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AN EQUIVALENT CHARACTERIZATION OF WEAK BMO MARTINGALE SPACES

BY

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Abstract. In this paper, we give an equivalent characterization of weak BMO martingale spaces due to Ferenc Weisz (1998).

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1. INTRODUCTION

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, and $\{\mathcal{F}_n\}_{n\geqslant 0}$ be an increasing sequence of sub-σ-algebras of $\mathcal{F}$ such that $\mathcal{F} = \sigma\big(\bigcup_{n\geqslant 0} \mathcal{F}_n\big)$. The expectation operator and the conditional expectation operator relative to $\mathcal{F}_n$ are denoted by $\mathbb{E}$ and $\mathbb{E}_n$, respectively. A sequence $f = (f_n)_{n\geqslant 0}$ of random variables such that f_n is $\mathcal{F}_n$-measurable is said to be a *martingale* if $\mathbb{E}(|f_n|) < \infty$ and $\mathbb{E}_n(f_{n+1}) = f_n$ for every $n \geqslant 0$.

The study of the space BMO (Bounded Mean Oscillation) began with the establishment of the so-called John–Nirenberg theorem in [11]. Basing mainly on the duality and something else, the space BMO plays a remarkable role both in classical analysis and martingale theory. For example, BMO is a good space in operator actions (see e.g. [14], Chapter 4). And the martingale space $BMO_r(\alpha)$ was first introduced by Herz in [4] as the dual of H_p^s $(0 < p \leqslant 1)$ associated with the dyadic filtration (see Example 2.1 below). With the help of atomic decomposition, Weisz extended this result in [15] to a general case. Let $\mathcal{T}$ be the set of all stopping times with respect to $\{\mathcal{F}_n\}_{n\geqslant 0}$. The martingale space $BMO_r(\alpha)$ ([16], p. 8; or [15]) for $1 \leqslant r < \infty$ and $\alpha \geqslant 0$ is defined as

$$BMO_r(\alpha) = \{f = (f_n)_{n\geqslant 0} : \|f\|_{BMO_r(\alpha)} < \infty\},$$

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where

$$\|f\|_{BMO_r(\alpha)} = \sup_{\nu\in\mathcal{T}} \mathbb{P}(\nu<\infty)^{-1/r-\alpha}\|f-f^\nu\|_r.$$

We present two well-known results (see [16] or [15]). If $0<p\leqslant 1$ and $\alpha=\frac{1}{p}-1$, then $BMO_2(\alpha)$ is the dual space of the Hardy space H_p^s. If the stochastic basis $\{\mathcal{F}_n\}_{n\geqslant 1}$ is regular, then $BMO_r(\alpha)=BMO_1(\alpha)$. And recently, Yi et al. proved in [18] that $BMO_E(\alpha)=BMO_1(\alpha)$, where $\alpha=0$ and E is a rearrangement invariant Banach function space.

In the present paper, we consider a weak BMO martingale space. To characterize the dual of the weak Hardy martingale space $H^s_{p,\infty}$, Weisz in [17] first introduced and studied the weak BMO martingale space. Let us recall the definition. We also refer the reader to [12] and [13] for some new results related to weak BMO martingales spaces.

DEFINITION 1.1. Let $1\leqslant r<\infty, \alpha r+1>0$. The space $w\mathcal{BMO}_r(\alpha)$ is defined as the set of all martingales $f\in L_r$ with the norm

$$\|f\|_{w\mathcal{BMO}_r(\alpha)} = \int_0^\infty \frac{t_\alpha^r(x)}{x}dx<\infty,$$

where

$$t^r_\alpha(x) = x^{-1/r-\alpha}\sup_{\nu\in\mathcal{T}:P(\nu<\infty)\leqslant x}\|f-f^\nu\|_r.$$

In the very recent paper [8], the generalized BMO martingale space is introduced as the dual of Hardy–Lorentz martingale space. Strongly motivated by [8], Definition 1.1, we introduce the following new weak BMO martingale space by stopping time sequences.

DEFINITION 1.2. Let $1\leqslant r<\infty$ and $\alpha\geqslant 0$. The weak BMO martingale space $wBMO_r(\alpha)$ is defined by

$$wBMO_r(\alpha) = \{f\in L_r : \|f\|_{wBMO_r(\alpha)}<\infty\},$$

where

$$\|f\|_{wBMO_r(\alpha)} = \sup\frac{\sum_{k\in\mathbb{Z}} 2^k\mathbb{P}(\nu_k<\infty)^{1-1/r}\|f-f^{\nu_k}\|_r}{\sup_k 2^k\mathbb{P}(\nu_k<\infty)^{1+\alpha}}$$

and the supremum is taken over all stopping time sequences $\{\nu_k\}_{k\in\mathbb{Z}}$ such that $2^k\mathbb{P}(\nu_k<\infty)^{1+\alpha}\in\ell_\infty$.

It is a very natural question: what is the relationship between $w\mathcal{BMO}_r(\alpha)$ and $wBMO_r(\alpha)$? The paper fully answers this question. Our main result can be described as follows. We simply put $w\mathcal{BMO}=w\mathcal{BMO}(0)$ and $wBMO = wBMO(0)$.

THEOREM 1.1. *Let* $1 \leqslant r < \infty$ *and* $\alpha \geqslant 0$. *If the stochastic basis* $\{\mathcal{F}_n\}_{n \geqslant 0}$ *is regular*, *then*

$$w\mathcal{BMO}_r(\alpha) = wBMO_r(\alpha)$$

with equivalent norms. In particular,

$$w\mathcal{BMO}_r = wBMO_r$$

with equivalent norms.

In this paper, the set of integers and the set of nonnegative integers are always denoted by $\mathbb{Z}$ and $\mathbb{N}$, respectively. We use C to denote a positive constant which may vary from line to line. The symbol $\subset$ means the continuous embedding.

2. PRELIMINARIES

Firstly, we give the definition of Lorentz spaces. We denote by $L_0(\Omega, \mathcal{F}, \mathbb{P})$, or simply $L_0(\Omega)$, the space of all measurable functions on $(\Omega, \mathcal{F}, \mathbb{P})$. For any $f \in L_0(\Omega)$, we define the distribution function of f by

$$\lambda_s(f) = \mathbb{P}\big(\{\omega \in \Omega : |f(\omega)| > s\}\big), \quad s \geqslant 0.$$

Moreover, denote by $\mu_t(f)$ the decreasing rearrangement of f defined by

$$\mu_t(f) = \inf\{s \geqslant 0 : \lambda_s(f) \leqslant t\}, \quad t \geqslant 0,$$

with the convention that $\inf \emptyset = \infty$.

DEFINITION 2.1. Let $0 < p < \infty$ and $0 < q \leqslant \infty$. Then, the *Lorentz space* $L_{p,q}(\Omega)$ consists of measurable functions such that $\|f\|_{p,q} < \infty$, where

$$\|f\|_{p,q} = \left[\int_0^\infty \big(t^{1/p}\mu_t(f)\big)^q \frac{dt}{t}\right]^{1/q}, \quad 0 < q < \infty,$$

and

$$\|f\|_{p,\infty} = \sup_{0 \leqslant t < \infty} t^{1/p}\mu_t(f), \quad q = \infty.$$

REMARK 2.1. *We refer the reader to* [2] *for the following basic properties.*

(1) *If* $p = q$, *then* $L_{p,q}(\Omega)$ *becomes* $L_p(\Omega)$.

(2) *If* $0 < p_1 \leqslant p_2 < \infty$ *and* $0 < q \leqslant \infty$, *then* $\|f\|_{p_1,q} \leqslant C\|f\|_{p_2,q}$, *where* C *depends on* p_1, p_2 *and* q. *This is due to* $\mathbb{P}(\Omega) = 1$.

(3) *If* $0 < p < \infty$ *and* $0 < q_1 \leqslant q_2 \leqslant \infty$, *then* $\|f\|_{p,q_2} \leqslant C\|f\|_{p,q_1}$, *where* C *depends on* q_1, q_2 *and* p.

Denote by $\mathcal{M}$ the set of all martingales $f = (f_n)_{n\geqslant 0}$ relative to $\{\mathcal{F}_n\}_{n\geqslant 0}$ such that $f_0 = 0$. For $f \in \mathcal{M}$, denote its martingale difference by $d_n f = f_n - f_{n-1}$ ($n \geqslant 0$, with the convention $f_{-1} = 0$). Then the maximal function and the conditional quadratic variation of a martingale f are respectively defined by

$$f_n^* = \sup_{0\leqslant i\leqslant n} |f_i|, \quad f^* = \sup_{n\geqslant 0} |f_n|,$$

$$s_n(f) = \Big(\sum_{i=1}^{n} \mathbb{E}_{i-1}|d_i f|^2\Big)^{1/2}, \quad s(f) = \Big(\sum_{i=1}^{\infty} \mathbb{E}_{i-1}|d_i f|^2\Big)^{1/2}.$$

Then we define *martingale Hardy–Lorentz spaces* as follows.

DEFINITION 2.2. Let $0 < p < \infty$ and $0 < q \leqslant \infty$. Define

$$H_{p,q}^* = \{f \in \mathcal{M} : \|f\|_{H_{p,q}^*} = \|f^*\|_{p,q} < \infty\},$$
$$H_{p,q}^s = \{f \in \mathcal{M} : \|f\|_{H_{p,q}^s} = \|s(f)\|_{p,q} < \infty\}.$$

If $p = q$, then the martingale Hardy–Lorentz spaces recover the martingale Hardy spaces H_p^* and H_p^s (see [16]).

Recall that the stochastic basis $\{\mathcal{F}_n\}_{n\geqslant 0}$ is said to be *regular* if there exists a positive constant $R > 0$ such that

$$f_n \leqslant R f_{n-1}, \quad \forall n > 0, \tag{2.1}$$

holds for all nonnegative martingales $f = (f_n)_{n\geqslant 0}$. Condition (2.1) can be replaced by several other equivalent conditions (see [14], Chapter 7). We refer the reader to [14], p. 265, for examples for regular stochastic basis. Here, we give a special case.

EXAMPLE 2.1. Let $\big((0,1], \mathcal{F}, \mu\big)$ be a probability space such that μ is the Lebesgue measure and subalgebras $\{\mathcal{F}_n\}_{n\geqslant 0}$ are generated as follows:

$$\mathcal{F}_n = \text{a } \sigma\text{-algebra generated by atoms } \left(\frac{j}{2^n}, \frac{j+1}{2^n}\right], j = 0, \ldots, 2^n - 1.$$

Then $\{\mathcal{F}_n\}_{n\geqslant 0}$ is regular. And all martingales with respect to such $\{\mathcal{F}_n\}_{n\geqslant 0}$ are called *dyadic martingales*.

The method of atomic decompositions plays an important role in martingale theory (see, for example, [3]–[5], [16], [17]). The atomic decompositions of Hardy–Lorentz martingale spaces $H_{p,q}^s$ and martingale inequalities are given in [6] and [8]. We also mention that Hardy–Lorentz spaces with variable exponents were investigated very recently in [9] and [10]. Let us first introduce the concept of an atom (see [16], p. 14).

DEFINITION 2.3. Let $0 < p < \infty$ and $p < r \leqslant \infty$. A measurable function a is called a $(1, p, r)$-*atom* (or $(3, p, r)$-*atom*) if there exists a stopping time $\nu \in \mathcal{T}$ such that $a_n = \mathbb{E}_n(a) = 0$ if $\nu \geqslant n$, and

$$\|s(a)\|_r \text{ (or } \|a^*\|_r) \leqslant \mathbb{P}(\nu < \infty)^{1/r-1/p}.$$

REMARK 2.2. *Let $0 < p < r \leqslant \infty$ and $0 < q \leqslant r$. If a is a $(1, p, r)$-atom, then $\|a\|_{H^s_{p,q}} \leqslant C$. Choose p_1, p_2 such that $\frac{1}{p} = \frac{1}{r} + \frac{1}{p_1}$, $\frac{1}{q} = \frac{1}{r} + \frac{1}{q_1}$. By Hölder's inequality, we have (ν is the stopping time corresponding to the atom a)*

$$\begin{aligned}\|a\|_{H^s_{p,q}} &= \|s(a)\chi_{\{\nu<\infty\}}\|_{p,q} \leqslant C\|s(a)\|_{r,r}\|\chi_{\{\nu<\infty\}}\|_{p_1,q_1} \\ &\leqslant C\mathbb{P}(\nu<\infty)^{1/r-1/p}\Big(\int_0^\infty t^{q_1/p_1-1}\chi_{(0,\mathbb{P}(\nu<\infty))}dt\Big)^{1/q_1} \leqslant C.\end{aligned}$$

Similarly, we have $\|a\|_{H^*_{p,q}} \leqslant C$ for a $(3, p, r)$-atom a. If $p = q$, then $C = 1$.

The following result is from [8]. And the result about the Hardy space $H^*_{p,q}$ follows from the combining of Theorem 3.3 and Lemma 5.1 in [8].

THEOREM 2.1. *If $f = (f_n)_{n\geqslant 0} \in H^s_{p,q}$ for $0 < p < \infty, 0 < q \leqslant \infty$, then there exist a sequence $(a^k)_{k\in\mathbb{Z}}$ of $(1, p, \infty)$-atoms and a positive number A satisfying $\mu_k = A \cdot 2^k\mathbb{P}(\nu_k < \infty)^{1/p}$ (where ν_k is the stopping time corresponding to a^k) such that*

$$f_n = \sum_{k\in\mathbb{Z}} \mu_k a^k_n \ \ a.e., \qquad n \in \mathbb{N}, \tag{2.2}$$

and

$$\|\{\mu_k\}\|_{l_q} \leqslant C\|f\|_{H^s_{p,q}}.$$

Conversely, if the martingale f has the above decomposition, then $f \in H^s_{p,q}$ and $\|f\|_{H^s_{p,q}} \approx \inf \|\{\mu_k\}\|_{l_q}$, where the infimum is taken over all the above decompositions.

Moreover, if the stochastic basis $\{\mathcal{F}_n\}_{n\geqslant 0}$ is regular and if we replace $H^s_{p,q}$, $(1, p, \infty)$-atoms by $H^*_{p,q}$, $(3, p, \infty)$-atoms, then the conclusions above still hold.

LEMMA 2.1 ([1], Lemma 1.2). *Let $0 < p < \infty$ and let the nonnegative sequence $\{\mu_k\}$ be such that $\{2^k\mu_k\} \in l^q, 0 < q \leqslant \infty$. Further, suppose the nonnegative function φ satisfies the following property: there exists $0 < \varepsilon < \min(1, q/p)$ such that, given an arbitrary integer k_0, we have $\varphi \leqslant \psi_{k_0} + \eta_{k_0}$, where ψ_{k_0} and η_{k_0} satisfy*

$$2^{k_0 p}\mathbb{P}(\psi_{k_0} > 2^{k_0})^{\varepsilon} \leqslant C \sum_{k=-\infty}^{k_0-1} (2^k\mu_k^{\varepsilon})^p,$$

$$2^{k_0\varepsilon p}\mathbb{P}(\eta_{k_0} > 2^{k_0}) \leqslant C \sum_{k=k_0}^{\infty} (2^{k\varepsilon}\mu_k)^p.$$

Then $\varphi \in L_{p,q}$ and $\|\varphi\|_{p,q} \leqslant C\|\{2^k\mu_k\}\|_{l_q}$.

3. A JOHN–NIRENBERG THEOREM

In this section, we prove a John–Nirenberg theorem when the stochastic basis $\{\mathcal{F}_n\}_{n\geqslant 0}$ is regular. The main idea and method are similar to those of [8]. The following lemma can be found in [5], [16]. In fact, it follows from Theorem 7.14 in [5] and Corollary 5.13 in [16].

LEMMA 3.1. *Suppose that* $0 < q \leqslant \infty$ *and the stochastic basis* $\{\mathcal{F}_n\}_{n\geqslant 0}$ *is regular.*

If $0 < p < \infty$, *then* $H^*_{p,q}$ *and* $H^s_{p,q}$ *are equivalent.*

If $1 < p < \infty$, *then* $H^*_{p,q}$, $H^s_{p,q}$ *and* $L_{p,q}$ *are all equivalent.*

L_p is not dense in $L_{p,\infty}$. This fact is mentioned in [17], p. 143 (see also [2], Remark 1.4.14). Hence, to describe the duality, we need the following definition from [7], Remark 1.7.

DEFINITION 3.1. Let a measurable set $A_k \subset \Omega$ satisfy $\mathbb{P}(A_k) \to 0$ as $k \to \infty$. Define $\mathcal{L}_{p,\infty}$ as the set of all $f \in L_{p,\infty}$ having the absolute continuous quasi-norm defined by

$$\mathcal{L}_{p,\infty} = \{f \in L_{p,\infty} : \lim_{k\to\infty} \|f\chi_{A_k}\|_{p,\infty} = 0\}.$$

$\mathcal{L}_{p,\infty}$ is a closed subspace of $L_{p,\infty}$ and $L_p \subset \mathcal{L}_{p,\infty} \subset L_{p,\infty}$ (see [7]). Now we define

$$\mathcal{H}^s_{p,\infty} = \{f = (f_n)_{n\geqslant 0} : s(f) \in \mathcal{L}_{p,\infty}\},$$

which is a closed subspace of $H^s_{p,\infty}$. Similarly, we define $\mathcal{H}^*_{p,\infty}$.

REMARK 3.1. (1) *According to* [7], *Remark* 2.2, *we can conclude that* $H^s_2 = L_2$ *is dense in* $\mathcal{H}^s_{p,\infty}$.

(2) *If the stochastic basis* $\{\mathcal{F}_n\}_{n\geqslant 0}$ *is regular*, *then*, *by the same argument of Remark* 2.2 *in* [7], L_∞ *is dense in* $\mathcal{H}^*_{p,\infty}$.

LEMMA 3.2. *Let* $0 < p \leqslant 1$. *If the stochastic basis* $\{\mathcal{F}_n\}_{n\geqslant 0}$ *is regular*, *then*

$$(\mathcal{H}^*_{p,\infty})^* = wBMO_1(\alpha), \quad \alpha = \frac{1}{p} - 1.$$

Proof. Let $g \in wBMO_1(\alpha)$. Define

$$\phi_g(f) = \mathbb{E}(fg), \quad f \in L_\infty.$$

Then, by Theorem 2.1, we find that (ν_k is the stopping time corresponding to the atom a^k for every $k \in \mathbb{Z}$)

$$\begin{aligned}|\phi_g(f)| &\leqslant \sum_{k\in\mathbb{Z}} |\mu_k| \mathbb{E}\big(a^k(g-g^{\nu_k})\big) \leqslant \sum_{k\in\mathbb{Z}} |\mu_k| \|a^k\|_\infty \|g-g^{\nu_k}\|_1 \\ &\leqslant C \sum_{k\in\mathbb{Z}} |\mu_k| \|(a^k)^*\|_\infty \|g-g^{\nu_k}\|_1 \\ &\leqslant C \sum_{k\in\mathbb{Z}} |\mu_k| \mathbb{P}(\nu_k<\infty)^{-1/p} \|g-g^{\nu_k}\|_1 \\ &= C\cdot A \sum_{k\in\mathbb{Z}} 2^k \|g-g^{\nu_k}\|_1.\end{aligned}$$

By the definition of $\|\cdot\|_{wBMO_r(\alpha)}$, we obtain

$$\begin{aligned}|\phi_g(f)| &\leqslant C\cdot A \sup_k 2^k \mathbb{P}(\nu_k<\infty)^{1/p} \|g\|_{wBMO_1(\alpha)} \\ &\leqslant C\|f\|_{H^*_{p,\infty}} \|g\|_{wBMO_1(\alpha)}.\end{aligned}$$

Since the stochastic basis $\{\mathcal{F}_n\}_{n\geqslant 0}$ is regular, L_∞ is dense in $\mathcal{H}^*_{p,\infty}$ (see Remark 3.1(2)). Then ϕ_g can be uniquely extended to be a continuous linear functional on $\mathcal{H}^*_{p,\infty}$.

Conversely, let $\phi \in (\mathcal{H}^*_{p,\infty})^*$. Since L_2 is dense in $\mathcal{H}^*_{p,\infty}$ (see Remark 3.1(2)), there exists $g \in L_2 \subset L_1$ such that

$$\phi(f) = \mathbb{E}(fg), \quad f \in L_\infty.$$

Let $\{\nu_k\}_{k\in\mathbb{Z}}$ be a stopping time sequence satisfying $\{2^k \mathbb{P}(\nu_k<\infty)^{1/p}\}_{k\in\mathbb{Z}} \in l_\infty$ and let

$$h_k = \operatorname{sign}(g-g^{\nu_k}), \quad a^k = \frac{1}{2}(h_k - h_k^{\nu_k}) \mathbb{P}(\nu_k<\infty)^{-1/p}.$$

Then a^k is a $(3,p,\infty)$-atom. Let $f^N = \sum_{k=-N}^N 2^{k+1} \mathbb{P}(\nu_k<\infty)^{1/p} a^k$, where N is an arbitrary nonnegative integer. By Theorem 2.1, we have $f^N \in H^*_{p,\infty}$ and

$$\|f^N\|_{H^*_{p,\infty}} \leqslant C \sup_k 2^k \mathbb{P}(\nu_k<\infty)^{1/p}.$$

Consequently,

$$\begin{aligned}\sum_{k=-N}^N 2^k \|g-g^{\nu_k}\|_1 &= \sum_{k=-N}^N 2^k \mathbb{E}\big(h_k(g-g^{\nu_k})\big) = \sum_{k=-N}^N 2^k \mathbb{E}\big((h_k - h_k^{\nu_k})g\big) \\ &= \mathbb{E}(f^N g) = \phi(f^N) \leqslant \|f^N\|_{H^*_{p,\infty}} \|\phi\| \\ &\leqslant C \sup_k 2^k \mathbb{P}(\nu_k<\infty)^{1/p} \|\phi\|.\end{aligned}$$

Thus we have

$$\frac{\sum_{k=-N}^{N} 2^k \|g - g^{\nu_k}\|_1}{\sup_k 2^k \mathbb{P}(\nu_k < \infty)^{1/p}} \leqslant C\|\phi\|.$$

This implies $\|g\|_{wBMO_1(\alpha)} \leqslant C\|\phi\|$. The proof is complete. ■

LEMMA 3.3. *Let* $0 < p \leqslant 1$, $1 < r < \infty$. *If the stochastic basis* $\{\mathcal{F}_n\}_{n\geqslant 0}$ *is regular, then*

$$(\mathcal{H}^*_{p,\infty})^* = wBMO_r(\alpha), \quad \alpha = \frac{1}{p} - 1.$$

Proof. By Hölder's inequality, we have $\|f\|_{wBMO_1(\alpha)} \leqslant \|f\|_{wBMO_r(\alpha)}$ for any $f \in wBMO_r(\alpha)$. Let $g \in wBMO_r(\alpha) \subset L_r$. We define

$$\phi_g(f) = \mathbb{E}(fg), \quad \forall f \in L_{r'}.$$

Then, by Lemma 3.2, we have

$$|\phi_g(f)| \leqslant C\|f\|_{H^s_{p,\infty}} \|g\|_{wBMO_1(\alpha)} \leqslant C\|f\|_{H^s_{p,\infty}} \|g\|_{wBMO_r(\alpha)}.$$

It follows from Remark 3.1(2) that $L_{r'}$ is dense in $\mathcal{H}^*_{p,\infty}$. Thus ϕ_g can be uniquely extended to be a continuous linear functional on $\mathcal{H}^*_{p,\infty}$.

Conversely, if $\phi \in (\mathcal{H}^*_{p,\infty})^*$, by Doob's maximal inequality, we have $L_{r'} = H^*_{r',r'} \subset \mathcal{H}^*_{p,\infty}$. Then $(\mathcal{H}^*_{p,\infty})^* \subset (L_{r'})^* = L_r$. Thus there exists $g \in L_r$ such that

$$\phi(f) = \phi_g(f) = \mathbb{E}(fg), \quad \forall f \in L_{r'}.$$

Let $\{\nu_k\}_{k\in\mathbb{Z}}$ be a stopping time sequence such that $\{2^k \mathbb{P}(\nu_k < \infty)^{1/p}\}_{k\in\mathbb{Z}} \in l_\infty$ and N be an arbitrary nonnegative integer. Let

$$h_k = \frac{|g - g^{\nu_k}|^{r-1}\mathrm{sign}(g - g^{\nu_k})}{\|g - g^{\nu_k}\|_r^{r-1}}, \quad f = \sum_{k=-N}^{N} 2^k \mathbb{P}(\nu_k < \infty)^{1/r'}(h_k - h_k^{\nu_k}).$$

For an arbitrary integer k_0 which satisfies $-N \leqslant k_0 \leqslant N$ (for $k_0 \leqslant -N$, let $G = 0$ and $H = f$; for $k_0 > N$, let $H = 0$ and $G = f$), let

$$f = G + H,$$

where

$$G = \sum_{k=-N}^{k_0-1} 2^k \mathbb{P}(\nu_k < \infty)^{1/r'}(h_k - h_k^{\nu_k})$$

and

$$H = \sum_{k=k_0}^{N} 2^k \mathbb{P}(\nu_k < \infty)^{1/r'}(h_k - h_k^{\nu_k}).$$

Obviously, $\|h_k\|_{r'} = 1$, and $\|G\|_{r'} \leqslant 2\sum_{k=-N}^{k_0-1} 2^k \mathbb{P}(\nu_k < \infty)^{1/r'}$. By the sublinearity of the maximal operator $*$, we have $f^* \leqslant G^* + H^*$. Let $\varepsilon = p/r'$ $(0<\varepsilon<1)$. By Doob's maximal inequality, we have

$$\mathbb{P}(G^* > 2^{k_0}) \leqslant \frac{1}{2^{k_0 r'}} \|G^*\|_{r'}^{r'} \leqslant C \frac{1}{2^{k_0 r'}} \|G\|_{r'}^{r'} \leqslant C \frac{1}{2^{k_0 r'}} \Big(\sum_{k=-N}^{k_0-1} 2^k \mathbb{P}(\nu_k < \infty)^{1/r'} \Big)^{r'}.$$

On the other hand, $\{H^* > 0\} \subset \bigcup_{k=k_0}^{N} \{\nu_k < \infty\}$. Then, for each $0 < \varepsilon < 1$, we have

$$\begin{aligned} 2^{k_0 \varepsilon p} \mathbb{P}(H^* > 2^{k_0}) &\leqslant 2^{k_0 \varepsilon p} \mathbb{P}(H^* > 0) \leqslant 2^{k_0 \varepsilon p} \sum_{k=k_0}^{N} \mathbb{P}(\nu_k < \infty) \\ &\leqslant \sum_{k=k_0}^{N} 2^{k \varepsilon p} \mathbb{P}(\nu_k < \infty) = \sum_{k=k_0}^{N} \big(2^{k\varepsilon} \mathbb{P}(\nu_k < \infty)^{1/p} \big)^p \\ &\leqslant \sum_{k=k_0}^{\infty} \big(2^{k\varepsilon} \mathbb{P}(\nu_k < \infty)^{1/p} \big)^p. \end{aligned}$$

By Lemma 2.1, we have $f^* \in L_{p,\infty}$ and $\|f^*\|_{p,\infty} \leqslant C \|\{2^k \mathbb{P}(\nu_k < \infty)^{1/p}\}_{k\in\mathbb{Z}}\|_{l_\infty}$. Thus, $f \in H_{p,\infty}^*$ and

$$\|f\|_{H_{p,\infty}^*} \leqslant C \sup_k 2^k \mathbb{P}(\nu_k < \infty)^{1/p}.$$

Consequently,

$$\begin{aligned} \sum_{k=-N}^{N} 2^k \mathbb{P}(\nu_k < \infty)^{1-1/r} \|g - g^{\nu_k}\|_r &= \sum_{k=-N}^{N} 2^k \mathbb{P}(\nu_k < \infty)^{1/r'} \mathbb{E}\big(h_k (g - g^{\nu_k})\big) \\ &= \sum_{k=-N}^{N} 2^k \mathbb{P}(\nu_k < \infty)^{1/r'} \mathbb{E}\big((h_k - h_k^{\nu_k}) g\big) \\ &= \mathbb{E}(fg) = \varphi(f) \leqslant \|f\|_{H_{p,q}^*} \|\varphi\| \\ &\leqslant C \sup_k 2^k \mathbb{P}(\nu_k < \infty)^{1/p}. \end{aligned}$$

Thus we obtain

$$\frac{\sum_{k=-N}^{N} 2^k \mathbb{P}(\nu_k < \infty)^{1-1/r} \|g - g^{\nu_k}\|_r}{\sup_k 2^k \mathbb{P}(\nu_k < \infty)^{1/p}} \leqslant C \|\varphi\|.$$

Taking $N \to \infty$ and the supremum over all stopping time sequences satisfying $\{2^k \mathbb{P}(\nu_k < \infty)^{1/p}\}_{k\in\mathbb{Z}} \in l_\infty$, we get $\|g\|_{wBMO_r(\alpha)} \leqslant C\|\varphi\|$. ■

Now we formulate the weak version of the John–Nirenberg theorem, which directly follows from Lemmas 3.2 and 3.3.

THEOREM 3.1. *Let* $\alpha \geqslant 0$ *and* $1 \leqslant r < \infty$. *If the stochastic basis* $\{\mathcal{F}_n\}_{n\geqslant 0}$ *is regular*, *then*

$$wBMO_r(\alpha) = wBMO_1(\alpha)$$

with equivalent norms.

According to Lemma 3.1, Lemma 3.3 holds if we replace $\mathcal{H}^*_{p,\infty}$ by $\mathcal{H}^s_{p,\infty}$. Without regularity of stochastic basis $\{\mathcal{F}_n\}_{n\geqslant 0}$, we also get a duality result.

PROPOSITION 3.1. *Let* $0 < p \leqslant 1$. *Then* $\left(\mathcal{H}^s_{p,\infty}\right)^* = wBMO_2(\alpha)$ *with* $\alpha = 1/p - 1$.

Proof. Note that $H^s_2 = L_2$ is dense in $\mathcal{H}^s_{p,\infty}$ by Remark 3.1(1). The first part of the proof is similar to that of Lemma 3.2, and the converse part is similar to that of Lemma 3.3 with $r = 2$. We omit the proof. ■

4. PROOF OF THE MAIN THEOREM

In this section we complete the proof of Theorem 1.1.

Let $\overline{H}^s_{p,\infty}$ be the $H^s_{p,\infty}$ closure of H^s_∞. Since $H^s_\infty \subset H^s_2 = L_2$, using Remark 3.1(1), we have $\overline{H}^s_{p,\infty} \subset \mathcal{H}^s_{p,\infty}$. Then $(\mathcal{H}^s_{p,\infty})^* \subset (\overline{H}^s_{p,\infty})^*$.

LEMMA 4.1 ([17], Corollary 6). *Let* $0 < p < 2$. *Then the dual space of* $\overline{H}^s_{p,\infty}$ *is* $w\mathcal{BMO}_2(\alpha)$ *with* $\alpha = 1/p - 1$.

LEMMA 4.2 ([17], Corollary 8). *Suppose that the stochastic basis* $\{\mathcal{F}_n\}_{n\geqslant 0}$ *is regular and* $1 \leqslant r < \infty$. *If* $\alpha r + 1 > 0$ *for a fixed* α, *then*

$$w\mathcal{BMO}_r(\alpha) = w\mathcal{BMO}_2(\alpha)$$

with equivalent norms.

THEOREM 4.1. *Suppose that* $\alpha \geqslant 0$. *Then*

$$w\mathcal{BMO}_2(\alpha) = wBMO_2(\alpha)$$

with equivalent norms.

Proof. Let $p = \frac{1}{1+\alpha}$. Since $(\mathcal{H}^s_{p,\infty})^* \subset (\overline{H}^s_{p,\infty})^*$, it follows from Proposition 3.1 and Lemma 4.1 that

$$wBMO_2(\alpha) \subset w\mathcal{BMO}_2(\alpha).$$

To obtain

$$wBMO_2(\alpha) \supset w\mathcal{BMO}_2(\alpha),$$

we shall show that

$$C\|f\|_{w\mathcal{BMO}_2(\alpha)} \geqslant \|f\|_{wBMO_2(\alpha)}$$

for any $f \in w\mathcal{BMO}_2(\alpha)$. Suppose that $\{\nu_k\}_{k\in\mathbb{Z}}$ is an arbitrary stopping time sequence such that $\{2^k\mathbb{P}(\nu_k < \infty)^{1/p}\}_{k\in\mathbb{Z}} \in \ell_\infty$. Let

$$B = \sup_k 2^k\mathbb{P}(\nu_k < \infty)^{1/p}.$$

We can claim that

$$\sum_{k=-\infty}^{\infty} t_\alpha^2(B^p 2^{-kp}) \leqslant C\|f\|_{w\mathcal{BMO}_2(\alpha)}.$$

To this end, let $C_k = B2^{-kp}$. Then, for any $x \in (C_{k+1}, C_k)$, we have

$$C_{k+1}^{1/2+\alpha} t_\alpha^2(C_{k+1}) \leqslant x^{1/2+\alpha} t_\alpha^2(x) \leqslant C_k^{1/2+\alpha} t_\alpha^2(C_k).$$

We refer to [17], p. 144, for a more general case of the inequalities above. Hence,

$$\int_0^\infty \frac{t_\alpha^2(x)}{x}dx = \sum_{k=-\infty}^{\infty} \int_{C_{k+1}}^{C_k} \frac{t_\alpha^2(x)}{x}dx \geqslant (1-2^{-p})2^{-p(1/2+\alpha)} \sum_{k=-\infty}^{\infty} t_\alpha^2(B^p 2^{-kp}).$$

On the other hand, since $B^p 2^{-kp} \geqslant \mathbb{P}(\nu_k < \infty)$ for all k, we have

$$\begin{aligned}\sum_{k=-\infty}^{\infty} t_\alpha^2(B^p 2^{-kp}) &\geqslant \sum_{k=-\infty}^{\infty} \frac{2^k(B^p 2^{-kp})^{1/2}\|f - f^{\nu_k}\|_2}{B} \\ &\geqslant \sum_{k=-\infty}^{\infty} \frac{2^k\mathbb{P}(\nu_k < \infty)^{1/2}\|f - f^{\nu_k}\|_2}{B}.\end{aligned}$$

By the definition of $wBMO_2(\alpha)$, we complete the proof. ■

REMARK 4.1. *If one proves the dual space of* $\mathcal{H}_{p,\infty}^s$ *is* $w\mathcal{BMO}(\alpha)$, *then Theorem* 4.1 *holds. If one shows* $\mathcal{H}_{p,\infty}^s = \overline{H}_{p,\infty}^s$, *then Proposition* 3.1 *implies Theorem* 4.1. *We leave the proofs to the interested reader.*

Now we are ready to prove the main result of the paper.

Proof of Theorem 1.1. It directly follows from Theorems 3.1 and 4.1 and Lemma 4.2. ■

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