PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 34, Fasc. 2 (2014), pp. 199–214

ON DUAL GENERATORS FOR NON-LOCAL SEMI-DIRICHLET FORMS

BY

TOSHIHIRO UEMURA* (SUITA, OSAKA)

Abstract. Let $k(x, y)$ be a measurable function defined on $E \times E$ off the diagonal, where E is a locally compact separable metric space, and let m be a positive Radon measure on E with full support. In 2012, we showed that a quadratic form having k as a Lévy kernel becomes a lower bounded semi-Dirichlet form on $L^2(E; m)$ which is non-local and regular. Then there associates a Hunt process corresponding to the semi-Dirichlet form. In the case where $E = \mathbb{R}^d$, we will show that the dual form of the semi-Dirichlet form also produces a Hunt process by taking a killing. As a byproduct, a precise description of the infinitesimal generator of the dual form is also given.

2000 AMS Mathematics Subject Classification: Primary: 31C25; Secondary: 60J75, 47G20.

Key words and phrases: Semi-Dirichlet form, non-local operator, stable-like process.

1. INTRODUCTION

Let E be a locally compact separable metric space equipped with a metric d, m a positive Radon measure with full topological support, and $k(x, y)$ a nonnegative Borel measurable function on the space $E \times E \setminus \mathsf{diag}$, where diag denotes the diagonal set $\{(x, x) : \ x \in E\}$. For $n \in \mathbb{N}$, consider the (integro-differential) operator

$$(1.1) \qquad \mathcal{L}_n u(x) = \int_{d(x,y)>1/n} \big(u(y) - u(x)\big) k(x, y) m(dy), \quad x \in E,$$

for appropriate functions u. Under the conditions (A1) and (A2) below, we have shown in [4] that the finite limit

$$(1.2) \qquad \eta(u, v) := \lim_{n\to\infty} \eta^n(u, v) \ := \ \lim_{n\to\infty} (-\mathcal{L}_n u, v) = - \lim_{n\to\infty} \iint_{d(x,y)>1/n} \big(u(y) - u(x)\big) v(x) k(x, y) m(dx) m(dy)$$

---

* This work was supported by JSPS.

exists for any $u, v \in C_0^{\mathsf{lip}}(E)$ and then the limit produces a lower bounded semi-Dirichlet form $(\eta, \mathcal{F})$ on $L^2(E; m)$, where $C_0^{\mathsf{lip}}(E)$ is the set of all uniformly Lipschitz continuous functions on E with compact support. So there associates a Hunt process corresponding to the limit $(\eta, \mathcal{F})$. Moreover, set $k^*(x, y) := k(y, x)$ for $x, y \in E$ with $x \neq y$ and consider the operator $\mathcal{L}_n^*$ and the form η^* in (1.1) and (1.2) defined with k^* in place of k. Then the same conclusions hold as above for η^* under the same assumptions on k. The domain $\mathcal{F}^*$ in this case coincides with $\mathcal{F}$. That is, $(\eta^*, \mathcal{F})$ is also a lower bounded semi-Dirichlet form on $L^2(E; m)$.

On the other hand, we have noted in [4] that the dual form defined by

$$\hat{\eta}(u, v) := \eta(v, u), \quad u, v \in \mathcal{F},$$

may not produce a lower bounded semi-Dirichlet form in general. But, assuming a bit stronger conditions (A1$'$) and (A2$'$) below instead of (A1) and (A2), we have seen that the dual form can be written as

$$\widehat{\eta}(u, v) = \eta^*(u, v) - (u, Kv), \quad u, v \in \mathcal{F},$$

for some bounded function K, and $(\widehat{\eta}, \mathcal{F})$ is a lower bounded closed form. Furthermore, we have verified that, denoting the dual semigroup by $\{\widehat{T}_t\}$, the killed dual semigroup $\{e^{-\beta t}\widehat{T}_t\}$ is Markov for a large $\beta > 0$ in this case. In general, the killed dual semigroup may not be Markovian no matter how big β is and we gave an example in [4], Section 3, that the dual semigroup indeed could not be Markovian.

One of our objectives in this paper is to give a condition other than the conditions (A1$'$) and (A2$'$) for the (killed) dual semigroup to be Markov. Recently, Schilling and Wang [12] considered the (formal) dual operator of a Lévy-type operator on $\mathbb{R}^d$ and gave some description of the form of the dual under slightly different conditions on the kernel k.

The organization of the paper is as follows. In the next section, the notion of a lower bounded semi-Dirichlet form and some necessary results obtained in [4] are given. Under a bit stronger assumptions on the kernel k, we are able to describe precise forms of the generator and its dual on $L^2(E; m)$ of the form $(\eta, \mathcal{F})$, where $E = \mathbb{R}^d$ and $m(dx) = dx$ is Lebesgue measure in Section 3. We then try to apply the result to the case of stable-like generators to obtain a precise expression of the dual in the last section. We stress that the dual of a stable-like generator corresponds to a Hunt process by taking a killing to the "reversed stable-like process". This means that we could show that, for a higher order case, the dual semigroup is Markov if we take a β sufficiently large.

2. LOWER BOUNDED SEMI-DIRICHLET FORM

In this section we recall the notion of a lower bounded semi-Dirichlet form. The inner product and the norm in $L^2(E; m)$ are denoted by $(\cdot, \cdot)$ and $\|\cdot\|$, respectively. Let $\mathcal{F}$ be a dense linear subspace of $L^2(E; m)$ such that $u \wedge 1 \in \mathcal{F}$

whenever $u \in \mathcal{F}$. A (not necessarily symmetric) bilinear form η on $\mathcal{F}$ is called a *lower bounded closed form* if the following conditions (B1)–(B3) are satisfied. We set $\eta_\beta(u,v) := \eta(u,v) + \beta(u,v), u,v \in \mathcal{F}$ for $\beta \geqslant 0$. There exists a $\beta_0 \geqslant 0$ such that

(B1) (lower boundedness): for any $u \in \mathcal{F},\ \eta_{\beta_0}(u,u) \geqslant 0$;

(B2) (sector condition): for any $u, v \in \mathcal{F}$,

$$|\eta(u,v)| \leqslant K\sqrt{\eta_{\beta_0}(u,u)} \cdot \sqrt{\eta_{\beta_0}(u,u)}$$

for some constant $K \geqslant 1$;

(B3) (completeness): $\mathcal{F}$ is complete with respect to the norm $\eta_\alpha^{1/2}(\cdot,\cdot)$ for some or, equivalently, for all $\alpha > \beta_0$.

For a lower bounded closed form $(\eta, \mathcal{F})$ on $L^2(E;m)$, there exist unique semigroups $\{T_t; t > 0\}, \{\hat{T}_t; t > 0\}$ of linear operators on $L^2(E;m)$ satisfying

$$(2.1)\quad (T_t f, g) = (f, \hat{T}_t g),\ f,g \in L^2(E;m),\ \|T_t\| \leqslant e^{\beta_0 t},\ \|\hat{T}_t\| \leqslant e^{\beta_0 t},\ t > 0,$$

such that their Laplace transforms G_α and $\hat{G}_\alpha$ are determined for $\alpha > \beta_0$ by

$$G_\alpha f,\ \hat{G}_\alpha f \in \mathcal{F},\quad \eta_\alpha(G_\alpha f, u) = \eta_\alpha(u, \hat{G}_\alpha f) = (f,u),\quad f \in L^2(E;m),\ u \in \mathcal{F}$$

(see, e.g., [7]). Moreover, there associates the generator $\big(\mathcal{L}, \mathcal{D}(\mathcal{L})\big)$ (respectively, co-generator $\big(\hat{\mathcal{L}}, \mathcal{D}(\hat{\mathcal{L}})\big)$) on $L^2(E;m)$ so that both $\mathcal{D}(\mathcal{L})$ and $\mathcal{D}(\hat{\mathcal{L}})$ are dense in $\mathcal{F}$ with respect to the norm η_α for $\alpha > \beta_0$, respectively, $\eta(u,v) = -(\mathcal{L}u, v)$ for $u \in \mathcal{D}(\mathcal{L}), v \in \mathcal{F}$ and $(\mathcal{L}u, v) = (u, \hat{\mathcal{L}}v)$ for $u \in \mathcal{D}(\mathcal{L}), v \in \mathcal{D}(\hat{\mathcal{L}})$ (see, e.g., [10]). $\{T_t; t > 0\}$ is said to be *Markov* if $0 \leqslant T_t f \leqslant 1, t > 0$, whenever $f \in L^2(E;m)$, $0 \leqslant f \leqslant 1$. It was shown by Kunita [6] that the semigroup $\{T_t; t > 0\}$ is Markov if and only if

$$(2.2)\qquad Uu \in \mathcal{F} \quad \text{and} \quad \eta(Uu, u - Uu) \geqslant 0 \quad \text{for any } u \in \mathcal{F},$$

where Uu denotes the unit contraction of u: $Uu = (0 \vee u) \wedge 1$. A lower bounded closed form $(\eta, \mathcal{F})$ on $L^2(E;m)$ satisfying (2.2) is called a *lower bounded semi-Dirichlet form* on $L^2(E;m)$. The term "semi" is added to indicate that the dual semigroup $\{\hat{T}_t; t > 0\}$ may not be Markovian although it is positivity preserving (see [8], [4], [9]). Thus a quadratic form defined by

$$\hat{\eta}(u,v) = -(\hat{\mathcal{L}}u, v)\ (= \eta(v,u)),\quad u \in \mathcal{D}(\hat{\mathcal{L}}),\ v \in \mathcal{F},$$

may not become a semi-Dirichlet form in general. A lower bounded semi-Dirichlet form $(\eta, \mathcal{F})$ is said to be *regular* if $\mathcal{F} \cap C_0(E)$ is uniformly dense in $C_0(E)$ and η_α-dense in $\mathcal{F}$ for $\alpha > \beta_0$, where $C_0(E)$ denotes the space of continuous functions on E with compact support. Carrillo Menendez [2] constructed a Hunt process properly associated with any regular lower bounded semi-Dirichlet form on $L^2(E;m)$.

For the sake of reader's convenience, we now consider and show the limits of $\mathcal{L}_n$ and η^n defined by (1.1) and (1.2), respectively, for which the limit operator $\mathcal{L}$ (or the form η) corresponds to a lower bounded semi-Dirichlet form. We set, for $x, y \in E$ with $x \neq y$,

$$k_s(x,y) = \frac{1}{2}\big(k(x,y) + k(y,x)\big), \qquad k_a(x,y) = \frac{1}{2}\big(k(x,y) - k(y,x)\big), \qquad x \neq y,$$

where k_s (respectively, k_a) denotes the symmetrized function (respectively, anti-symmetrized function) of k. Set also for $u, v \in C_0^{\mathsf{lip}}(E)$

$$\mathcal{E}(u,v) = \iint_{x\neq y} \big(u(y) - u(x)\big)\big(v(y) - v(x)\big) k_s(x,y) m(dx) m(dy).$$

Suppose that

$$\text{(A1)} \qquad x \mapsto \int_{y\neq x} \big(1 \wedge d(x,y)^2\big) k_s(x,y) m(dy) \in L^2_{\mathsf{loc}}(E; m).$$

Then the pair $\big(\mathcal{E}, C_0^{\mathsf{lip}}(E)\big)$ is a closable symmetric bilinear form on $L^2(E;m)$ and the closure $(\mathcal{E}, \mathcal{F})$ on $L^2(E)$ becomes a regular symmetric Dirichlet form on $L^2(E;m)$ (see, e.g., [3], [13]). Here $\mathcal{F}$ is the closure of $C_0^{\mathsf{lip}}(E)$ with respect to the norm $\sqrt{\mathcal{E}_1(\cdot,\cdot)}$, i.e., $\mathcal{E}_1(u,v) := \mathcal{E}(u,v) + (u,v)$.

Note that under the condition (A1), all integrals appearing $\mathcal{L}_n u$ in (1.1) and $\eta^n(u,v)$ in (1.2) are absolute convergent for each $u, v \in C_0^{\mathsf{lip}}(E)$. Suppose further that

$$\text{(A2)} \qquad \sup_x \int_{\{y: k_s(x,y)\neq 0\}} \frac{k_a(x,y)^2}{k_s(x,y)} m(dy) < \infty.$$

We then have shown in [4] (see also [12]) that the finite limit

$$\eta(u,v) := \lim_{n\to\infty} \eta^n(u,v) = -\lim_{n\to\infty} \int_E \mathcal{L}_n u(x) v(x) m(dx), \qquad u, v \in C_0^{\mathsf{lip}}(E),$$

exists, η extends to $\mathcal{F} \times \mathcal{F}$ so that for each $\alpha > \beta_0$, for some positive numbers C_1, C_2,

$$C_1 \mathcal{E}_1(u,u) \leqslant \eta_\alpha(u,u) \leqslant C_2 \mathcal{E}_1(u,u) \quad \text{ for } u \in \mathcal{F}$$

and $(\eta, \mathcal{F})$ is a lower bounded semi-Dirichlet form on $L^2(E;m)$. Moreover, the limit η has the following form: for $u, v \in \mathcal{F}$,

$$\text{(2.3)} \qquad \eta(u,v) = \frac{1}{2}\mathcal{E}(u,v) + \iint_{x\neq y} \big(u(x) - u(y)\big) v(y) k_a(x,y) m(dx) m(dy).$$

In [4] we also succeeded to obtain a precise form of the generator $\mathcal{L}$ of the form $(\eta, \mathcal{F})$ under the following conditions (A1′) and (A2′) in place of (A1) and (A2), respectively:

(A1′) $$x \mapsto \int_{y\neq x} \big(1 \wedge d(x,y)\big) k_s(x,y) m(dy) \in L^2_{\mathsf{loc}}(E;m)$$

and

(A2′) $$\sup_{x\in E} \int_{y\neq x} |k_a(x,y)| m(dy) = \sup_{x\in E} \frac{1}{2} \int_{y\neq x} |k(x,y) - k(y,x)| m(dy) < \infty.$$

We find that the integrals

$$\mathcal{L}u(x) = \int_{y\neq x} \big(u(y) - u(x)\big) k(x,y) m(dy),$$
(2.4)
$$\mathcal{L}^* u(x) = \int_{y\neq x} \big(u(y) - u(x)\big) k^*(x,y) m(dy)$$

converge for $u \in C_0^{\mathsf{lip}}(E),\ x \in E$, where k^* is the "reversed kernel" of k as above, and in this case we get

(2.5) $$\eta(u,v) = -(\mathcal{L}u, v), \quad \eta^*(u,v) = -(\mathcal{L}^* u, v), \quad u,v \in C_0^{\mathsf{lip}}(E).$$

Furthermore,

$$K(x) := 2 \int_{y\neq x} k_a(x,y) m(dy) = \int_{y\neq x} \big(k(x,y) - k(y,x)\big) m(dy), \quad x \in E,$$

defines a bounded function on E, and then from the relations (2.3)–(2.5) it follows that

$$\hat{\eta}(u,v) = \eta^*(u,v) + (u, Kv), \quad u,v \in \mathcal{F},$$

which means that

(2.6)
$$\begin{aligned}\hat{\mathcal{L}}u(x) &= \mathcal{L}^* u(x) - u(x) \cdot K(x) \\ &= \int_{y\neq x} \big(u(y) - u(x)\big) k^*(x,y) m(dy) - u(x) K(x) \\ &= \int_{y\neq x} \big(u(y) - u(x)\big) k(y,x) m(dy) - u(x) \int_{y\neq x} \big(k(x,y) - k(y,x)\big) m(dy)\end{aligned}$$

is the dual operator of $\mathcal{L}$ on $L^2(E;m)$ for $u \in C_0^{\mathsf{lip}}(E)$. Thus, as noted in Section 1, we have verified that the killed dual semigroup $\{e^{-\beta t}\widehat{T}_t; t \geqslant 0\}$ is Markovian for a large $\beta > 0$ in this (lower order) case. For a higher order case, the killed dual semigroup may not be Markovian no matter how big β is.

On the other hand, Schilling and Wang considered in [12] the (formal) operator of a Lévy-type operator on $\mathbb{R}^d$ for a kernel k: for $u \in C_0^2(\mathbb{R}^d)$,

$$\begin{aligned}\mathcal{L}u(x) = &\int_{h\neq 0} \big(u(x+h) - u(x) - \nabla u(x) \cdot h \mathbf{1}_{B(1)}(h)\big) k(x, x+h) dh \\ &+ \frac{1}{2} \int_{0<|h|<1} \nabla u(x) \cdot h \big(k(x,x+h) - k(x,x-h)\big) dh,\end{aligned}$$

where

$$\int_{h\neq 0} (1\wedge |h|^2)k(x,x+h)dh<\infty$$

and

$$\int_{0<|h|<1} |h|\cdot |k(x,x+h)-k(x,x-h)|dh<\infty$$

for any $x\in\mathbb{R}^d$. Under some conditions on k they also gave a description of the (formal) dual $\hat{\mathcal{L}}$ of $\mathcal{L}$:

$$\begin{aligned}\hat{\mathcal{L}}u(x) &= \int_{h\neq 0} \big(u(x+h)-u(x)-\nabla u(x)\cdot h\mathbf{1}_{B(1)}(h)\big)k(x+h,x)dh \\ &\quad + \frac{1}{2}\int_{0<|h|<1} \nabla u(x)\cdot h\big(k(x{+}h,x){-}k(x{-}h,x)\big)dh + u(x)\kappa(dx).\end{aligned}$$

Here $\kappa(dx)$ is a signed measure on $\mathbb{R}^d$ which is the vague limit of the sequence of signed measures $\big\{-2\int_{|h|>1/n} k_a(x,y)dydx\big\}_{k\in\mathbb{N}}$. They also applied their result to the generator $\mathcal{L}=-\big(-\Delta\big)^{\alpha(x)/2}$ of Bass's stable-like process (see [1]).

3. GENERATORS OF THE SEMI-DIRICHLET FORM AND ITS DUAL

In this section, we first consider a precise expression of the infinitesimal generator of the semi-Dirichlet form described in the preceding section. To this end, we restrict ourselves to the case where $E=\mathbb{R}^d$ and $m(dx)=dx$ is the Lebesgue measure on $\mathbb{R}^d$. Let k be a kernel on $\mathbb{R}^d$ satisfying the condition (A1). Suppose the following conditions also hold. For any positive numbers r and R with $R-r\geqslant 1$,

(A3) $$x\mapsto \int_{B(1)^c} \mathbf{1}_{B(r)}(x+h)k_s(x,x+h)dh \in L^2\big(\mathbb{R}^d\setminus B(R)\big),$$

where $B(r)$ is an open ball with radius r at the origin ($B(r)=\{x\in\mathbb{R}^d\colon |x|<r\}$), and

(A4) $$\sup_{x\in\mathbb{R}^d}\int_{0<|h|<1} |h|\cdot |k(x,x+h)-k(x,x-h)|dh<\infty.$$

Following an argument developed in [14] (see also [11]), we see that the finite limit

$$\begin{aligned}\mathcal{L}u(x) &= \lim_{n\to\infty}\mathcal{L}_n u(x)\\ &= \int_{h\neq 0} \big(u(x{+}h){-}u(x){-}\nabla u(x){\cdot} h\mathbf{1}_{B(1)}(h)\big)k(x,x+h)dh + b(x){\cdot}\nabla u(x)\end{aligned}$$

exists for any $x\in\mathbb{R}^d$ and $u\in C_0^2(\mathbb{R}^d)$, and $\mathcal{L}$ sends $C_0^2(\mathbb{R}^d)$ into $L^2(\mathbb{R}^d)$ under (A1), (A3), and (A4), where

$$b(x):=\frac{1}{2}\int_{0<|h|<1} h\big(k(x,x+h)-k(x,x-h)\big)dh,\quad x\in\mathbb{R}^d.$$

On the other hand, we see that, for each $u, v \in C_0^2(\mathbb{R}^d)$ and $n \in \mathbb{N}$,

$$
\begin{aligned}
&(\mathcal{L}_n u, v) = \\
&= \iint_{1/n<|h|} \big(u(x+h) - u(x)\big) v(x) k(x, x+h) dh dx \\
&= \iint_{1/n<|h|} u(x+h) v(x) k(x, x+h) dh dx - \iint_{1/n<|h|} u(x) v(x) k(x, x+h) dh dx \\
&= \iint_{1/n<|h|} u(x) v(x+h) k(x+h, x) dh dx - \iint_{1/n<|h|} u(x) v(x) k(x, x+h) dh dx \\
&= \iint_{1/n<|h|} u(x) \big(v(x+h) - v(x)\big) k(x+h, x) dh dx \\
&\quad + \iint_{1/n<|h|} u(x) v(x) \big(k(x+h, x) - k(x, x+h)\big) dh dx \\
&=: (u, \mathcal{L}_n^* v) + (u, v K_n),
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{L}_n^* v(x) &:= \int_{1/n<|h|} \big(v(x+h) - v(x)\big) k(x+h, x) dh \\
&\Big(= \int_{1/n<|h|} \big(v(x+h) - v(x)\big) k^*(x, x+h) dh\Big) \\
K_n(x) &:= \int_{1/n<|h|} \big(k(x+h, x) - k(x, x+h)\big) dh, \quad x \in \mathbb{R}^d.
\end{aligned}
$$

In the third equality, we made a change of variables twice ($x \mapsto x - h$, and then $h \mapsto -h$). Therefore, if we can show that $\mathcal{L}_n^* v$ and K_n converge to finite limits, say $\mathcal{L}^* v$ and K, respectively, for appropriate functions v, then it follows that $\mathcal{L}^* + K$ is the dual operator $\hat{\mathcal{L}}$ of $\mathcal{L}$ on $L^2(\mathbb{R}^d)$.

We now give a sufficient condition on the kernel in order that $\mathcal{L}^*$ and K exist. To this end, we assume there exist nonnegative measurable functions $C(x, h)$ on $\mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\})$ and n on $\mathbb{R}^d \setminus \{0\}$ satisfying

$$
(3.1) \qquad \begin{cases} C(x,h) = C(x,-h),\ n(h) = n(-h) \text{ for } x \in \mathbb{R}^d,\ h \in \mathbb{R}^d \setminus \{0\}, \\ \text{such that } k(x,y) = C(x, y-x) n(y-x) \text{ for } x, y \in \mathbb{R}^d,\ x \neq y, \end{cases}
$$

$$
(\mathrm{A5}) \qquad \begin{cases} x \mapsto \displaystyle\int_{0<|h|<1} |h|^2 \big(C(x,h) + C(x+h,h)\big) n(h) dh \in L^2_{\mathsf{loc}}(\mathbb{R}^d), \\ M := \displaystyle\sup_{x \in \mathbb{R}^d} \int_{|h| \geqslant 1} \big(C(x,h) + C(x+h,h)\big) n(h) dh < \infty, \end{cases}
$$

and

$$\text{(A6)} \qquad \begin{cases} x \mapsto C(x,h) \in C^2(\mathbb{R}^d) \ \text{ for each } h \in \mathbb{R}^d \text{ with } 0 < |h| < 1, \\ x \mapsto \displaystyle\sum_{i,j=1}^{d} \int_{0<|h|<1} \left| \frac{\partial^2 C(x,h)}{\partial x_i \partial x_j} h_i h_j \right| n(h)dh \in L^\infty(\mathbb{R}^d). \end{cases}$$

In this case, (A2) becomes

$$\sup_{x \in \mathbb{R}^d} \int_{\{h: C(x,h)n(h) \neq 0\}} \frac{|C(x,h) - C(x+h,h)|^2}{C(x,h) + C(x+h,h)} n(h) < \infty.$$

Note also that (A5) and (A6) imply (A1) and (A3). In fact, noting that

$$k_s(x, x+h) = \frac{k(x,x+h) + k(x+h,x)}{2} = \frac{C(x,h) + C(x+h,h)}{2} \cdot n(h),$$

we see that (A5) implies (A1). For any positive numbers R, r with $R - r \geqslant 1$,

$$\begin{aligned}
&\int_{B(R)^c} \Big(\int_{B(1)^c} \mathbf{1}_{B(r)}(x+h) \big(C(x,h) + C(x+h,h)\big) n(h) dh \Big)^2 dx \\
&\leqslant M \int_{B(R)^c} \int_{B(1)^c} \mathbf{1}_{B(r)}(x+h) \big(C(x,h) + C(x+h,h)\big) n(h) dh dx \\
&= M \iint_{\mathbb{R}^d \times \mathbb{R}^d} \mathbf{1}_{B(R)^c}(x) \mathbf{1}_{B(1)^c}(h) \mathbf{1}_{B(r)}(x+h) \big(C(x,h) + C(x+h,h)\big) n(h) dx dh \\
&= M \iint_{\mathbb{R}^d \times \mathbb{R}^d} \mathbf{1}_{B(R)^c}(x+h) \mathbf{1}_{B(1)^c}(h) \mathbf{1}_{B(r)}(x) \big(C(x+h,h) + C(x,h)\big) n(h) dx dh \\
&\leqslant M \int_{B(r)} \int_{B(1)^c} \big(C(x+h,h) + C(x,h)\big) n(h) dh dx \leqslant M^2 \,\mathsf{Vol}\big(B(r)\big) < \infty.
\end{aligned}$$

This means that (A3) follows from (A6). Now we show that, under the conditions (A5) and (A6), $\mathcal{L}_n^* u$ and K_n have the finite limits for $u \in C_0^2(\mathbb{R}^d)$. For any $n \in \mathbb{N}$,

$$\begin{aligned}
\mathcal{L}_n^* u(x) &= \int_{|h|>1/n} \big(u(x+h) - u(x)\big) C(x+h,h) n(h) dh \\
&= \int_{|h|>1/n} \big(u(x+h) - u(x) - \nabla u(x) \cdot h \mathbf{1}_{B(1)}(h)\big) C(x+h,h) n(h) dh \\
&\quad + \int_{1/n<|h|<1} \nabla u(x) \cdot h C(x+h,h) n(h) dh \\
&=: (\mathrm{I})_n + (\mathrm{II})_n.
\end{aligned}$$

According to (A5), we easily see that $(\mathrm{I})_n$ converges to

$$\int_{h \neq 0} \big(u(x+h) - u(x) - \nabla u(x) \cdot h \mathbf{1}_{B(1)}(h)\big) C(x+h,h) n(h) dh,$$

and so belongs to $L^2(\mathbb{R}^d)$ for $u \in C_0^2(\mathbb{R}^d)$. As for $(\mathrm{II})_n$, first making a change of variables ($h \mapsto -h$) and then averaging, we find

$$(\mathrm{II})_n = \frac{1}{2} \int\limits_{1/n<|h|<1} \nabla u(x) \cdot h\big(C(x+h,h) - C(x-h,h)\big)n(h)dh,$$

and then the right-hand side converges to

$$\frac{1}{2} \int\limits_{0<|h|<1} \nabla u(x) \cdot h\big(C(x+h,h) - C(x-h,h)\big)n(h)dh \quad \text{as } n \to \infty.$$

The limit also belongs to $L^2(\mathbb{R}^d)$. Therefore, it follows that $\mathcal{L}_n^* u$ converges to

$$\begin{aligned} \mathcal{L}^* u(x) = &\int\limits_{h\neq 0} \big(u(x+h) - u(x) - \nabla u(x) \cdot h\mathbf{1}_{B(1)}(h)\big)C(x+h,h)n(h)dh \\ &+\frac{1}{2} \int\limits_{0<|h|<1} \nabla u(x) \cdot h\big(C(x+h,h) - C(x-h,h)\big)n(h)dh, \end{aligned}$$

which is in $L^2(\mathbb{R}^d)$.

We next consider the term K_n. Since $k(x, x+h) = C(x,h)n(h)$, we obtain

$$\begin{aligned} K_n(x) &= \int\limits_{1/n<|h|} \big(C(x+h,h) - C(x,h)\big)n(h)dh \\ &= \int\limits_{1/n<|h|<1} \big(C(x+h,h) - C(x,h)\big)n(h)dh \\ &\quad + \int\limits_{|h|\geqslant 1} \big(C(x+h,h) - C(x,h)\big)n(h)dh =: (\mathrm{I})_n + (\mathrm{II}). \end{aligned}$$

The second condition in (A5) means that (II) is a bounded function. Since the function $x \mapsto C(x,h)$ is in $C^2(\mathbb{R}^d)$ for each $h \in \mathbb{R}^d$ with $0 < |h| < 1$, we have

$$\begin{aligned} (\mathrm{I})_n = &\int\limits_{1/n<|h|<1} \big(C(x+h,h) - C(x,h) - \nabla_x C(x,h) \cdot h\big)n(h)dh \\ &+ \int\limits_{1/n<|h|<1} \nabla_x C(x,h) \cdot h\, n(h)dh. \end{aligned}$$

Since $h \mapsto \nabla_x C(x,h) \cdot h$ is an odd function on $\{1/n < |h| < 1\}$ for each $x \in \mathbb{R}^d$, the second term on the right-hand side disappears. By Taylor's expansion of the function $x \mapsto C(x,h)$, we get

$$C(x+h,h) - C(x,h) - \nabla C_x(x,h)\cdot h = \frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial^2}{\partial x_i \partial x_j} C(\theta x, h) h_i h_j, \;\; x \in \mathbb{R}^d,$$

for some $0 < \theta < 1$ and for each $h \in \mathbb{R}^d$ with $0 < |h| < 1$. Hence we infer from (A6) that

$$|(\mathrm{I})_n| \leqslant \frac{1}{2} \sup_{x \in \mathbb{R}^d} \sum_{i,j=1}^{d} \int_{0<|h|<1} \left| \frac{\partial^2}{\partial x_i \partial x_j} C(x,h) h_i h_j \right| n(h)dh < \infty,$$

and $(\mathrm{I})_n$ also converges to a bounded function

$$\int_{0<|h|<1} \big(C(x+h,h) - C(x,h) - \nabla_x C(x,h) \cdot h\big) n(h)dh, \qquad x \in \mathbb{R}^d.$$

Combining the estimates above, we see that the dual operator $\hat{\mathcal{L}}$ on $L^2(\mathbb{R}^d)$ is given for functions $u \in C_0^2(\mathbb{R}^d)$ as follows:

$$\begin{aligned}\hat{\mathcal{L}}u(x) &= \mathcal{L}^* u(x) + u(x) \cdot K(x) \\ &= \int_{h \neq 0} \big(u(x+h) - u(x) - \nabla u(x) \cdot h \mathbf{1}_{B(1)}(h)\big) C(x+h,h) n(h)dh \\ &\quad + \frac{1}{2} \int_{0<|h|<1} \nabla u(x) \cdot h \big(C(x+h,h) - C(x-h,h)\big) n(h)dh \\ &\quad + u(x) \cdot \int_{h \neq 0} \big(C(x+h,h) - C(x,h) - \nabla_x C(x,h) \cdot h \mathbf{1}_{B(1)}(h)\big) n(h)dh\end{aligned}$$

for $x \in \mathbb{R}^d$. Since $\mathcal{L}^*$ corresponds to η^*, and K is a bounded function, we have the following theorem.

THEOREM 3.1. *Assume* (A2′) *and* (A4)–(A6) *hold for a kernel* $k(x,y) = C(x, y-x)n(y-x)$ *satisfying* (3.1). *Let* $\{T_t; t > 0\}$ *and* $\{\hat{T}_t; t > 0\}$ *be the semigroups corresponding to the lower bounded semi-Dirichlet form* $(\eta, \mathcal{F})$ *on* $L^2(\mathbb{R}^d)$. *Then the following assertions hold*:

(i) *The operator* $\big(\mathcal{L}, C_0^2(\mathbb{R}^d)\big)$ (*respectively*, $\big(\hat{\mathcal{L}}, C_0^2(\mathbb{R}^d)\big)$) *coincides with the infinitesimal generator of the semigroup* $\{T_t; t > 0\}$ (*respectively*, $\{\hat{T}_t; t > 0\}$) *on* $L^2(\mathbb{R}^d)$ *restricted to* $C_0^2(\mathbb{R}^d)$, *where*

$$\mathcal{L}u(x) = \int_{h \neq 0} \big(u(x+h) - u(x) - \nabla u(x) \cdot h \mathbf{1}_{B(1)}(h)\big) C(x,h) n(h)dh,$$

$$\begin{aligned}\hat{\mathcal{L}}u(x) &= \mathcal{L}^* u(x) + u(x) \cdot K(x) \\ &= \int_{h \neq 0} \big(u(x+h) - u(x) - \nabla u(x) \cdot h \mathbf{1}_{B(1)}(h)\big) C(x+h,h) n(h)dh \\ &\quad + \frac{1}{2} \int_{0<|h|<1} \nabla u(x) \cdot h \big(C(x+h,h) - C(x-h,h)\big) n(h)dh \\ &\quad + u(x) \cdot \int_{h \neq 0} \big(C(x+h,h) - C(x,h) - \nabla_x C(x,h) \cdot h \mathbf{1}_{B(1)}(h)\big) n(h)dh\end{aligned}$$

for $x \in \mathbb{R}^d$ *and* $u \in C_0^2(\mathbb{R}^d)$.

(ii) *Put* $\beta_1 := \operatorname{ess\,sup}_{x\in\mathbb{R}^d} K^+(x)$, *where* K^+ *is the positive part of* $K = K^+ - K^-$, *and define a quadratic form*

$$\hat\eta(u,v) := -(\hat{\mathcal{L}}u, v) \quad \text{for } u,v \in C_0^2(\mathbb{R}^d).$$

Then $(\hat\eta_\beta, \mathcal{F})$, *which is the dual of* $(\eta_\beta, \mathcal{F})$, *is a lower bounded semi-Dirichlet form on* $L^2(\mathbb{R}^d)$ *provided that* $\beta \geqslant \beta_1$. (*The constant appearing in the definition of the lower bounded closed form should be taken as* $\beta_0 + \beta_1$ *in place of* β_0.)

REMARK 3.1. (1) *Since* $\hat\eta(u,v) = -(\hat{\mathcal{L}}u, v)$ *and* $\hat{\mathcal{L}}u = \mathcal{L}^* u + K\cdot u$ *for* $u \in \mathcal{D}(\hat{\mathcal{L}})$ *and* $v \in \mathcal{F}$, *we find that*

$$\hat\eta_\beta(u,u) = \eta^*_{\beta_0+\varepsilon}(u,u) + (K^- u, u) + \big((\beta_1 - K^+)u, u\big) \geqslant 0$$

for any $u \in \mathcal{F}$ *and any* $\beta > \beta_0 + \beta_1$ *with* $\beta - \beta_0 - \beta_1 = \varepsilon > 0$. *This means that it is lower bounded. The sector condition is verified easily by using the property of the form* η^*. *The Markovian nature is shown as follows*: *for* $u \in \mathcal{F}$,

$$\begin{aligned}\hat\eta_{\beta_1}(u, u - Uu) &= \eta^*(u, u - Uu) + (K^- u, u - Uu)\\ &\quad + \big((\beta_1 - K^+)u, u - Uu\big) \geqslant 0.\end{aligned}$$

(2) *The drift term of* $\mathcal{L}u$ *disappears in the expression since the function* $h \mapsto \nabla u(x)\cdot h\, C(x,h)$ *is an odd function for* $x \in \mathbb{R}^d$ *and* $h \in \mathbb{R}^d$ *with* $0 < |h| < 1$.

We apply the following conditions when the function $C(x,h)$ *does not satisfy the symmetric condition* ($C(x,h) = C(x,-h),\ x \in \mathbb{R}^d, h \in \mathbb{R}^d \setminus \{0\}$) *in* (3.1):

$$\sup_{x\in\mathbb{R}^d} \int_{0<|h|<1} |h|\cdot|C(x,h) - C(x,-h)|n(dh) < \infty,$$

$$\sup_{x\in\mathbb{R}^d} \int_{0<|h|<1} |h|\cdot|C(x+h,-h) - C(x-h,-h)|n(dh) < \infty,$$

and

$$\sup_{x\in\mathbb{R}^d} \int_{0<|h|<1} |\nabla_x C(x,h)\cdot h - \nabla_x C(x,-h)\cdot h| n(h)dh < \infty.$$

In this case, *the operators* $\mathcal{L}$ *and* $\hat{\mathcal{L}}$, *and the function* K *have the following forms*:

$$\begin{aligned}\mathcal{L}u(x) &= \int_{h\neq 0} \big(u(x+h) - u(x) - \nabla u(x)\cdot h \mathbf{1}_{B(1)}(h)\big) C(x,h) n(h)dh\\ &\quad + \frac{1}{2}\int_{0<|h|<1} \nabla u(x)\cdot h\big(C(x,h) - C(x,-h)\big)n(h)dh,\\ \hat{\mathcal{L}}u(x) &= \int_{h\neq 0} \big(u(x+h) - u(x) - \nabla u(x)\cdot h \mathbf{1}_{B(1)}(h)\big) C(x+h,-h) n(h)dh\\ &\quad + \frac{1}{2}\int_{0<|h|<1} \nabla u(x)\cdot h\big(C(x+h,-h) - C(x-h,h)\big)n(h)dh,\end{aligned}$$

$$K(x) = \int_{h\neq 0} \big(C(x+h,h) - C(x,h) - \nabla_x C(x,h)\cdot h\mathbf{1}_{B(1)}(h)\big)n(h)dh$$
$$+ \frac{1}{2}\int_{0<|h|<1} \big(\nabla_x C(x,h) - \nabla_x C(x,-h)\big)\cdot h\, n(h)dh$$

for $x \in \mathbb{R}^d$ *and* $u \in C_0^2(\mathbb{R}^d)$.

(3) *In Theorem* 2.1 *of* [12], *Schilling and Wang obtained a similar result under slightly weaker assumptions on the kernel than ours. They also derived the closed expression of the form of the dual operator by using the so-called "symmetric principal value" due to Zhi-Ming Ma et al.* [5]. *Moreover*, *the dual operator is then represented as the sum of a non-local operator and a killing/creation which is obtained through the vague limit of some sequence of bounded* (*signed*) *measures. They also claimed that the dual semigroup is sub-Markov if the killing/creation term is non-positive. But in our case*, *the dual operator/form always is sub-Markov by taking a killing and it seems that the dual operator hardly satisfies the sub-Markov property unless the killing/creation vanishes.*

4. DUAL OF GENERATORS OF STABLE-LIKE PROCESSES

In this section, we apply the result obtained in the preceding section to the case of Bass's stable-like processes [1]. Take $\alpha \in C_b^2(\mathbb{R}^d)$. Assume there exist positive numbers $\underline{\alpha}$ and $\overline{\alpha}$ such that $0 < \underline{\alpha} \leqslant \alpha(x) \leqslant \overline{\alpha} < 2,\ x \in \mathbb{R}^d$. Then the generator of stable-like process is given by

$$-(-\Delta)^{-\alpha(x)/2}u(x) = \int_{h\neq 0} \big(u(x+h) - u(x) - \nabla u(x)\cdot h\mathbf{1}_B(h)\big)\frac{w(x)}{|h|^{d+\alpha(x)}}dh$$

for $u \in C_0^2(\mathbb{R}^d)$, where $B = \{h \in \mathbb{R}^d :\ |h| < 1\}$, the unit ball at the origin, and w is a function chosen so that $-(-\Delta)^{-\alpha(x)/2}e^{iux} = -|u|^{\alpha(x)}e^{iux}$. Note that the function w is given by

$$w(x) = \frac{\Gamma\big((1+\alpha(x))/2\big)\Gamma\big((\alpha(x)+d)/2\big)\sin\big(\pi\alpha(x)/2\big)}{2^{1-\alpha(x)}\pi^{d/2+1}}, \quad x \in \mathbb{R}^d.$$

Since α belongs to $C_b^2(\mathbb{R}^d)$ and satisfies $0 < \underline{\alpha} \leqslant \alpha(x) \leqslant \overline{\alpha} < 2$, it follows that w also belongs to $C_b^2(\mathbb{R}^d)$ and $0 < \underline{w} \leqslant w(x) \leqslant \overline{w} < \infty$ for some constants $\underline{w}$ and $\overline{w}$. Let us define

$$C(x,h) := w(x)|h|^{\overline{\alpha}-\alpha(x)}, \quad n(h) = |h|^{-d-\overline{\alpha}}, \quad \text{for } x,h \in \mathbb{R}^d,\ h \neq 0,$$

and put

$$k(x,x+h) := w(x)|h|^{-d-\alpha(x)} := C(x,h)n(h) = w(x)|h|^{\overline{\alpha}-\alpha(x)}\cdot|h|^{-d-\overline{\alpha}}.$$

We now check all the conditions in Theorem 3.1. Since

$$C(x,h) - C(x+h,h) = w(x)|h|^{\overline{\alpha}-\alpha(x)} - w(x+h)|h|^{\overline{\alpha}-\alpha(x+h)}$$

and $\alpha \in C_b^2(\mathbb{R}^d)$, we find that

$$\begin{aligned}
&\int_{h\neq 0} \frac{|C(x,h)-C(x+h,h)|^2}{C(x,h)+C(x+h,h)}\, n(h)dh \\
&\quad\leqslant \frac{4\overline{w}^2}{\underline{w}^2}\int_{h\neq 0} \frac{\big||h|^{\overline{\alpha}-\alpha(x)} - |h|^{\overline{\alpha}-\alpha(x+h)}\big|^2}{|h|^{\overline{\alpha}-\alpha(x)} + |h|^{\overline{\alpha}-\alpha(x+h)}}\, |h|^{-d-\overline{\alpha}}dh \\
&\qquad + \frac{4}{\underline{w}^2}\int_{h\neq 0} \frac{|w(x)-w(x+h)|^2\cdot |h|^{2\overline{\alpha}-2\alpha(x)}}{|h|^{\overline{\alpha}-\alpha(x)} + |h|^{\overline{\alpha}-\alpha(x+h)}}\, |h|^{-d-\overline{\alpha}}dh \\
&\quad\leqslant \frac{4\overline{w}^2}{\underline{w}^2}\int_{0<|h|<1} \frac{\big||h|^{\overline{\alpha}-\alpha(x)} - |h|^{\overline{\alpha}-\alpha(x+h)}\big|^2}{|h|^{\overline{\alpha}-\alpha(x)} + |h|^{\overline{\alpha}-\alpha(x+h)}}\, |h|^{-d-\overline{\alpha}}dh \\
&\qquad + \frac{4}{\underline{w}^2}\int_{0<|h|<1} \frac{|w(x)-w(x+h)|^2\cdot |h|^{2\overline{\alpha}-2\alpha(x)}}{|h|^{\overline{\alpha}-\alpha(x)} + |h|^{\overline{\alpha}-\alpha(x+h)}}\, |h|^{-d-\overline{\alpha}}dh \\
&\qquad + \left(\frac{8\overline{w}^2}{\underline{w}^2} + 4\frac{(\overline{w}-\underline{w})^2}{\underline{w}^2}\right)\int_{|h|\geqslant 1} |h|^{-d-\underline{\alpha}}dh \\
&\quad =: \text{(I)} + \text{(II)} + \text{(III)}.
\end{aligned}$$

It is clear that the term (III) is finite. We first consider the term (I). Since, for each $x, h \in \mathbb{R}^d$ with $0 < |h| < 1$,

$$\begin{aligned}
\big||h|^{\overline{\alpha}-\alpha(x)} - |h|^{\overline{\alpha}-\alpha(x+h)}\big| &= |h|^{\overline{\alpha}}\cdot\Big|\int_{\alpha(x+h)}^{\alpha(x)} |h|^{-t}(\ln|h|)dt\Big| \\
&\leqslant |h|^{\overline{\alpha}}\cdot|\alpha(x)-\alpha(x+h)|\cdot|h|^{-\alpha(x)\vee\alpha(x+h)}\big(\log(1/|h|)\big) \\
&\leqslant \|\nabla\alpha\|_\infty\cdot|h|^{\overline{\alpha}+1-\alpha(x)\vee\alpha(x+h)}\big(\log(1/|h|)\big),
\end{aligned}$$

it follows that

$$\begin{aligned}
\text{(I)} &\leqslant \frac{4\overline{w}^2}{\underline{w}^2}\int_{0<|h|<1} \frac{\|\nabla\alpha\|_\infty^2\cdot|h|^{2\overline{\alpha}+2-2(\alpha(x)\vee\alpha(x+h))}\big(\log(1/|h|)\big)^2}{|h|^{\overline{\alpha}-\alpha(x)} + |h|^{\overline{\alpha}-\alpha(x+h)}}\cdot|h|^{-d-\overline{\alpha}}dh \\
&\leqslant \frac{4\overline{w}^2}{\underline{w}^2}\cdot\|\nabla\alpha\|_\infty^2\int_{0<|h|<1} |h|^{-d+2}\cdot\frac{|h|^{-2(\alpha(x)\vee\alpha(x+h))}}{|h|^{-\alpha(x)} + |h|^{-\alpha(x+h)}}\cdot\big(\log(1/|h|)\big)^2dh \\
&\leqslant \frac{4\overline{w}^2}{\underline{w}^2}\cdot\|\nabla\alpha\|_\infty^2\int_{0<|h|<1} |h|^{-d+2}\cdot|h|^{-(\alpha(x)\vee\alpha(x+h))}\big(\log(1/|h|)\big)^2dh \\
&\leqslant \frac{4\overline{w}^2}{\underline{w}^2}\cdot\|\nabla\alpha\|_\infty^2\int_{0<|h|<1} |h|^{-d+2-\overline{\alpha}}\big(\log(1/|h|)\big)^2dh < \infty.
\end{aligned}$$

Since $w \in C_b^2(\mathbb{R}^d)$, we obtain

$$|w(x) - w(x+h)| \leqslant \|\nabla w\|_\infty \cdot |h|, \quad x, h \in \mathbb{R}^d \text{ with } 0 < |h| < 1.$$

Then

$$\begin{aligned}(\mathrm{II}) &\leqslant \frac{4\|\nabla w\|_\infty^2}{\underline{w}^2} \int_{0<|h|<1} \frac{|h|^2 \cdot |h|^{2\overline{\alpha}-2\alpha(x)}}{|h|^{\overline{\alpha}-\alpha(x)} + |h|^{\overline{\alpha}-\alpha(x+h)}} |h|^{-d-\overline{\alpha}} dh \\ &\leqslant \frac{4\|\nabla w\|_\infty^2}{\underline{w}^2} \int_{0<|h|<1} |h|^{-d+2-\overline{\alpha}} dh < \infty.\end{aligned}$$

Therefore, the condition (A2) holds. Since $C(x,h) = C(x,-h)$ for any $x \in \mathbb{R}^d$ and $h \in \mathbb{R}^d \setminus \{0\}$, we find that, for $x, h \in \mathbb{R}^d$ with $0 < |h| < 1$,

$$k(x, x+h) - k(x, x-h) = \big(C(x,h) - C(x,-h)\big) n(h) = 0,$$

and hence (A4) is automatically satisfied. Next we see that (A5) is satisfied. We have

$$\begin{aligned}&\int_{0<|h|<1} |h|^2 \big(C(x,h) + C(x+h,h)\big) n(h) dh \\ &\qquad \leqslant \overline{w} \int_{0<|h|<1} |h|^2 (|h|^{\overline{\alpha}-\alpha(x)} + |h|^{\overline{\alpha}-\alpha(x+h)}) |h|^{-d-\overline{\alpha}} dh \\ &\qquad \leqslant 2\,\overline{w} \int_{0<|h|<1} |h|^{2-d-\overline{\alpha}} dh < \infty.\end{aligned}$$

This means that $x \mapsto \int_{0<|h|<1} |h|^2 \big(C(x,h) + C(x+h,h)\big) n(h) dh$ is a bounded function, and hence is in $L^2_{\mathrm{loc}}(\mathbb{R}^d)$. Moreover,

$$\begin{aligned}&\int_{|h|\geqslant 1} \big(C(x,h) + C(x+h,h)\big) n(h) dh \\ &\leqslant \overline{w} \int_{|h|\geqslant 1} (|h|^{\overline{\alpha}-\alpha(x)} + |h|^{\overline{\alpha}-\alpha(x+h)}) |h|^{-d-\overline{\alpha}} dh \leqslant 2\,\overline{w} \int_{|h|\geqslant 1} |h|^{-d-\underline{\alpha}} dh < \infty.\end{aligned}$$

Thus these estimates imply (A5). We finally consider (A6). The first condition in (A6) holds since α belongs to $C^2(\mathbb{R}^d)$. Therefore, it is enough to show the second condition in (A6). For $x \in \mathbb{R}^d$ and $0 < |h| < 1$, we have

$$\begin{aligned}\frac{\partial C(x,h)}{\partial x_i} &= \frac{\partial w(x)}{\partial x_i} \cdot |h|^{\overline{\alpha}-\alpha(x)} + w(x) \cdot \frac{\partial \alpha(x)}{\partial x_i} \cdot |h|^{\overline{\alpha}-\alpha(x)} \log(1/|h|), \\ \frac{\partial^2 C(x,h)}{\partial x_i^2} &= \bigg[\frac{\partial^2 w(x)}{\partial x_i^2} + 2 \frac{\partial w(x)}{\partial x_i} \frac{\partial \alpha(x)}{\partial x_i} \log(1/|h|) + w(x) \frac{\partial^2 \alpha(x)}{\partial x_i^2} \log(1/|h|) \\ &\qquad + w(x) \Big(\frac{\partial \alpha(x)}{\partial x_i} \Big)^2 \big(\log(1/|h|)\big)^2 \bigg] \cdot |h|^{\overline{\alpha}-\alpha(x)}\end{aligned}$$

and

$$\frac{\partial^2 C(x,h)}{\partial x_i \partial x_j} = \left[\frac{\partial^2 w(x)}{\partial x_i \partial x_j} + \left(\frac{\partial w(x)}{\partial x_i}\frac{\partial \alpha(x)}{\partial x_j} + \frac{\partial w(x)}{\partial x_j}\frac{\partial \alpha(x)}{\partial x_i}\right)\log(1/|h|)\right.$$
$$\left. + w(x)\frac{\partial^2 \alpha(x)}{\partial x_i \partial x_j}\log(1/|h|) + w(x)\frac{\partial \alpha(x)}{\partial x_i}\frac{\partial \alpha(x)}{\partial x_j}\big(\log(1/|h|)\big)^2\right] \cdot |h|^{\overline{\alpha}-\alpha(x)}$$

for $i, j = 1, 2, \ldots, d$. Then we find

$$\int_{0<|h|<1} \left|\frac{\partial^2 C(x,h)}{\partial x_i \partial x_j} h_i h_j\right| n(h)dh$$
$$\leqslant C \int_{0<|h|<1} \Big(1 + \log(1/|h|) + \big(\log(1/|h|)\big)^2\Big)|h|^{2-d-\overline{\alpha}} dh < \infty.$$

Hence this gives us the second condition in (A6).

Summarizing the calculations done above, we can state the following

PROPOSITION 4.1. *Let $\alpha \in C_b^2(\mathbb{R}^d)$ be a function taking values in the interval $[\underline{\alpha}, \overline{\alpha}]$ for some $0 < \underline{\alpha} \leqslant \overline{\alpha} < 2$. Then the dual operator $\widehat{(-\Delta)^{\alpha(x)/2}}$ of the stable-like generator $(-\Delta)^{\alpha(x)/2}$ on $L^2(\mathbb{R}^d)$ has the following form for $u \in C_0^2(\mathbb{R}^d)$:*

$$-\widehat{(-\Delta)^{\alpha(x)/2}}u(x) = \int_{h\neq 0} \big(u(x+h) - u(x) - \nabla u(x)\cdot h\mathbf{1}_{B(1)}(h)\big)\frac{w(x+h)}{|h|^{d+\alpha(x+h)}}dh$$
$$+\frac{1}{2}\int_{0<|h|<1} \nabla u(x)\cdot h\left(\frac{w(x+h)}{|h|^{d+\alpha(x+h)}} - \frac{w(x)}{|h|^{d+\alpha(x)}}\right)dh$$
$$+ u(x)\int_{h\neq 0}\Big(w(x+h)|h|^{\overline{\alpha}\alpha(x+h)} - w(x)|h|^{\overline{\alpha}-\alpha(x)}$$
$$-\nabla_x\big(w(x)|h|^{\overline{\alpha}-\alpha(x)}\big)\cdot h\mathbf{1}_{B(1)}(h)\Big)\frac{dh}{|h|^{d+\overline{\alpha}}}$$
$$=: \mathcal{L}^* u(x) + u(x)\cdot K(x), \quad x \in \mathbb{R}^d,$$

where

$$\mathcal{L}^* u(x) = \int_{h\neq 0} \big(u(x+h) - u(x) - \nabla u(x)\cdot h\mathbf{1}_{B(1)}(h)\big)\frac{w(x+h)}{|h|^{d+\alpha(x+h)}}dh$$
$$+\frac{1}{2}\int_{0<|h|<1} \nabla u(x)\cdot h\left(\frac{w(x+h)}{|h|^{d+\alpha(x+h)}} - \frac{w(x)}{|h|^{d+\alpha(x)}}\right)dh,$$
$$K(x) = \int_{h\neq 0}\Big(w(x+h)|h|^{\overline{\alpha}-\alpha(x+h)} - w(x)|h|^{\overline{\alpha}-\alpha(x)}$$
$$-\nabla_x\big(w(x)|h|^{\overline{\alpha}-\alpha(x)}\big)\cdot h\mathbf{1}_{B(1)}(h)\Big)\frac{dh}{|h|^{d+\overline{\alpha}}}$$

for $u \in C_0^2(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$. Moreover, the dual $\widehat{(-\Delta)^{\alpha(x)/2}}$ corresponds to a Hunt process by taking a killing to the Hunt process associated with the lower *bounded semi-Dirichlet form generated by $\mathcal{L}^*$, which we call "reversed stable-like process". Note that the killing rate is given by $\beta := \sup_{x\in\mathbb{R}^d} K^+(x)$, where K^+ is the positive part of $K(x)$.*

Note that, for a closed form $(\eta, \mathcal{F})$ on $L^2(E; m)$ and a positive number β,

$$\tilde{\eta}(u, v) := \eta(u, v) + \beta \int uvdm, \quad u, v \in \mathcal{F},$$

also defines a closed form on L^2. Then it is known that the semigroup $\{\tilde{T}_t\}$ associated with $\tilde{\eta}$ is given by $\tilde{T}_t = e^{-\beta t}T_t$, where $\{T_t\}$ is the semigroup corresponding to $\mathcal{E}$. In this case, $\tilde{\eta}$ is called the *killed form* with killing rate β with respect to the form η.

Acknowledgement. The author thanks the anonymous referees for the helpful comments.

REFERENCES

[1] R. F. Bass, *Uniqueness in law for pure jump Markov processes*, Probab. Theory Related Fields 79 (1988), pp. 271–287.

[2] S. Carrillo Menendez, *Processus de Markov associé à une forme de Dirichlet non symétrique*, Z. Wahrsch. Verw. Gebiete 33 (1975), pp. 139–154.

[3] M. Fukushima, Y. Oshima, and M. Takeda, *Dirichlet Forms and Symmetric Markov Processes*, second edition and revised edition, de Gruyter, 2011.

[4] M. Fukushima and T. Uemura, *Jump-type Hunt processes generated by lower bounded semi-Dirichlet forms*, Ann. Probab. 40 (2012), pp. 858–889.

[5] Z.-C. Hu, Z.-M. Ma, and W. Sun, *Extensions of Lévy–Khintchine formula and Beurling–Deny formula in semi-Dirichlet forms setting*, J. Funct. Anal. 239 (2006), pp. 179–213.

[6] H. Kunita, *Sub-Markov semi-groups in Banach lattices*, in: *Proceedings of the International Conference on Functional Analysis and Related Topics*, University of Tokyo Press, Tokyo 1969, pp. 332–343.

[7] Z.-M. Ma and M. Röckner, *Introduction to the Theory of (Non-symmetric) Dirichlet Forms*, Springer, 1992.

[8] Y. Ōshima, *Lectures on Dirichlet spaces*, Lecture Notes at Erlangen University, 1988.

[9] Y. Ōshima, *Semi-Dirichlet Forms and Markov Processes*, de Gruyter, 2013.

[10] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer, 1983.

[11] R. L. Schilling and T. Uemura, *On the Feller property of Dirichlet forms generated by pseudo differential operators*, Tohoku Math. J. 59 (2007), pp. 401–422.

[12] R. L. Schilling and J. Wang, *Lower bounded semi-Dirichlet forms associated with Lévy type operators*, to appear in: *Festschrift Masatoshi Fukushima*.

[13] T. Uemura, *On symmetric stable-like processes: some path properties and generators*, J. Theoret. Probab. 17 (2004), pp. 541–555.

[14] T. Uemura, *A remark on non-local operators with variable order*, Osaka J. Math. 46 (2009), pp. 503–514.

Department of Mathematics
Kansai University
Yamate, Suita, Osaka 564-8680, Japan
E-mail: t-uemura@kansai-u.ac.jp

Received on 9.1.2013;
revised version on 17.12.2013