and $F_0(1/t)$ (see (2.2)) are twice differentiable for all $t > t_0 > 1$.

Put $\mu(t) = t\left(\log F_0(1/t)\right)'$, $t \geqslant t_0$, and introduce the conditions

$$\int_{t_0}^{\infty} |\mu'(t)|\, dt < \infty \tag{2.15}$$

and

(2.16)
$$\frac{1}{T}\int\limits_{t_0}^{T}\left|\big(\log a(t)\big)''\right|dt\to 0 \quad \text{as } T\to\infty.$$

Note that (2.15) is a mild version of condition (1.6), and it obviously holds if $\mu(\cdot)$ is monotone at infinity (since $\mu(t)\to 0$ as $t\to\infty$) as in the case $F_0(1/t) = c\,\log^{\delta} t$ in which $t\,\mu'(t) = -\delta/\log^2 t$.

THEOREM 2.3. *Let* (2.15), (2.16) *and* (1.8) *hold true. Then we have for any* $u_0 \geqslant 1/a(1)$ (*see the notation in Theorem* 2.1)

(2.17)
$$-\log V(r) = \alpha\, l(h) + \int\limits_{u_0}^{h} -\log F_0(u/h)\,db(u) - (\alpha/2)\log(1/r)$$
$$-\big(b(u_0)-1/2\big)\log F_0(r) + \big(C_\alpha + o(1)\big)\,b(h) \quad \text{as } r\to 0.$$

Now consider the example which follows from Theorem 2.3.

COROLLARY 2.3. *Let* $\lambda_n \sim e^{d-(n/c)^{\delta}}$ *with some constants* $d,\ c>0$ *and* $0<\delta<2$. *If* $\mathbf{E}\log^{1/\delta}(1+X)<\infty$ *and* (2.7) *holds, then, as* $r\to 0$,

(2.18)
$$-\log V(r) = c\,s^{1/\delta}\left(\frac{\alpha\delta}{1+\delta}\,s + \frac{\alpha}{\delta}\,\log s - \log F_0(r) + \kappa + o(1)\right) - \frac{\alpha}{2}\,s$$

with $s=\log(1/r)$, $\kappa = \alpha\,(d-1)+\alpha\log(\alpha\,c) - \log\Gamma(1+\alpha) - \gamma\,\nu$, *where*

$$\nu = \int\limits_0^1 \frac{1-(1-u)^{1/\delta}}{u}\,du.$$

Note that Theorem 1.1 does not work in the case $a(n) = e^{d-(n/c)^{\delta}},\ 1<\delta<2$.

3. AUXILIARY RESULTS

We start with several auxiliary results.

Let $\{\lambda_n\}$ be a positive non-increasing sequence, $Z=\sum_{n\geqslant 1}\lambda_n X_n$, and $V(r) = \mathbf{P}(Z<r)$. Assuming that $V(\infty)=1$, put for $u>0$

(3.1)
$$\lambda(u) = \mathbf{E}e^{-u\,Z}, \quad L(u) = \log\lambda(u),$$
$$m(u) = -L'(u), \quad \sigma^2(u) = L''(u), \quad Q(u) = uL'(u) - L(u), \quad \tau(u) = u\,\sigma(u).$$

LEMMA 3.1. *Let* (1.3), (1.5) *and* (2.1) *hold true. Then*

(3.2)
$$-\log V(r) = Q(h) + \log\tau(h) + O(1) \quad \textit{as } r\to 0,$$

where $h = h(r)$ *is the unique solution of the equation*

$$m(h) = r. \tag{3.3}$$

Lemma 3.1 follows from Theorem 3 and the Lemma of [14] (recall that (1.8) implies (1.5)).

Let us continue. At first we show that if (1.8) (along with (1.2), (2.2) and (2.1)) holds, then (see the notation in (3.1)), as $h \to \infty$,

$$-L(h) = \sum_{1 \leqslant j \leqslant N} \big(-\log f(a_j\, h)\big) + o\big(b(h)\big), \tag{3.4}$$

$$h\, m(h) \sim \tau^2(h) = h^2\, \sigma^2(h) \sim \alpha\, b(h) \tag{3.5}$$

provided that the integer $N = N(h)$ satisfies the condition $h\, a_{N+1} < 1 \leqslant h\, a_N$, and hence $N \leqslant b(h) \leqslant N + 1$.

Let $\epsilon = \epsilon(h) > 0$ tend to zero slowly enough together with h and let parameters $M = M(h)$ and $R = R(h)$ be such that

$$h\, a_{R+1} < 1/\epsilon \leqslant h\, a_R, \qquad h\, a_{M+1} \leqslant \epsilon < h\, a_M, \tag{3.6}$$

which (see (1.2)), in particular, implies that $R \leqslant b(h\,\epsilon) \leqslant R + 1$, $M \leqslant b(h/\epsilon) \leqslant M + 1$, and, by standard properties of slowly varying functions, we get $R \sim N \sim M \sim b(h)$ as $h \to \infty$.

We have (recall that $f(u) = \mathbf{E}e^{-uX}$)

$$-L(h) = \Big(\sum_{1 \leqslant j \leqslant R} + \sum_{R < j \leqslant N} + \sum_{N < j \leqslant M} + \sum_{j > M} \Big)\big(-\log f(\lambda_j\, h)\big) = I_1 + \ldots + I_4 \tag{3.7}$$

(if $R = N$ or/and $N = M$, the reasoning is only simplified).

Now, by (1.8), arguing as in [11] ((27), etc.), again, one gets

$$I_4 = o\,\big(b(h)\big) \quad \text{as } h \to \infty. \tag{3.8}$$

It is well known that (2.2) implies, as $t \to \infty$,

$$f(t) \sim l_\alpha(t) = \Gamma(1 + \alpha)\, t^{-\alpha}\, F_0(1/t), \tag{3.9}$$

$$t\, \big(\log f(t)\big)' \to -\alpha, \quad t^2\, \big(\log f(t)\big)'' \to \alpha. \tag{3.10}$$

Taking into account (3.9) and (2.1), we obtain

$$I_1 = \sum_{1 \leqslant j \leqslant R} \big(-\log f(a_j\, h)\big) + o\,\big(b(h)\big) \quad \text{as } h \to \infty.$$

Moreover, as $h \to \infty$,

$$I_2 + I_3 \leqslant (M - R)\big(-\log f(h\, \lambda_{R+1})\big) = o\,\big(b(h)\, |\log f(1/\epsilon)|\big) = o\,\big(b(h)\big).$$

Combining these estimates, (3.7) and (3.8), we obtain (3.4).

By using (3.10), the condition (3.5) can be verified similarly.

Let a function $h_* = h_*(r)$ tend to infinity and satisfy the condition

$$h_*/b(h_*) \sim \alpha/r \quad \text{as } r \to 0. \tag{3.11}$$

We infer by (3.5) and (3.11) that the solution h of the equation (3.3) satisfies the condition

$$h \sim h_*, \quad h\,r \sim \alpha\, b(h_*) \quad \text{as } r \to 0. \tag{3.12}$$

Now we show that (see (3.1))

$$Q(h) = -h_*\,r - L(h_*) + o\big(b(h_*)\big) \quad \text{as } r \to 0. \tag{3.13}$$

Indeed, $Q(h) = -h\,r - L(h)$. Since, by (3.5) and (3.12),

$$-h_*\,r - L(h_*) - Q(h) = -\frac{(h_* - h)^2}{2}\,\sigma^2(\tilde{h})\,\Big|_{\tilde{h}\in(h,h_*)} = o\big(b(h_*)\big) \quad \text{as } r \to 0,$$

and (3.13) follows.

Using (3.13), (3.11) and (3.4) one easily gets

$$Q(h) = -\sum_{1\leqslant j\leqslant N_*} \log f(a_j\,h_*) - \big(\alpha + o\,(1)\big)\,b(h_*) \quad \text{as } r \to 0, \tag{3.14}$$

where $N_* = [b(h_*)]$, and therefore (see (1.2)) $h_*\,a_{N_*+1} \leqslant 1 \leqslant h_*\,a_{N_*}$.

Next we change the sum in (3.14) by the appropriate integral. The Euler–MacLaurin summation formula of first order gives

(3.15)

$$\sum_{j=1}^{N_*} \log f(h_*\,a_j) = \int_1^{N_*} \log f\big(h_*\,a(u)\big)\,du + \frac{1}{2}\,\big(\log f(h_*\,a_1) + \log f(h_*\,a_{N_*})\big) + \Sigma_1,$$

where

$$\Sigma_1 = \sum_{j=1}^{N_*-1} \int_0^1 \frac{2t-1}{2}\,\Big(\log f\big(h_*\,a(t+j)\big)\Big)'\,dt.$$

Obviously,

$$|\Sigma_1| \leqslant \frac{1}{2}\int_1^{N_*} \Big(\log f\big(h_*\,a(u)\big)\Big)'\,du = \frac{1}{2}\,\log\big(f(h_*\,a_{N_*})/f(h_*\,a_1)\big). \tag{3.16}$$

4. PROOFS OF THEOREMS 2.1–2.3

P r o o f o f T h e o r e m 2.1. Let the assumption (2.3) hold true. Then from (3.16), by (2.2) and (3.9), it follows that

$$\Sigma_1 = o\,(1)\, b(h_*) \quad \text{as } r \to 0, \tag{4.1}$$

and, moreover, $-\log f(h_*\, a_1) \sim \alpha\, \log h_* = o\,(1)\, b(h_*)$,

$$\begin{aligned} 0 \leqslant \int_{N_*}^{b(h_*)} -\log f\big(h_*\, a(u)\big)\, du &\leqslant -\log f(h_*\, a_{N_*}) \leqslant -\log f(a_{N_*}/a_{N_*+1}) \\ &\sim \alpha\,\big(\log\,(1/a_{N_*+1}) - \log\,(1/a_{N_*})\big) = o\,(1)\, b(h_*). \end{aligned} \tag{4.2}$$

Thus (2.3) implies

$$Q(h) = \int_{1}^{b(h_*)} -\log f\big(h_*\, a(u)\big)\, du - \big(\alpha + o\,(1)\big)\, b(h_*) \quad \text{as } r \to 0. \tag{4.3}$$

We have by (3.9) (irrespective of (2.3)), for any $u_0 \geqslant 1/a_1$ as $r \to 0$,

$$\begin{aligned} &\int_{1}^{b(h_*)} \log f\big(h_*\, a(u)\big)\, du \\ &\quad = \big(b(u_0) - 1\big) \log f(h_*) + \int_{u_0}^{h_*} \log f(h_*/u)\, db(u) + o\,\big(b(h_*)\big) \\ &\quad = \big(b(u_0) - 1\big) \log f(h_*) + \int_{u_0}^{h_*} \log F_0(u/h_*)\, db(u) \\ &\qquad + \alpha \int_{u_0}^{h_*} \log\,(u/h_*)\, db(u) + \big(\log \Gamma(1+\alpha) + o\,(1)\big)\, b(h_*). \end{aligned} \tag{4.4}$$

Next (see the notation before Theorem 2.1),

$$\begin{aligned} \int_{u_0}^{h_*} -\log\,(u/h_*)\, db(u) &= l(h_*) - b(u_0)\,(\log h_* - \log u_0), \\ l(h_*) &= l(h_*/\alpha) + \big(\log \alpha + o\,(1)\big)\, b(h_*) \quad \text{as } r \to 0. \end{aligned} \tag{4.5}$$

Combining (4.3)–(4.5) and using (3.2), (3.3), (3.9) and (2.3), (2.6), we easily obtain (2.5) (with $h = h_*/\alpha$), and complete the proof of Theorem 2.1. ∎

P r o o f o f T h e o r e m 2.2. Assuming (2.12), we return to (3.14) and (3.15), provided that $h_* = c\,\alpha\, |\log r|/r$ (and thus (3.11) is satisfied). Observe that the conditions (4.1) and (4.2) still hold.

Let us verify (4.1). We have, taking R such that $h_* a_R \geqslant 1/\epsilon > h_* a_{R+1}$, where $\epsilon = \epsilon(r)$ tends to zero slowly enough,

$$\Sigma_1 = \Big(\sum_{1 \leqslant j \leqslant [\epsilon N_*]} + \sum_{[\epsilon N_*] < j \leqslant R} + \sum_{R < j < N_*} \Big) \int_0^1 \frac{2t-1}{2} \Big(\log f\big(h_* \, a(t+j)\big)\Big)' \, dt$$
$$= I_1 + I_2 + I_3.$$

Then, as earlier in (3.16),

$$|I_1| \leqslant \frac{1}{2} \log \big(f(h_* \, a_{[\epsilon N_*]})/f(h_* \, a_1)\big), \quad |I_3| \leqslant \frac{1}{2} \log \big(f(h_* \, a_{N_*})/f(h_* \, a_{R+1})\big)$$

and, due to (2.12), $I_1 + I_3 = o\,(1)\, b(h_*)$ as $r \to 0$.

Now, if $\epsilon N_* \leqslant j \leqslant R$, then by (3.10) uniformly in $t \in [0,1]$, as $r \to 0$,

$$\Big(\log f\big(h_* \, a(t+j)\big)\Big)' = \big(s \, \log' f(s)\big)\big|_{s = h_* \, a(t+j)} \Big(\log \big(1/a(t+j)\big)\Big)' \to -\alpha/c,$$

which, keeping in mind that $\int_0^1 \big((2t-1)/2\big)\, dt = 0$, leads to $I_2 = o\,(1)\, b(h_*)$ as $r \to 0$. Hence, under the condition (2.12) we get, as $r \to 0$,

$$Q(h) = \int_1^{b(h_*)} -\log f\big(h_* \, a(u)\big)\, du + \alpha \big(1/(2c) - 1 + o\,(1)\big)\, b(h_*). \tag{4.6}$$

Since

$$\log \big(f(h_*)/f(1/r)\big) \sim \alpha \, \log r \, h_* \sim \alpha \, \log b(h_*) = o\,\big(b(h_*)\big) \quad \text{as } r \to 0 \tag{4.7}$$

(see (3.9) and (3.11)), using (4.6) instead of (4.3), one can obtain (2.13) in just the same way as (2.5). Thus, Theorem 2.2 is proved. ■

P r o o f o f T h e o r e m 2.3. We have (see (3.9), (3.6), etc.), putting $R_* = R(h_*)$,

$$\begin{aligned} \sum_{1 \leqslant j \leqslant N_*} \big(-\log f(a_j \, h_*)\big) &= \sum_{1 \leqslant j \leqslant R_*} \big(-\log f(a_j \, h_*)\big) + o\,(1)\, b(h_*) \\ &= \sum_{1 \leqslant j \leqslant R_*} \big(-\log l_\alpha(a_j \, h_*)\big) + o\,(1)\, b(h_*) \quad \text{as } r \to 0. \end{aligned} \tag{4.8}$$

Applying the Euler–MacLaurin summation formula of second order to estimate the last sum in (4.8), we find

$$\begin{aligned} &\sum_{1 \leqslant j \leqslant R_*} \big(-\log l_\alpha(a_j \, h_*)\big) \\ &= \int_1^{R_*} \Big(-\log l_\alpha\big(h_* \, a(u)\big)\Big)\, du + \frac{1}{2} \Big(-\log l_\alpha\big(h_* \, a(1)\big) - \log l_\alpha\big(h_* \, a(R_*)\big)\Big) + \Sigma_2, \end{aligned} \tag{4.9}$$

where

$$\Sigma_2 = \sum_{j=1}^{R_*-1} \int_0^1 \frac{t-t^2}{2} \Big(\log l_\alpha\big(h_*\, a(t+j)\big)\Big)'' \, dt.$$

Next,

$$|\Sigma_2| \leqslant \frac{1}{8}(A_1 + A_2), \tag{4.10}$$

where

$$A_1 = \int_1^{R_*} \big| \big(\log a(u)\big)'' \big| \, \big|\mu_\alpha\big(h_*\, a(u)\big)\big| \, du,$$

$$A_2 = \int_1^{R_*} \big| \big(\log a(u)\big)' \big| \, \Big| \Big(\mu\big(h_*\, a(u)\big)\Big)' \Big| \, du.$$

But $A_1 = o\,(R_*) = o\,\big(b(h_*)\big)$ as $r \to 0$, by (2.16), and

$$A_2 \leqslant \sup_{1 \leqslant u \leqslant R_*} \big| \big(\log a(u)\big)' \big| \int_{h_*\, a_{R_*}}^{h_*\, a_1} |\mu'(s)| \, ds = o\,\big(b(h_*)\big) \quad \text{as } r \to 0,$$

since due to (2.15) the integral above tends to zero (recall that $h_*\, a_{R_*} \geqslant 1/\epsilon$) and, by virtue of (2.16), as $r \to 0$,

$$\sup_{1 \leqslant u \leqslant R_*} \big| \big(\log a(u)\big)' \big| \leqslant \sup_{1 \leqslant u \leqslant R_*} \Big(|\log a(1)| + \int_1^u \big| \big(\log a(t)\big)'' \big| \, dt \Big) = O\,\big(b(h_*)\big). \tag{4.11}$$

Moreover, (3.9) and (2.16) imply in (4.9), as $r \to 0$,

$$-\log l_\alpha\big(h_*\, a(1)\big) = -\log l_\alpha(h_*) + O\,(1)$$

and

$$-\log l_\alpha\big(h_*\, a(R_*)\big) = O\,\Big(\log 1/\epsilon + \log\big(a(R_*)/a(R_*+1)\big)\Big) = o\,\big(b(h_*)\big)$$

because, similarly to (4.11),

$$\begin{aligned} \log\big(a(R_*)/a(R_*+1)\big) &= \int_{R_*}^{R_*+1} \big| \big(\log a(t)\big)' \big| \, dt \\ &\leqslant \sup_{R_* \leqslant u \leqslant R_*+1} \big| \big(\log a(u)\big)' \big| = o\,\big(b(h_*)\big) \quad \text{as } r \to 0. \end{aligned}$$

Therefore, using (3.9), (4.2) and (4.8)–(4.10), one easily obtains

$$\sum_{1\leqslant j\leqslant N_*}\big(-\log f(a_j\,h_*)\big)=\int_1^{R_*}\Big(-\log l_\alpha\big(h_*\,a(u)\big)\Big)\,du-\frac{1}{2}\,\log l_\alpha(h_*)+o\,\big(b(h_*)\big)$$
$$=\int_1^{R_*}\Big(-\log f\big(h_*\,a(u)\big)\Big)\,du-\frac{1}{2}\,\log f(h_*)+o\,\big(b(h_*)\big)$$
$$=\int_1^{b(h_*)}\Big(-\log f\big(h_*\,a(u)\big)\Big)\,du-\frac{1}{2}\,\log f(h_*)+o\,\big(b(h_*)\big)\qquad\text{as } r\to 0.$$

Applying here (4.4), (4.5) and (4.7), we find that the conditions (2.15), (2.16) and (1.8) (see also (3.14), (3.2) and (3.9)) imply (2.17). Thus, Theorem 2.3 is proved. ■

5. PROOFS OF COROLLARIES 2.1–2.3

Proof of Corollary 2.1. In order to derive the corollary from Theorem 2.1 we have to estimate suitably two first summands on the right-hand side of (2.5).

So, let (2.4) and (2.6) hold, and let $I(x)=\int_1^x e^x\,x^{\delta-1}\,dx$. Then

(5.1)
$$\int_{u_0}^{h} b(u)\,du/u=\int_{u_0}^{h} e^{(c^{-1}\log u)^{1/\delta}}\,du/u=c\,\delta\,I\big(\log b(h)\big)+O\,(1)\qquad\text{as } r\to 0.$$

Let $M=[\delta/(\delta-1)]$ be the integer part of $\delta/(\delta-1)$, and therefore

$$M=k\geqslant 1\Leftrightarrow (k+1)/k<\delta\leqslant k/(k-1).$$

Further, we need the following result (see, for instance, [8], (6.5)).

LEMMA 5.1. *Let* $y(x)=1+\sum_{k\geqslant 1}c_k\,x^k$. *Then* $y^{1/\delta}(x)=1+\sum_{l\geqslant 1}b_l\,x^l$, *where*

$$b_\nu=\sum\prod_{l=0}^{s-1}(1/\delta-l)\prod_{m=1}^{\nu}\frac{{c_m}^{k_m}}{k_m!},\qquad s=k_1+\ldots+k_\nu,\tag{5.2}$$

and the summation is taken over all integers $k_m\geqslant 0$ *with* $1\cdot k_1+\ldots+\nu\cdot k_\nu=\nu$.

Put $s=|\log r|,\ \tilde s=(s/c)^{1/\delta}$ (that is, $e^{\tilde s}=b(1/r)$), $\tau=\tilde s/s=\tilde s^{1-\delta}/c$.

Next we show that one can define the function h from (2.6) by means of the equality

$$\log h=s\,\big(1+\sum_{l=1}^{M}c_l\,\tau^l\big),\tag{5.3}$$

where $c_1=1$ and $c_{l+1},\ 1\leqslant l\leqslant M-1$, satisfy the equation $c_{l+1}=b_l$ (see (5.2)).

In particular, $c_2 = b_1 = c_1/\delta = 1/\delta$, $c_3 = b_2 = c_2/\delta + (1/\delta)(1/\delta - 1)\, c_1^2/2 = (3-\delta)/(2\delta^2)$.

We have

$$\log b(h) = \tilde{s}\,\Big(1 + \sum_{k=1}^{M} c_k\,\tau^k\Big)^{1/\delta} = \tilde{s}\,\Big(1 + \sum_{l=1}^{M} b_l\,\tau^l + O\,(\tau^{M+1})\Big) \quad \text{as } r \to 0, \tag{5.4}$$

where, by virtue of (5.3) and Lemma 5.1 with $y(x) = 1 + \sum_{k=1}^{M} c_k\, x^k$, the coefficients b_l satisfy (5.2). Hence,

$$\begin{aligned} \log h - \log b(h) &= s\,\Big(1 + \sum_{k=1}^{M} c_k\,\tau^k\Big) - s\,\tau\,\Big(1 + \sum_{l=1}^{M-1} b_l\,\tau^l + O\,(\tau^M)\Big) \\ &= s + O\,(s\,\tau^{M+1}) = s + o\,(1) \quad \text{as } r \to 0, \end{aligned}$$

and (2.6) follows.

Now we examine the asymptotic behavior of $I\big(\log b(h)\big)$ (see (5.1)).

Put $Y_M = \tilde{s}\,\big(1 + \sum_{l=1}^{M} b_l\,\tau^l\big)$. Note that due to (5.4) we have as $r \to 0$

$$\begin{gathered} e^{Y_M} \sim e^{Y_{M-1}} \sim b(h), \\ I\big(\log b(h)\big) = I(Y_M) + O\,\big(\tilde{s}\,\tau^{M+1}\, b(h)\,\tilde{s}^{\delta-1}\big) = I(Y_M) + o\,\big(b(h)\big). \end{gathered} \tag{5.5}$$

We will study the cases $\delta > 2$, $\delta = 2$ and $1 < \delta < 2$ (i.e., $M = 1, M = 2$ and $M > 2$) separately.

In the first case we have $Y_M = Y_1 = \tilde{s} + \tilde{s}\,\tau/\delta$.

Put $\Delta = \tilde{s}\,\tau/\delta = \tilde{s}^{2-\delta}/(c\,\delta),\ k = 2 + [1/(\delta-2)]$. Then we have

$$I(Y_1) = I(\tilde{s}) + \sum_{l=1}^{k-1} \frac{\Delta^l}{l!}\, I^{(l)}(\tilde{s}) + \frac{\Delta^k}{k!}\, I^{(k)}(\tilde{s} + \theta\,\Delta), \qquad 0 < \theta < 1. \tag{5.6}$$

But

$$I^{(l)}(t) = e^t\, t^{\delta-1}\,\big(1 + O\,(1/t)\big), \qquad l \geqslant 2,\ t \to \infty,$$

and, in addition, we have $I^{(k)}(\tilde{s} + \theta\,\Delta) \sim b(1/r)\,\tilde{s}^{\delta-1}$ and $\Delta^k\,\tilde{s}^{\delta-1} = o\,(1)$ as $r \to 0$. Hence,

$$\begin{aligned} \sum_{l=1}^{k-1} \frac{\Delta^l}{l!}\, I^{(l)}(\tilde{s}) &= e^{\tilde{s}}\,\tilde{s}^{\delta-1} \sum_{l=1}^{k-1} \Delta^l/l! + O\,(\tilde{s}^{\delta-2}\,\Delta^2) \\ &= e^{\tilde{s}}\,\tilde{s}^{\delta-1}\,(e^{\Delta} - 1) + o\,\big(b(1/r)\big) \quad \text{as } r \to 0. \end{aligned} \tag{5.7}$$

Taking into account (5.5)–(5.7) and the relation

$$I(\tilde{s}) = e^{\tilde{s}}\,\tilde{s}^{\delta-1}\,\Big(1 + \sum_{l=1}^{[\delta-1]} (-1)^l \prod_{k=1}^{l} (\delta - k)\,\tilde{s}^{-l}\Big) + o\,\big(b(1/r)\big) \quad \text{as } r \to 0, \tag{5.8}$$

one easily gets for $\delta > 2$, as $r \to 0$,

(5.9)

$$I\big(\log b(h)\big) = e^{\tilde{s}}\, \tilde{s}^{\delta-1} \left(e^{\tilde{s}^{2-\delta}/(c\,\delta)} + \sum_{l=1}^{[\delta-1]} (-1)^l \prod_{k=1}^{l} (\delta - k)\, \tilde{s}^{-l}\right) + o\left(b(1/r)\right).$$

Now consider the case $\delta = 2$. We have

$$Y_M = Y_2 = \tilde{s} + \tilde{s}\,\tau/2 + \tilde{s}\,\tau^2/8 = \tilde{s} + \frac{1}{2c} + \frac{1}{8\,c^2\,\tilde{s}},$$
$$e^{Y_2} = e^{\tilde{s}+1/(2c)} \left(1 + \frac{1}{8\,c^2\,\tilde{s}} + O\left(1/\tilde{s}^2\right)\right) \quad \text{as } r \to 0.$$

Thus, as $r \to 0$,

$$I(Y_2) = e^{Y_2}\,(Y_2 - 1) + O\,(1) = b(1/r)\, e^{1/(2c)} \left(\tilde{s} - 1 + \frac{1}{2c} + \frac{1}{8\,c^2} + O\left(1/\tilde{s}\right)\right),$$

and, therefore, for $\delta = 2$ we have

$$\text{(5.10)} \quad I\big(\log b(h)\big) = b(1/r)\, e^{1/(2c)} \left(\tilde{s} - 1 + \frac{1}{2c} + \frac{1}{8\,c^2} + o\,(1)\right) \quad \text{as } r \to 0.$$

It remains to examine the case $\delta < 2$. Here (see (5.4)), as $r \to 0$,

$$I(Y_M) = e^{Y_M}\, Y_M^{\delta-1} + o\left(b(h)\right),$$
$$Y_M^{\delta-1} = \tilde{s}^{\delta-1}\,(1 + \nu\,\tau) + O\,(\tau),\ \nu = (\delta - 1)/\delta, \quad \tilde{s}^{\delta-1}\,(1 + \nu\,\tau) = \tilde{s}^{\delta-1} + \nu/c.$$

Hence, by (5.5), for $1 < \delta < 2$ we have

$$\text{(5.11)} \qquad I\big(\log b(h)\big) = e^{Y_M} \left(\tilde{s}^{\delta-1} + \frac{\delta - 1}{c\,\delta} + o\,(1)\right).$$

Thus, under the condition (2.4) the required asymptotics for the first summand on the right-hand side of (2.5) follow from (5.1) and (5.9)–(5.11).

Now evaluate the second one. We can assume without loss of generality (see [1]) that under the condition (2.7)

$$\text{(5.12)} \qquad -\log F_0(e^{-t}) = g(t) + o\,(1), \quad \text{where } t\,g'(t) \to -\gamma,\ t \to \infty.$$

Let us put

$$\text{(5.13)} \quad J(h) = \int_{u_0}^{h} -\log F_0(u/h)\, db(u), \quad \mu(t) = b(e^t), \quad k = \log u_0, \quad \tau = \log h.$$

If $R = R(h)$ tends to infinity slowly enough as $r \to 0$, then (see (1.2) and [1])

$$J(h) = \int_k^{\tau-R} g(\tau - y)\, d\mu(y) + o\big(b(h)\big). \tag{5.14}$$

Set $\epsilon = \delta\,(c/\tau)^{1/\delta},\ Q = \epsilon\,\tau$ and

$$J_1 = \int_R^Q g(u)\, d\big(\mu(\tau) - \mu(\tau - u)\big), \quad J_2 = -\int_Q^{\tau-k} g(u)\, d\mu(\tau - u),$$
$$\tilde J_1 = -\int_R^Q \big(\mu(\tau) - \mu(\tau - u)\big)\, dg(u), \quad \tilde J_2 = \int_Q^{\tau-k} \mu(\tau - u)\, dg(u).$$

We have

$$\int_k^{\tau-R} g(\tau - y)\, d\mu(y) = -\int_R^{\tau-k} g(u)\, d\mu(\tau - u) = J_1 + J_2, \tag{5.15}$$

and

$$J_1 = \big(\mu(\tau) - \mu(\tau - Q)\big)\, g(Q) - \big(\mu(\tau) - \mu(\tau - R)\big)\, g(R) + \tilde J_1,$$
$$J_2 = \mu(\tau - Q)\, g(Q) - \mu(k)\, g(\tau - k) + \tilde J_2,$$

whence

$$J_1 + J_2 = \tilde J_1 + \tilde J_2 + \big(g(Q) + o\,(1)\big)\, b(h) \quad \text{as } r \to 0. \tag{5.16}$$

Let us write

$$\omega(u) = \frac{1 - (1 - u)^{1/\delta}}{u}, \quad u \in (0, 1]. \tag{5.17}$$

Then (recall (2.4) and (5.13)) $\mu(\tau - y)/\mu(\tau) = e^{-\omega(y/\tau)\, y/Q}$, and therefore

$$\tilde J_1/\mu(\tau) = -\int_{R/Q}^{1} (1 - e^{-\omega(\epsilon\, y)\, \delta\, y})\, \big(Q\, y\, g'(Q\, y)\big)\, dy/y,$$
$$\tilde J_2/\mu(\tau) = \int_1^{(\tau-k)/Q} e^{-\omega(\epsilon\, y)\, \delta\, y}\, \big(Q\, y\, g'(Q\, y)\big)\, dy/y.$$

From (5.12) and the dominated convergence theorem it follows that

$$\tilde J_1/\mu(\tau) \to \gamma \int_0^1 (1 - e^{-y})\, dy/y, \quad \tilde J_2/\mu(\tau) \to -\gamma \int_1^\infty e^{-y}\, dy/y \quad \text{as } \tau \to \infty.$$

Thus (see (5.13), (5.3) and (5.12)), we have, as $r \to 0$,

$$\tilde{J}_1 + \tilde{J}_2 = \big(\gamma\,\mathcal{E} + o\,(1)\big)\, b(h), \tag{5.18}$$

and

$$g(Q) = g(\tilde{s}^{\delta-1}) + \int\limits_{\tilde{s}^{\delta-1}}^{Q} t g'(t)\, dt/t = -\log F_0(\tilde{s}^{\delta-1}) - \gamma \log{(c\delta)} + o\,(1). \tag{5.19}$$

The relations (5.13)–(5.19) imply the relevant asymptotics for the second summand on the right-hand side of (2.5). Thus, the proof of Corollary 2.1 is complete. ∎

P r o o f o f C o r o l l a r y 2.2. For the proof we use Theorem 2.2 for $a(u) = e^{d-u/c}$, $u \geqslant 1$ (that is, $b(t) = c\,(d + \log t)$, $t \geqslant 1/a(1)$).

Set $s = \log{(1/r)},\ h = s\,e^s,\ \tau = \log h = s + \log s$. Then we have

$$l(h) = c \int\limits_{u_0}^{h} (d + \log t)\, dt/t = c\,(d\,s + s\log s + s^2/2) + o\,(s) \quad \text{as } r \to 0. \tag{5.20}$$

Further (see (5.12)–(5.14) with $k = R = \log s$), as $r \to 0$,

$$\begin{aligned} J(h) &= c \int\limits_{u_0}^{h} -\log F_0(u/h)\, du/u = c \int\limits_{\log s}^{s} g(\tau - y)\, dy + o\,(s) \\ &= c \int\limits_{\log s}^{s} g(t)\, dt + o\,(s) = c\Big(-Rg(R) + sg(s) - \int\limits_{\log s}^{s} t g'(t)\, dt\Big) + o\,(s) \\ &= c\,s\,\big(\gamma + g(s) + o\,(1)\big) = c\,s\,\big(\gamma - \log F_0(r) + o\,(1)\big). \end{aligned} \tag{5.21}$$

The relation (2.14) follows from (2.13), (5.20) and (5.21), i.e., Corollary 2.2 is established. ∎

P r o o f o f C o r o l l a r y 2.3. Let us substitute $b(t) = c\,(c + \log t)^{1/\delta}$ and $h = c\,s^{1/\delta}\, e^s$ with $s = \log{(1/r)}$ in (2.17). Then we have, as $r \to 0$,

$$l(h) = c\,s^{1/\delta} \left(\frac{\delta}{1+\delta}\, s + \frac{1}{\delta}\, \log s + d + \log c + o\,(1) \right)$$

and (see (5.12)–(5.14))

$$J(h) = \left(g(\tau - k) - \int\limits_{R/\tau}^{1-k/\tau} \left(1 - \frac{\mu\big(\tau\,(1-u)\big)}{\mu(\tau)} \right) dg(\tau\, u) \right) b(h) + o\,(s^{1/\delta}),$$

where $g(\tau - k) = g(s) + o\,(1) = -\log F_0(r) + o\,(1)$ and the integral tends to $\gamma\,\nu$. Consequently, (2.18) follows. ∎

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Leonid V. Rozovsky
St. Petersburg Chemical-Pharmaceutical Academy
Department of Mathematics
Prof. Popova Street, 14
197376, St. Petersburg, Russia
E-mail: L_rozovsky@mail.ru

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