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TANAKA FORMULA FOR STRICTLY STABLE PROCESSES

BY

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Abstract. For symmetric Lévy processes, if the local times exist, the Tanaka formula has already been constructed via the techniques in the potential theory by Salminen and Yor (2007). In this paper, we study the Tanaka formula for arbitrary strictly stable processes with index $\alpha \in (1, 2)$, including spectrally positive and negative cases in a framework of Itô's stochastic calculus. Our approach to the existence of local times for such processes is different from that of Bertoin (1996).

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1. INTRODUCTION

First, we begin with the definition of a local time for a real-valued stochastic process $X = (X_t)_{t \geqslant 0}$.

DEFINITION 1.1. A family of random variables $L = \{L_t^a : a \in \mathbb{R}, t \geqslant 0\}$ is a *local time* of X if, for any bounded Borel measurable function $f : \mathbb{R} \to [0, \infty)$ and $t \geqslant 0$,

$$\int_0^t f(X_s) ds = \int_{\mathbb{R}} f(a) L_t^a da \text{ a.s.},$$

which is called an *occupation time formula*.

It is known that there exist local times of Brownian motions (cf. Berman [3]), and necessary and sufficient conditions of the existence for Lévy processes can be found in Bertoin [4]. They considered the local time as the Radon–Nikodym derivative of the occupation time measure μ_t defined by

$$\int_0^t f(X_s) ds = \int_{\mathbb{R}} f(a) \mu_t(da)$$

for each Borel measurable function $f : \mathbb{R} \to [0, \infty)$ and $t \geqslant 0$. The existence of almost surely jointly continuous local times was studied by Trotter [13] for Brownian motions, and by Boylan [5] for stable processes with index $\alpha \in (1,2)$, and necessary and sufficient conditions for the almost sure joint continuity of local times were given by Barlow [2].

Let $B = (B_t)_{t \geqslant 0}$ be a Brownian motion on $\mathbb{R}$. It is well known that the Tanaka formula for a Brownian motion holds, that is

$$|B_t - a| - |B_0 - a| = \int_0^t \operatorname{sgn}(B_s - a) dB_s + L_t^a,$$

where L^a denotes the local time of the Brownian motion B at level a. For more details, see, for example, Chung and Williams [6]. By Lévy's characterization, $\int_0^{\cdot} \operatorname{sgn}(B_s - a) dB_s$ is another Brownian motion. By setting $a = 0$, we know the process $|B|$ is the reflection of the Brownian motion, which means that $(|B|, L^0)$ is the solution of the Skorokhod problem for the Brownian motion. By the Doob–Meyer decomposition, the local time L^a can be understood as the unique bounded variation process which is the difference of the positive submartingale $|B - a|$ and the martingale $\int_0^{\cdot} \operatorname{sgn}(B_s - a) dB_s + |B_0 - a|$. Thus, in this paper we say that the *Tanaka formula holds* if the local time can be expressed as the difference of such processes because it has a gap between the Skorokhod solution and local times in the case of jump processes. The Tanaka formula has already been studied by Yamada [15] for symmetric stable processes with index $\alpha \in (1,2)$, and by Salminen and Yor [10] for symmetric Lévy processes when local times exist. On the other hand, Watanabe [14], Tanaka [12] and Engelbert et al. [7] focused on the solution of the Skorokhod problem.

In [10], they constructed the Tanaka formula for a symmetric Lévy process X by using the continuous resolvent density u^p:

$$v(X_t - a) = v(a) + M_t^a + L_t^a,$$

where $v(x) := \lim_{p \to 0} \{u^p(0) - u^p(x)\}$ which is called a *renormalized zero resolvent*, and M^a is a martingale. But the existence of the limit is not clear in the asymmetric case and the representation for the martingale part is not given. In [16], Yano obtained a renormalized zero resolvent for an asymmetric Lévy process under some conditions, and a harmonic function for the killed process upon hitting zero is associated with the Tanaka formula of the local time at level zero because the local time of such processes at level zero becomes zero.

In this paper, we shall show the Tanaka formula for arbitrary strictly stable processes, including spectrally positive and negative cases with index $\alpha \in (1,2)$, using Itô's stochastic calculus. Although our formula might give an insight in a reflection problem of jump type processes, the martingale part in the formula is not suitable to become the same law of the original process. Thus, there exists a gap between the Skorokhod solution and the existence of local times in case of jump processes.

In Section 2, we shall prepare the Itô formula, the operator associated with the remainder term of the Itô formula and the moments about stable processes. In Section 3, we will construct the Tanaka formula for strictly stable processes.

2. PRELIMINARIES

Let $C^n_{1+,b}(\mathbb{R})$ be the elements of $C^n(\mathbb{R})$ such that all derivatives of any order $k = 1, \ldots, n$ are bounded, and $C^n_c(\mathbb{R})$ be the functions in $C^n(\mathbb{R})$ having compact support. Let $\mathcal{S}(\mathbb{R})$ be the Schwartz space of rapidly decreasing functions on $\mathbb{R}$.

A Lévy process $Y = (Y)_{t \geqslant 0}$ is characterized by the triplet (b, a, ν), where b is the drift parameter, a is the variance of the Gaussian part of Y_1 and ν is the Lévy measure on $\mathbb{R} \setminus \{0\}$ satisfying the following integrability condition:

$$\int_{\mathbb{R}\setminus\{0\}} (|h|^2 \wedge 1)\nu(dh) < \infty.$$

Note that in the following we always use the standard truncation function χ defined by $\chi(h) = \mathbf{1}_{\{|h| \leqslant 1\}}$ for $h \in \mathbb{R}$ (see Sato [11], Remark 8.4).

In this paper, we are concerned with a one-dimensional strictly stable process $X = (X_t)_{t \geqslant 0}$ with index $\alpha \in (1, 2)$. This process X is characterized by the triplet $(b_\alpha, 0, \nu_\alpha)$, where the Lévy measure ν_α is given by

$$\nu_\alpha(dh) = \begin{cases} c_+ |h|^{-\alpha-1} dh & \text{on } (0, \infty), \\ c_- |h|^{-\alpha-1} dh & \text{on } (-\infty, 0), \end{cases}$$

c_+ and c_- being non-negative constants such that $c_+ + c_- > 0$, and the drift parameter b_α is given by

$$b_\alpha = -\int_{|h|>1} h \nu_\alpha(dh) = -\frac{c_+ - c_-}{\alpha - 1}.$$

The Lévy–Khintchine representation for X is given as

$$\mathbb{E}[e^{iuX_t}] = e^{t\eta(u)}, \quad u \in \mathbb{R},$$

where the function η is the Lévy symbol of X defined as

$$\eta(u) = -d|u|^\alpha \left\{1 - i\beta \operatorname{sgn}(u) \tan\left(\frac{\pi\alpha}{2}\right)\right\},$$

$d > 0$ and $\beta \in [-1, 1]$ being given by

$$d = \frac{c_+ + c_-}{2c(\alpha)}, \qquad \beta = \frac{c_+ - c_-}{c_+ + c_-}$$

with

$$c(\alpha) = \frac{1}{\pi}\Gamma(\alpha+1)\sin\left(\frac{\pi\alpha}{2}\right)$$

and the left-continuous signum function determined as

$$\operatorname{sgn}(x) := \begin{cases} 1 & \text{if } x > 0, \\ -1 & \text{if } x \leqslant 0. \end{cases}$$

Throughout this paper, we assume that X is *càdlàg*. By the Lévy–Itô decomposition (see Applebaum [1], Theorem 2.4.16), X can be represented by

$$\begin{aligned} X_t &= X_0 - t \int_{|h|>1} h\nu_\alpha(dh) + \int_0^t \int_{|h|\leqslant 1} h\tilde{N}(ds,dh) + \int_0^t \int_{|h|>1} hN(ds,dh) \\ &= X_0 + \int_0^t \int_{|h|\leqslant 1} h\tilde{N}(ds,dh) + \int_0^t \int_{|h|>1} h\tilde{N}(ds,dh), \end{aligned} \tag{2.1}$$

where N is the jump measure of X which is a Poisson random measure with the intensity measure $ds\nu_\alpha(dh)$ and the compensated measure $\tilde{N}$ is defined by

$$\tilde{N}(ds,dh) := N(ds,dh) - ds\nu_\alpha(dh).$$

REMARK 2.1. *The integral* $\int_0^t \int_{|h|>1} h\tilde{N}(ds,dh)$ *is represented by*

$$\int_0^t \int_{|h|>1} h\tilde{N}(ds,dh) = \int_0^t \int_{|h|>1} hN(ds,dh) - \int_0^t \int_{|h|>1} h\nu_\alpha(dh)ds,$$

since we have

$$\int_0^t \int_{|h|>1} |h|\nu_\alpha(dh)ds = \frac{t(c_+ + c_-)}{\alpha - 1} < \infty.$$

Applying the Itô formula ([1], Theorem 4.4.7) to (2.1), we have the formula for strictly stable processes with index $\alpha \in (1,2)$.

PROPOSITION 2.1. *For each* $f \in C^2(\mathbb{R})$, *the following formula holds*:

$$\begin{aligned} f(X_t) - f(X_0) = &- \int_0^t \int_{|h|>1} f'(X_s)h\nu_\alpha(dh)ds \\ &+ \int_0^t \int_{|h|>1} \{f(X_{s-}+h) - f(X_{s-})\}N(ds,dh) \\ &+ \int_0^t \int_{|h|\leqslant 1} \{f(X_{s-}+h) - f(X_{s-})\}\tilde{N}(ds,dh) \\ &+ \int_0^t \int_{|h|\leqslant 1} \{f(X_s+h) - f(X_s) - f'(X_s)h\}\nu_\alpha(dh)ds. \end{aligned}$$

Then, we can rewrite Proposition 2.1 as follows:

PROPOSITION 2.2. *For each $f \in C^2_{1+,b}(\mathbb{R})$, the following formula holds*:

$$\begin{aligned} f(X_t) - f(X_0) = &\int_0^t \int_{|h|\leqslant 1} \{f(X_{s-}+h) - f(X_{s-})\}\tilde{N}(ds,dh) \\ &+ \int_0^t \int_{|h|>1} \{f(X_{s-}+h) - f(X_{s-})\}\tilde{N}(ds,dh) \\ &+ \int_0^t \int_{\mathbb{R}\setminus\{0\}} \{f(X_s+h) - f(X_s) - f'(X_s)h\}\nu_\alpha(dh)ds. \end{aligned}$$

P r o o f. It is sufficient to show that

$$\mathbb{E}\Big[\int_0^t \int_{|h|>1} |f(X_s+h) - f(X_s)|\, \nu_\alpha(dh)ds\Big] < \infty.$$

Since $f \in C^2_{1+,b}(\mathbb{R})$, it follows from the mean value theorem that

$$\begin{aligned} &\mathbb{E}\Big[\int_0^t \int_{|h|>1} |f(X_s+h) - f(X_s)|\nu_\alpha(dh)ds\Big] \\ &\qquad = \mathbb{E}\Big[\int_0^t \int_{|h|>1} \Big|\int_0^1 f'(X_s+\theta h)hd\theta\Big|\nu_\alpha(dh)ds\Big] \\ &\qquad \leqslant Kt \int_{|h|>1} |h|\nu_\alpha(dh) = \frac{Kt(c_+ + c_-)}{\alpha - 1} < \infty, \end{aligned}$$

where K is a positive constant such that $|f'(x)| \leqslant K$ for all $x \in \mathbb{R}$. ■

REMARK 2.2. *The integral $\int_0^t \int_{|h|>1}\{f(X_{s-}+h) - f(X_{s-})\}\tilde{N}(ds,dh)$ is represented by*

$$\begin{aligned} &\int_0^t \int_{|h|>1} \{f(X_{s-}+h) - f(X_{s-})\}\tilde{N}(ds,dh) \\ &\qquad = \int_0^t \int_{|h|>1} \{f(X_{s-}+h) - f(X_{s-})\}N(ds,dh) \\ &\qquad\quad - \int_0^t \int_{|h|>1} \{f(X_s+h) - f(X_s)\}\nu_\alpha(dh)ds, \end{aligned}$$

and then this integral is a martingale (see Ikeda and Watanabe [8], pp. 61–63).

REMARK 2.3. *Since $f \in C^2_{1+,b}(\mathbb{R})$, it follows from the mean value theorem that*

$$\begin{aligned}\mathbb{E}\Big[\int_0^t \int_{|h|\leqslant 1} |f(X_s+h)-f(X_s)|^2 \nu_\alpha(dh)ds\Big] \\ &= \mathbb{E}\Big[\int_0^t \int_{|h|\leqslant 1} \Big|\int_0^1 f'(X_s+\theta h)hd\theta\Big|^2 \nu_\alpha(dh)ds\Big] \\ &\leqslant K^2 t \int_{|h|\leqslant 1} |h|^2 \nu_\alpha(dh) = \frac{K^2 t(c_+ + c_-)}{2-\alpha} < \infty,\end{aligned}$$

where K is a positive constant such that $|f'(x)| \leqslant K$ for all $x \in \mathbb{R}$. Thus, the integral $\int_0^t \int_{|h|\leqslant 1} \{f(X_{s-}+h)-f(X_{s-})\}\tilde{N}(ds,dh)$ with respect to the compensated Poisson random measure is a square integrable martingale (see [1], Theorem 4.2.3, or [8], pp. 61–63).

We consider the operator $\mathcal{L}$ defined by

$$\mathcal{L}f(x) := \int_{\mathbb{R}\setminus\{0\}} \{f(x+h)-f(x)-f'(x)h\}\nu_\alpha(dh)$$

for all $f \in C^2_{1+,b}(\mathbb{R})$ and $x \in \mathbb{R}$. Using the operator $\mathcal{L}$, we can rewrite the Itô formula (Proposition 2.2) as follows:

$$\begin{aligned}f(X_t)-f(X_0) &= \int_0^t \int_{|h|\leqslant 1} \{f(X_{s-}+h)-f(X_{s-})\}\tilde{N}(ds,dh) \\ &\quad+ \int_0^t \int_{|h|>1} \{f(X_{s-}+h)-f(X_{s-})\}\tilde{N}(ds,dh) \\ &\quad+ \int_0^t \mathcal{L}f(X_s)ds\end{aligned} \tag{2.2}$$

for each $f \in C^2_{1+,b}(\mathbb{R})$.

Using the Fourier transform of $f \in L^1(\mathbb{R})$ defined by

$$\mathcal{F}[f](u) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-iux} f(x)dx \quad \text{for } u \in \mathbb{R},$$

and the inverse Fourier transform of $f \in L^1(\mathbb{R})$ defined by

$$\mathcal{F}^{-1}[f](x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{iux} f(u)du \quad \text{for } x \in \mathbb{R},$$

we obtain also the following representation of the operator $\mathcal{L}$ on $\mathcal{S}(\mathbb{R})$:

PROPOSITION 2.3 ([1], Theorem 3.3.3). *For each* $f \in \mathcal{S}(\mathbb{R})$ *and* $x \in \mathbb{R}$,

$$\mathcal{L}f(x) = \mathcal{F}^{-1}\big[\eta(u)\mathcal{F}[f](u)\big](x).$$

We will use the following moments of stable processes.

LEMMA 2.1 ([11], Example 25.10). *Let* Z *be a stable process of index* $\alpha \in (0,2)$ *with the triplet* $(b,0,\nu_\alpha)$. *Then, for all* $t \geqslant 0$,

$$\mathbb{E}[|Z_t|^\gamma] < \infty \quad \text{if } 0 < \gamma < \alpha,$$
$$\mathbb{E}[|Z_t|^\gamma] = \infty \quad \text{if } \gamma \geqslant \alpha.$$

REMARK 2.4. *Since* $\mathbb{E}[|X_t|] < \infty$, *it follows from* (2.1) *that* X *is a martingale.*

Moreover, we will use the following negative-order moments.

LEMMA 2.2. *Let* Z *be a stable process of index* $\alpha \in (0,2)$ *with the triplet* $(b,0,\nu_\alpha)$. *Then, for all* $t > 0$ *and* $x \in \mathbb{R}$,

$$\mathbb{E}[|Z_t - x|^{-\gamma}] \leqslant S(\alpha,\gamma)t^{-\gamma/\alpha} \quad \text{if } 0 < \gamma < 1,$$

where the constant $S(\alpha,\gamma)$ *does not depend on* x.

Proof. By the monotone convergence theorem, we have

$$\lim_{\varepsilon \downarrow 0} \mathbb{E}[|Z_t - x|^{-\gamma} e^{-\varepsilon|Z_t - x|}] = \mathbb{E}[|Z_t - x|^{-\gamma}]. \tag{2.3}$$

Since $|e^{t\eta(\cdot)}| = e^{-dt|\cdot|^\alpha} \in L^1(\mathbb{R})$ for $t > 0$, the transition density p_t is given by

$$p_t(y) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-iuy}e^{t\eta(u)}du$$

for each $t > 0$ and $y \in \mathbb{R}$. Since $|\cdot|^{-\gamma}e^{-\varepsilon|\cdot|} \in L^1(\mathbb{R})$ for $\varepsilon > 0$, it follows from Fubini's theorem that for all $\varepsilon > 0$,

$$\begin{aligned}\mathbb{E}[|Z_t - x|^{-\gamma}e^{-\varepsilon|Z_t - x|}] &= \int_{\mathbb{R}} |y - x|^{-\gamma}e^{-\varepsilon|y-x|}p_t(y)dy \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{t\eta(u)}\int_{\mathbb{R}} |y-x|^{-\gamma}e^{-\varepsilon|y-x|}e^{-iuy}dydu \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{t\eta(u)-iux}\int_{\mathbb{R}} |z|^{-\gamma}e^{-\varepsilon|z|-iuz}dzdu,\end{aligned}$$

by putting $z = y - x$.

Now, we will make use of the identity

$$\int_0^\infty z^{\xi-1}e^{-xz}dz = \Gamma(\xi)x^{-\xi} \tag{2.4}$$

for all $\xi > 0$ and $\Re x > 0$, where $\Re x$ is the real part of x. By the relations $1-\gamma > 0$ and $\Re(\varepsilon \pm iu) = \varepsilon > 0$, we have

$$\begin{aligned}\int_{\mathbb{R}} |z|^{-\gamma}e^{-\varepsilon|z|-iuz}dz &= \int_0^\infty z^{-\gamma}e^{-(\varepsilon+iu)z}dz + \int_0^\infty z^{-\gamma}e^{-(\varepsilon-iu)z}dz\\ &= \Gamma(1-\gamma)(\varepsilon+iu)^{\gamma-1} + \Gamma(1-\gamma)(\varepsilon-iu)^{\gamma-1}.\end{aligned}$$

We then have for $u \neq 0$,

$$\begin{aligned}\lim_{\varepsilon\downarrow 0}\{(\varepsilon+iu)^{\gamma-1} + (\varepsilon-iu)^{\gamma-1}\} &= (iu)^{\gamma-1} + (-iu)^{\gamma-1}\\ &= 2|u|^{\gamma-1}\cos\left(\frac{\pi(\gamma-1)}{2}\right).\end{aligned}$$

By $-1 < \gamma - 1 < 0$ and $0 < \alpha < 2$, we have for $t > 0$,

$$|\varepsilon \pm iu|^{\gamma-1}|e^{t\eta(u)-iux}| \leqslant |u|^{\gamma-1}e^{-dt|u|^\alpha} \in L^1(\mathbb{R}).$$

Hence, it follows from the dominated convergence theorem that, for all $t > 0$ and $x \in \mathbb{R}$,

$$\begin{aligned}&\lim_{\varepsilon\downarrow 0}\mathbb{E}[|Z_t - x|^{-\gamma}e^{-\varepsilon|Z_t-x|}]\\ &\quad= \frac{\Gamma(1-\gamma)}{2\pi}\lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}}\{(\varepsilon+iu)^{\gamma-1} + (\varepsilon-iu)^{\gamma-1}\}e^{t\eta(u)-iux}du\\ &\quad= \frac{\Gamma(1-\gamma)}{\pi}\cos\left(\frac{\pi(\gamma-1)}{2}\right)\int_{\mathbb{R}}|u|^{\gamma-1}e^{t\eta(u)-iux}du\\ &\quad\leqslant \frac{\Gamma(1-\gamma)}{\pi}\cos\left(\frac{\pi(\gamma-1)}{2}\right)\int_{\mathbb{R}}|u|^{\gamma-1}e^{-dt|u|^\alpha}du\\ &\quad= \frac{\Gamma(1-\gamma)t^{-\gamma/\alpha}}{\pi}\cos\left(\frac{\pi(\gamma-1)}{2}\right)\int_{\mathbb{R}}|v|^{\gamma-1}e^{-d|v|^\alpha}dv < \infty,\end{aligned}$$

by putting $v = t^{1/\alpha}u$, and the required result follows from (2.3). ■

3. MAIN RESULTS

Before we establish the Tanaka formula for general strictly stable processes, we need the following lemma. In case of $\beta = 0$, that is, if X is a symmetric stable process with index $\alpha \in (1, 2)$, the following result is shown in Komatsu [9]:

LEMMA 3.1. *Let*

$$F(x) := D(\alpha)\{1 - \beta \operatorname{sgn}(x)\}|x|^{\alpha-1},$$

where

$$D(\alpha) = \frac{c(-\alpha)}{d\,\{1 + \beta^2 \tan^2(\pi\alpha/2)\}}.$$

Then, we have for all $\phi \in C_c^\infty(\mathbb{R})$ *and* $x \in \mathbb{R}$,

$$\mathcal{L}(F * \phi)(x) = \phi(x),$$

where $F * \phi$ *is given by the convolution of* F *and* ϕ:

$$(F * \phi)(x) := \int_{\mathbb{R}} F(y)\phi(x - y)dy.$$

Proof. Let $F_\phi := F * \phi$ for $\phi \in C_c^\infty(\mathbb{R})$. We know that $F_\phi \in C^\infty(\mathbb{R})$. By using integration by parts, we have for all $n \in \mathbb{N}$,

$$F_\phi^{(n)}(x) = \int_{\mathbb{R}} F(y)\phi^{(n)}(x - y)dy = \int_{\mathbb{R}} F'(y)\phi^{(n-1)}(x - y)dy,$$

where the weak derivative F' is given by

$$F'(y) = (\alpha - 1)D(\alpha)\{\operatorname{sgn}(y) - \beta\}|y|^{\alpha-2} \quad \text{for } y \in \mathbb{R} \setminus \{0\},$$

with $F'(0) = 0$. Since for $n \in \mathbb{N}$,

$$\begin{aligned}\sup_{x\in\mathbb{R}} |F_\phi^{(n)}(x)| &= \sup_{x\in\mathbb{R}} \Big|\int_{\mathbb{R}} F'(y)\phi^{(n-1)}(x - y)dy\Big| \\ &\leqslant 2D(\alpha) \sup_{x\in\mathbb{R}} \int_{\mathbb{R}} |y|^{\alpha-2}|\phi^{(n-1)}(x - y)|dy < \infty,\end{aligned}$$

we have $F_\phi \in C_{1+,b}^\infty(\mathbb{R})$.

Set $F_\varepsilon(x) := F(x)e^{-\varepsilon|x|}$ for $\varepsilon > 0$ and $F_{\varepsilon,\phi} := F_\varepsilon * \phi \in C^\infty(\mathbb{R})$. By using the inequality

$$1 + |x|^2 \leqslant 2(1 + |x - y|^2)(1 + |y|^2) \quad \text{for all } x, y \in \mathbb{R},$$

we have for all $k, n \in \mathbb{N}$,

$$\begin{aligned}&\sup_{x\in\mathbb{R}} (1 + |x|^2)^k |F_{\varepsilon,\phi}^{(n)}(x)| \\ &\leqslant 2^k \sup_{x\in\mathbb{R}} \int_{\mathbb{R}} (1 + |x - y|^2)^k (1 + |y|^2)^k |F_\varepsilon(y)\phi^{(n)}(x - y)|dy \\ &\leqslant 2^{k+1} D(\alpha) \sup_{x\in\mathbb{R}} (1 + |x|^2)^k |\phi^{(n)}(x)| \int_{\mathbb{R}} (1 + |y|^2)^k |y|^{\alpha-1} e^{-\varepsilon|y|} dy < \infty,\end{aligned}$$

since $\phi \in C_c^\infty(\mathbb{R}) \subset \mathcal{S}(\mathbb{R})$. Thus, we have $F_{\varepsilon,\phi} \in \mathcal{S}(\mathbb{R})$. By Proposition 2.3, we obtain

$$\begin{aligned}(3.1)\quad & \mathcal{L}F_{\varepsilon,\phi}(x)\\ &= \int_{\mathbb{R}\setminus\{0\}} \{F_{\varepsilon,\phi}(x+h) - F_{\varepsilon,\phi}(x) - F'_{\varepsilon,\phi}(x)h\}\nu_\alpha(dh)\\ &= \mathcal{F}^{-1}\big[\eta(u)\mathcal{F}[F_{\varepsilon,\phi}](u)\big](x) = \sqrt{2\pi}\mathcal{F}^{-1}\big[\eta(u)\mathcal{F}[F_\varepsilon](u)\mathcal{F}[\phi](u)\big](x).\end{aligned}$$

Since $|x|e^{-|x|} \leqslant 1$ for all $x \in \mathbb{R}$, we have for $x \neq 0$,

$$\begin{aligned}|F'_\varepsilon(x)| &= \big|D(\alpha)\{\mathrm{sgn}(x) - \beta\}\{(\alpha-1)|x|^{\alpha-2}e^{-\varepsilon|x|} - \varepsilon|x|^{\alpha-1}e^{-\varepsilon|x|}\}\big|\\ &\leqslant 2D(\alpha)|x|^{\alpha-2}e^{-\varepsilon|x|} + 2\varepsilon D(\alpha)|x|^{\alpha-1}e^{-\varepsilon|x|} \leqslant 4D(\alpha)|x|^{\alpha-2}.\end{aligned}$$

By using integration by parts, it follows from Fubini's theorem that

$$\begin{aligned}&|F_{\varepsilon,\phi}(x+h) - F_{\varepsilon,\phi}(x) - F'_{\varepsilon,\phi}(x)h|\\ &\qquad= \Big|\int_{\mathbb{R}} F_\varepsilon(y)\{\phi(x+h-y) - \phi(x-y) - \phi'(x-y)h\}dy\Big|\\ &\qquad= \Big|\int_{\mathbb{R}} F'_\varepsilon(y)h\int_0^1\{\phi(x+\theta h-y) - \phi(x-y)\}d\theta dy\Big|\\ &\qquad= \Big|\int_0^1\int_{\mathbb{R}} F'_\varepsilon(y)h\{\phi(x+\theta h-y) - \phi(x-y)\}dyd\theta\Big|\\ &\qquad\leqslant C_1(\alpha)|h|,\end{aligned}$$

where $C_1(\alpha)$ is a constant given by

$$C_1(\alpha) = 8D(\alpha)\sup_{x\in\mathbb{R}}\int_{\mathbb{R}} |y|^{\alpha-2}|\phi(x-y)|dy.$$

Similarly, by using integration by parts, it follows from Fubini's theorem that

$$\begin{aligned}&|F_{\varepsilon,\phi}(x+h) - F_{\varepsilon,\phi}(x) - F'_{\varepsilon,\phi}(x)h|\\ &\qquad= \Big|\int_{\mathbb{R}} F'_\varepsilon(y)h^2\int_0^1 \phi'(x+\theta h-y)(1-\theta)d\theta dy\Big|\\ &\qquad= \Big|\int_0^1\int_{\mathbb{R}} F'_\varepsilon(y)h^2\phi'(x+\theta h-y)(1-\theta)dyd\theta\Big|\\ &\qquad\leqslant C_2(\alpha)|h|^2,\end{aligned}$$

where $C_2(\alpha)$ is a constant given by

$$C_2(\alpha) = 2D(\alpha)\sup_{x\in\mathbb{R}}\int_{\mathbb{R}} |y|^{\alpha-2}|\phi'(x-y)|dy.$$

By $1 < \alpha < 2$, we know that

$$\int_{\mathbb{R}\setminus\{0\}} (|h|^2 \wedge |h|)\nu_\alpha(dh) = \frac{c_+ + c_-}{2-\alpha} + \frac{c_+ + c_-}{\alpha - 1} < \infty.$$

Hence, it follows from the dominated convergence theorem that

$$\begin{aligned}(3.2)\quad \lim_{\varepsilon\downarrow 0} \mathcal{L}F_{\varepsilon,\phi}(x) &= \lim_{\varepsilon\downarrow 0} \int_{\mathbb{R}\setminus\{0\}} \{F_{\varepsilon,\phi}(x+h) - F_{\varepsilon,\phi}(x) - F'_{\varepsilon,\phi}(x)h\}\nu_\alpha(dh) \\ &= \int_{\mathbb{R}\setminus\{0\}} \{F_\phi(x+h) - F_\phi(x) - F'_\phi(x)h\}\nu_\alpha(dh) = \mathcal{L}F_\phi(x).\end{aligned}$$

Now, by using (2.4), we have

$$\mathcal{F}[|x|^{\alpha-1}e^{-\varepsilon|x|}](u) = \frac{\Gamma(\alpha)}{\sqrt{2\pi}}\{(\varepsilon + iu)^{-\alpha} + (\varepsilon - iu)^{-\alpha}\},$$

and

$$\mathcal{F}[|x|^{\alpha-1}\mathrm{sgn}(x)e^{-\varepsilon|x|}](u) = \frac{\Gamma(\alpha)}{\sqrt{2\pi}}\{(\varepsilon + iu)^{-\alpha} - (\varepsilon - iu)^{-\alpha}\}.$$

We then have for $u \neq 0$,

$$\begin{aligned}\lim_{\varepsilon\downarrow 0} \mathcal{F}[|x|^{\alpha-1}e^{-\varepsilon|x|}](u) &= \frac{\Gamma(\alpha)}{\sqrt{2\pi}}|u|^{-\alpha}\{i^{-\alpha} + (-i)^{-\alpha}\} \\ &= \frac{2\Gamma(\alpha)}{\sqrt{2\pi}}|u|^{-\alpha}\cos\left(\frac{\pi\alpha}{2}\right),\end{aligned}$$

and

$$\begin{aligned}\lim_{\varepsilon\downarrow 0} \mathcal{F}[|x|^{\alpha-1}e^{-\varepsilon|x|}\mathrm{sgn}(x)](u) &= \frac{\Gamma(\alpha)}{\sqrt{2\pi}}|u|^{-\alpha}\mathrm{sgn}(u)\{i^{-\alpha} - (-i)^{-\alpha}\} \\ &= -i\frac{2\Gamma(\alpha)}{\sqrt{2\pi}}|u|^{-\alpha}\mathrm{sgn}(u)\sin\left(\frac{\pi\alpha}{2}\right).\end{aligned}$$

Thus, we have for $u \neq 0$,

$$\begin{aligned}(3.3)\quad & \lim_{\varepsilon\downarrow 0} \eta(u)\mathcal{F}[F_\varepsilon](u) \\ &= \left(-c(-\alpha)\frac{d|u|^\alpha\{1 - i\beta\mathrm{sgn}(u)\tan(\pi\alpha/2)\}}{d\{1+\beta^2\tan^2(\pi\alpha/2)\}}\right) \\ &\quad \times \lim_{\varepsilon\downarrow 0}\{\mathcal{F}[|x|^{\alpha-1}e^{-\varepsilon|x|}](u) - \beta\mathcal{F}[|x|^{\alpha-1}e^{-\varepsilon|x|}\mathrm{sgn}(x)](u)\} \\ &= \left\{-\frac{c(-\alpha)|u|^\alpha}{1 + i\beta\mathrm{sgn}(u)\tan(\pi\alpha/2)}\right\} \\ &\quad \times \frac{2\Gamma(\alpha)}{\sqrt{2\pi}}|u|^{-\alpha}\left\{\cos\left(\frac{\pi\alpha}{2}\right) + i\beta\mathrm{sgn}(u)\sin\left(\frac{\pi\alpha}{2}\right)\right\} \\ &= -\frac{2\Gamma(\alpha)c(-\alpha)}{\sqrt{2\pi}}\cos\left(\frac{\pi\alpha}{2}\right) = \frac{1}{\sqrt{2\pi}}.\end{aligned}$$

By $\phi \in C_c^\infty(\mathbb{R}) \subset \mathcal{S}(\mathbb{R})$, we have

$$\left|\frac{u}{\varepsilon \pm iu}\right|^\alpha \mathcal{F}[\phi](u) \leqslant \mathcal{F}[\phi](u) \in \mathcal{S}(\mathbb{R}) \subset L^1(\mathbb{R}).$$

Hence, it follows from (3.1), the dominated convergence theorem and (3.3) that

$$\begin{aligned}\lim_{\varepsilon\downarrow 0} \mathcal{L}F_{\varepsilon,\phi}(x) &= \lim_{\varepsilon\downarrow 0} \sqrt{2\pi}\mathcal{F}^{-1}\big[\eta(u)\mathcal{F}[F_\varepsilon](u)\mathcal{F}[\phi](u)\big](x)\\ &= \sqrt{2\pi}\mathcal{F}^{-1}\big[\lim_{\varepsilon\downarrow 0}\eta(u)\mathcal{F}[F_\varepsilon](u)\mathcal{F}[\phi](u)\big](x)\\ &= \mathcal{F}^{-1}\big[\mathcal{F}[\phi](u)\big](x) = \phi(x),\end{aligned}$$

and the required result follows from (3.2). ∎

We introduce a mollifier as follows:

DEFINITION 3.1. A function ρ on $\mathbb{R}$ is a *mollifier* if ρ satisfies the following requirements:

(i) $\rho(x) \geqslant 0$ for all $x \in \mathbb{R}$; (ii) $\rho \in C_c^\infty(\mathbb{R})$;

(iii) $\operatorname{supp}\rho = [-1, 1]$; (iv) $\int_{\mathbb{R}} \rho(x)dx = 1$.

Furthermore, we define $\rho_n(x) := n\rho(nx)$ for all $n \in \mathbb{N}$. The family of functions $(\rho_n)_{n\in\mathbb{N}}$ has the property that $\rho_n \to \delta_0$ as $n \to \infty$, where δ_0 is the Dirac distribution at zero, in the sense of Schwartz distributions, that means

$$\Big|\int_{\mathbb{R}} \rho_n(x)\phi(x)dx - \phi(0)\Big| \to 0 \quad \text{as } n \to \infty$$

for all $\phi \in \mathcal{S}(\mathbb{R})$.

Now let us state our main theorem which we call the *Tanaka formula* for arbitrary strictly stable processes with index $\alpha \in (1, 2)$.

THEOREM 3.1. *Let F be the same as in Lemma* 3.1. *Then, for all $a \in \mathbb{R}$ and $t \geqslant 0$, we have*

$$F(X_t - a) - F(X_0 - a) = M_t^a + L_t^a,$$

where the process M^a given by

$$M_t^a = \int_0^t \int_{\mathbb{R}\setminus\{0\}} \{F(X_{s-} - a + h) - F(X_{s-} - a)\}\tilde{N}(ds, dh)$$

is a square integrable martingale and the process L^a is the local time of X at a.

P r o o f. Let $F_n := F * \rho_n$ for all $n \in \mathbb{N}$. By the same argument as in the proof of Lemma 3.1, we have $F_n \in C^\infty_{1+,b}(\mathbb{R})$. By the Itô formula (2.2), we get for all $a \in \mathbb{R}$ and $t \geqslant 0$,

$$F_n(X_t - a) - F_n(X_0 - a) = H^{a,n}_t + K^{a,n}_t + V^{a,n}_t, \tag{3.4}$$

where

$$H^{a,n}_t = \int_0^t \int_{|h|\leqslant 1} \{F_n(X_{s-} - a + h) - F_n(X_{s-} - a)\}\tilde{N}(ds, dh),$$

$$K^{a,n}_t = \int_0^t \int_{|h|>1} \{F_n(X_{s-} - a + h) - F_n(X_{s-} - a)\}\tilde{N}(ds, dh),$$

$$V^{a,n}_t = \int_0^t \mathcal{L}F_n(X_s - a)ds.$$

In view of $F_n \in C^\infty_{1+,b}(\mathbb{R})$, we see from Remark 2.3 that the process $H^{a,n}$ is well defined and then is a square integrable martingale, and from Remark 2.2 and the proof of Proposition 2.2 that the process $K^{a,n}$ is well defined and then is a martingale. Since $F_n \in C^\infty_{1+,b}(\mathbb{R})$, it follows from the same argument as in the proof of Proposition 2.2 that

$$\int_0^t \int_{|h|>1} |F_n(X_s + h) - F_n(X_s)|\nu_\alpha(dh)ds \leqslant G_1(n)\frac{t(c_+ + c_-)}{\alpha - 1} < \infty,$$

where $G_1(n)$ is a positive constant such that $|F'_n(x)| \leqslant G_1(n)$ for all $x \in \mathbb{R}$, and it follows from Taylor's theorem that

$$\begin{aligned}
\int_0^t \int_{|h|\leqslant 1} &|F_n(X_s + h) - F_n(X_s) - F'_n(X_s)h|\nu_\alpha(dh)ds \\
&= \int_0^t \int_{|h|\leqslant 1} \Big|\int_0^1 F''_n(X_s + \theta h)(1-\theta)h^2 d\theta\Big|\nu_\alpha(dh)ds \\
&\leqslant \frac{G_2(n)t}{2} \int_{|h|\leqslant 1} h^2\nu_\alpha(dh) = G_2(n)\frac{t(c_+ + c_-)}{2(2-\alpha)} < \infty,
\end{aligned}$$

where $G_2(n)$ is a positive constant such that $|F''_n(x)| \leqslant G_2(n)$ for all $x \in \mathbb{R}$. Thus, the process $V^{a,n}$ is well defined and then is bounded on $[0, t]$ for all $t \geqslant 0$.

First we will show that $F_n(X_t - a), F(X_t - a) \in L^2(\mathbb{P})$ and $F_n(X_t - a)$ converges to $F(X_t - a)$ in $L^2(\mathbb{P})$ as $n \to \infty$. By using the inequality

$$|x + y|^{\alpha-1} \leqslant |x|^{\alpha-1} + |y|^{\alpha-1} \quad \text{for all } x, y \in \mathbb{R}, \tag{3.5}$$

we have for $x \in \mathbb{R}$,

$$0 \leqslant F_n(x) = \int_{\mathbb{R}} F(x-y)\rho_n(y)dy$$
$$\leqslant \int_{-1/n}^{1/n} 2D(\alpha)(|x|^{\alpha-1} + |y|^{\alpha-1})\rho_n(y)dy \leqslant 2D(\alpha)(|x|^{\alpha-1}+1).$$

By Lemma 2.1, we have

$$\begin{aligned}(3.6)\qquad \mathbb{E}[|F_n(X_t-a)|^2] &\leqslant 4D(\alpha)^2\mathbb{E}[(|X_t-a|^{\alpha-1}+1)^2]\\ &\leqslant 4D(\alpha)^2\mathbb{E}[(|X_t|^{\alpha-1}+|a|^{\alpha-1}+1)^2]\\ &\leqslant 12D(\alpha)^2(\mathbb{E}[|X_t|^{2\alpha-2}]+|a|^{2\alpha-2}+1) < \infty,\end{aligned}$$

since $0 < 2\alpha - 2 < \alpha$. Similarly, by Lemma 2.1, we have

$$\begin{aligned}(3.7)\qquad \mathbb{E}[|F(X_t-a)|^2] &\leqslant 4D(\alpha)^2\mathbb{E}[(|X_t|^{\alpha-1}+|a|^{\alpha-1})^2]\\ &\leqslant 8D(\alpha)^2(\mathbb{E}[|X_t|^{2\alpha-2}]+|a|^{2\alpha-2}) < \infty.\end{aligned}$$

It follows that

$$\begin{aligned}&|F_n(X_t-a)-F(X_t-a)|^2\\ &\quad\leqslant 2|F_n(X_t-a)|^2 + 2|F(X_t-a)|^2\\ &\quad\leqslant 24D(\alpha)^2(|X_t|^{2\alpha-2}+|a|^{2\alpha-2}+1) + 16D(\alpha)^2(|X_t|^{2\alpha-2}+|a|^{2\alpha-2}),\end{aligned}$$

and then we infer from Lemma 2.1 that the above right-hand side not depending on n is integrable. Hence, it follows from the dominated convergence theorem that

$$\begin{aligned}(3.8)\qquad \lim_{n\to\infty} \mathbb{E}[|F_n(X_t-a)-F(X_t-a)|^2]&\\ = \mathbb{E}[\lim_{n\to\infty} |F_n(X_t-a)-F(X_t-a)|^2] &= 0.\end{aligned}$$

Next, we will show that $K^{a,n}$ is a square integrable martingale, $M^{a,n}$ is well defined and then is a square integrable martingale, and $H_t^{a,n} + K_t^{a,n}$ converges to M_t^a in $L^2(\mathbb{P})$ as $n \to \infty$. By using the inequalities in the Appendix, we have for $x \neq 0$ and $h \in \mathbb{R}$,

$$\begin{aligned}|F(x+h)-F(x)|^2 &= \big|D(\alpha)\{|x+h|^{\alpha-1} - \beta|x+h|^{\alpha-1}\mathrm{sgn}(x+h)\}\\ &\qquad - D(\alpha)\{|x|^{\alpha-1} - \beta|x|^{\alpha-1}\mathrm{sgn}(x)\}\big|^2\\ &\leqslant 2D(\alpha)^2\left||x+h|^{\alpha-1} - |x|^{\alpha-1}\right|^2\\ &\qquad + 2D(\alpha)^2\left||x+h|^{\alpha-1}\mathrm{sgn}(x+h) - |x|^{\alpha-1}\mathrm{sgn}(x)\right|^2\\ &= 2D(\alpha)^2|x|^{2\alpha-2}\left||1+x^{-1}h|^{\alpha-1}-1\right|^2\\ &\qquad + 2D(\alpha)^2|x|^{2\alpha-2}\left||1+x^{-1}h|^{\alpha-1}\mathrm{sgn}(1+x^{-1}h)-1\right|^2\\ &\leqslant 20D(\alpha)^2\{(|x|^{\alpha-\varepsilon-2}|h|^{\alpha+\varepsilon}) \wedge |h|^{2\alpha-2}\}\end{aligned}$$

if $\alpha - 2 \leqslant \varepsilon \leqslant 2 - \alpha$. Now, choose ε_0 such that $0 < \varepsilon_0 < (\alpha - 1) \wedge (2 - \alpha)$. Then it follows from Jensen's inequality, Fubini's theorem and Lemma 2.2 that for $s > 0$,

$$\begin{aligned}
(3.9)\quad & \mathbb{E}[|F_n(X_s + h - a) - F_n(X_s - a)|^2] \\
&\leqslant \mathbb{E}\Big[\int_{\mathbb{R}} \rho_n(y)|F(X_s + h - a - y) - F(X_s - a - y)|^2 dy\Big] \\
&\leqslant 20D(\alpha)^2 \int_{\mathbb{R}} \rho_n(y)|h|^{2\alpha-2}\mathbf{1}_{\{|h|>1\}}dy \\
&\quad + 20D(\alpha)^2 \int_{\mathbb{R}} \rho_n(y)\mathbb{E}[|X_s - a - y|^{\alpha-\varepsilon_0-2}|h|^{\alpha+\varepsilon_0}]\mathbf{1}_{\{|h|\leqslant 1\}}dy \\
&\leqslant 20D(\alpha)^2|h|^{2\alpha-2}\mathbf{1}_{\{|h|>1\}} \\
&\quad + 20D(\alpha)^2 S(\alpha, -\alpha + \varepsilon_0 + 2)s^{(\alpha-\varepsilon_0-2)/\alpha}|h|^{\alpha+\varepsilon_0}\mathbf{1}_{\{|h|\leqslant 1\}},
\end{aligned}$$

by $0 < -\alpha + \varepsilon_0 + 2 < 1$. Similarly, it follows from Lemma 2.2 that for $s > 0$,

$$\begin{aligned}
(3.10)\quad & \mathbb{E}[|F(X_s + h - a) - F(X_s - a)|^2] \\
&\leqslant 20D(\alpha)^2|h|^{2\alpha-2}\mathbf{1}_{\{|h|>1\}} \\
&\quad + 20D(\alpha)^2\mathbb{E}[|X_s - a|^{\alpha-\varepsilon_0-2}|h|^{\alpha+\varepsilon_0}]\mathbf{1}_{\{|h|\leqslant 1\}} \\
&\leqslant 20D(\alpha)^2|h|^{2\alpha-2}\mathbf{1}_{\{|h|>1\}} \\
&\quad + 20D(\alpha)^2 S(\alpha, -\alpha + \varepsilon_0 + 2)s^{(\alpha-\varepsilon_0-2)/\alpha}|h|^{\alpha+\varepsilon_0}\mathbf{1}_{\{|h|\leqslant 1\}}.
\end{aligned}$$

By $\alpha > 2\alpha - 2$ and $\varepsilon_0 > 0$, we know that

$$\int_{\mathbb{R}\setminus\{0\}} (|h|^{\alpha+\varepsilon_0} \wedge |h|^{2\alpha-2})\nu_\alpha(dh) = \frac{c_+ + c_-}{\varepsilon_0} + \frac{c_+ + c_-}{2 - \alpha} < \infty,$$

and

$$\int_0^t s^{(\alpha-\varepsilon_0-2)/\alpha}ds = \frac{\alpha}{2\alpha - \varepsilon_0 - 2}t^{(2\alpha-\varepsilon_0-2)/\alpha} < \infty.$$

Then it follows that

$$\begin{aligned}
(3.11)\quad & \int_0^t \int_{|h|\leqslant 1} \mathbb{E}[|F_n(X_s - a + h) - F_n(X_s - a)|^2]\nu_\alpha(dh)ds \\
&\leqslant 20D(\alpha)^2 S(\alpha, -\alpha + \varepsilon_0 + 2)\int_0^t \int_{|h|\leqslant 1} s^{(\alpha-\varepsilon_0-2)/\alpha}|h|^{\alpha+\varepsilon_0}\nu_\alpha(dh)ds \\
&= C_3(\alpha)t^{(2\alpha-\varepsilon_0-2)/\alpha} < \infty,
\end{aligned}$$

where $C_3(\alpha)$ is a constant given by

$$C_3(\alpha) = 20D(\alpha)^2 S(\alpha, -\alpha+\varepsilon_0+2)\frac{\alpha(c_+ + c_-)}{\varepsilon_0(2\alpha-\varepsilon_0-2)},$$

and

$$(3.12)\quad \int_0^t \int_{|h|>1} \mathbb{E}[|F_n(X_s-a+h)-F_n(X_s-a)|^2]\nu_\alpha(dh)ds$$
$$\leqslant 20D(\alpha)^2 \int_0^t \int_{|h|>1} |h|^{2\alpha-2}\nu_\alpha(dh)ds = C_4(\alpha)t<\infty,$$

where $C_4(\alpha)$ is a constant given by

$$C_4(\alpha) = 20D(\alpha)^2\frac{c_+ + c_-}{2-\alpha}.$$

Moreover, it follows that

$$(3.13)\quad \int_0^t \int_{\mathbb{R}\setminus\{0\}} \mathbb{E}[|F(X_s-a+h)-F(X_s-a)|^2]\nu_\alpha(dh)ds$$
$$\leqslant C_3(\alpha)t^{(2\alpha-\varepsilon_0-2)/\alpha}+C_4(\alpha)t<\infty.$$

Recall that $H^{a,n}$ is a square integrable martingale. By Itô's isometry (see [1], p. 223) and (3.11), we have

$$(3.14)\quad \mathbb{E}[|H_t^{a,n}|^2] = \int_0^t \int_{|h|\leqslant 1} \mathbb{E}[|F_n(X_s-a+h)-F_n(X_s-a)|^2]\nu_\alpha(dh)ds$$
$$\leqslant C_3(\alpha)t^{(2\alpha-\varepsilon_0-2)/\alpha}.$$

From (3.12) it follows that the martingale $K^{a,n}$ is square integrable. Moreover, by Itô's isometry and (3.12), we have

$$(3.15)\quad \mathbb{E}[|K_t^{a,n}|^2] = \int_0^t \int_{|h|>1} \mathbb{E}[|F_n(X_s-a+h)-F_n(X_s-a)|^2]\nu_\alpha(dh)ds$$
$$\leqslant C_4(\alpha)t.$$

From (3.13) it follows that the process M^a is well defined and then is a square integrable martingale. Moreover, by Itô's isometry and (3.13), we have

$$(3.16)\quad \mathbb{E}[|M_t^{a}|^2] = \int_0^t \int_{\mathbb{R}\setminus\{0\}} \mathbb{E}[|F(X_s-a+h)-F(X_s-a)|^2]\nu_\alpha(dh)ds$$
$$\leqslant C_3(\alpha)t^{(2\alpha-\varepsilon_0-2)/\alpha}+C_4(\alpha)t.$$

Since the process $H^{a,n} + K^{a,n} - M^a$ is a square integrable martingale, it follows from Itô's isometry that

$$\begin{aligned}\mathbb{E}[|H_t^{a,n} + K_t^{a,n} - M_t^a|^2] = \int_0^t \int_{\mathbb{R}\setminus\{0\}} &\mathbb{E}[|F_n(X_s + h - a) - F_n(X_s - a) \\ &- \{F(X_s + h - a) - F(X_s - a)\}|^2]\nu_\alpha(dh)ds.\end{aligned}$$

By (3.8), we see that for $a, h \in \mathbb{R}$ and $s \geqslant 0$,

$$\begin{aligned}\lim_{n\to\infty} \mathbb{E}[|F_n(X_s + h - a) - F_n(X_s - a) - \{F(X_s + h - a) - F(X_s - a)\}|^2] \\ \leqslant 2 \lim_{n\to\infty} \mathbb{E}[|F_n(X_s + h - a) - F(X_s + h - a)|^2] \\ + 2 \lim_{n\to\infty} \mathbb{E}[|F_n(X_s - a) - F(X_s - a)|^2] \\ = 0.\end{aligned}$$

By (3.9) and (3.10), we have

$$\begin{aligned}\mathbb{E}[|F_n(X_s + h - a) - F_n(X_s - a) - \{F(X_s + h - a) - F(X_s - a)\}|^2] \\ \leqslant 2\mathbb{E}[|F_n(X_s + h - a) - F_n(X_s - a)|^2] \\ + 2\mathbb{E}[|F(X_s + h - a) - F(X_s - a)|^2] \\ \leqslant 80D(\alpha)^2 |h|^{2\alpha-2} \mathbf{1}_{\{|h|>1\}} \\ + 80D(\alpha)^2 S(\alpha, -\alpha + \varepsilon_0 + 2) s^{(\alpha-\varepsilon_0-2)/\alpha} |h|^{\alpha+\varepsilon_0} \mathbf{1}_{\{|h|\leqslant 1\}},\end{aligned}$$

and then it follows that the above right-hand side not depending on n is integrable in (s, h). Hence, we infer from the dominated convergence theorem that

$$\begin{aligned}\lim_{n\to\infty} \mathbb{E}[|H_t^{a,n} + K_t^{a,n} - M_t^a|^2] \\ = \lim_{n\to\infty} \int_0^t \int_{\mathbb{R}\setminus\{0\}} \mathbb{E}[|F_n(X_s + h - a) - F_n(X_s - a) \\ - \{F(X_s + h - a) - F(X_s - a)\}|^2]\nu_\alpha(dh)ds \\ = \int_0^t \int_{\mathbb{R}\setminus\{0\}} \lim_{n\to\infty} \mathbb{E}[|F_n(X_s + h - a) - F_n(X_s - a) \\ - \{F(X_s + h - a) - F(X_s - a)\}|^2]\nu_\alpha(dh)ds \\ = 0.\end{aligned} \tag{3.17}$$

Finally, we will show that $L_t^a := F(X_t - a) - F(X_0 - a) - M_t^a$ is the local time of X. It is sufficient to show that an occupation time formula holds for $g \in$

$C_c(\mathbb{R})$. By Lemma 3.1 and Fubini's theorem, we have for ω-wise,

$$\int_{\mathbb{R}} g(a)V_t^{a,n}(\omega)da = \int_{\mathbb{R}} g(a)\int_0^t \rho_n\big(X_s(\omega)-a\big)dsda = \int_0^t (g*\rho_n)\big(X_s(\omega)\big)ds. \tag{3.18}$$

It follows from the Cauchy–Schwarz inequality, Fubini's theorem and (3.4) that for $g \in C_c(\mathbb{R})$,

$$\begin{aligned}\mathbb{E}\big[\big|\int_{\mathbb{R}} g(a)V_t^{a,n}da\big|^2\big] &\leqslant \mathbb{E}\big[C_5\int_{\mathbb{R}} |g(a)||V_t^{a,n}|^2da\big] \\ &\leqslant 3C_5\int_{\mathbb{R}} |g(a)|\mathbb{E}[|F_n(X_t-a)|^2]da \\ &\quad + 3C_5\int_{\mathbb{R}} |g(a)|\mathbb{E}[|F_n(X_0-a)|^2]da \\ &\quad + 3C_5\int_{\mathbb{R}} |g(a)|\mathbb{E}[|H_t^{a,n}+K_t^{a,n}|^2]da,\end{aligned}$$

where $C_5 = \int_{\mathbb{R}} |g(x)|dx$. From (3.6), (3.14) and (3.15) it follows that the above right-hand side is bounded by

$$\begin{aligned}&36C_5^2D(\alpha)^2(\mathbb{E}[|X_t|^{2\alpha-2}]+1) + 36C_5D(\alpha)^2\int_{\mathbb{R}} |g(a)||a|^{2\alpha-2}da \\ &\quad + 36C_5^2D(\alpha)^2(\mathbb{E}[|X_0|^{2\alpha-2}]+1) + 36C_5D(\alpha)^2\int_{\mathbb{R}} |g(a)||a|^{2\alpha-2}da \\ &\quad + 6C_3(\alpha)C_5^2t^{(2\alpha-\varepsilon_0-2)/\alpha} + 6C_4(\alpha)C_5^2t.\end{aligned}$$

Similarly, it follows from the Cauchy–Schwarz inequality and Fubini's theorem that for $g \in C_c(\mathbb{R})$,

$$\begin{aligned}\mathbb{E}\big[\big|\int_{\mathbb{R}} g(a)L_t^a da\big|^2\big] &\leqslant \mathbb{E}\big[C_5\int_{\mathbb{R}} |g(a)||L_t^a|^2da\big] \\ &\leqslant 3C_5\int_{\mathbb{R}} |g(a)|\mathbb{E}[|F(X_t-a)|^2]da \\ &\quad + 3C_5\int_{\mathbb{R}} |g(a)|\mathbb{E}[|F(X_0-a)|^2]da \\ &\quad + 3C_5\int_{\mathbb{R}} |g(a)|\mathbb{E}[|M_t^a|^2]da.\end{aligned}$$

From (3.7) and (3.16) it follows that the above right-hand side is bounded by

$$\begin{aligned}&24C_5^2D(\alpha)^2\mathbb{E}[|X_t|^{2\alpha-2}] + 24C_5D(\alpha)^2\int_{\mathbb{R}} |g(a)||a|^{2\alpha-2}da \\ &\quad + 24C_5^2D(\alpha)^2\mathbb{E}[|X_0|^{2\alpha-2}] + 24C_5D(\alpha)^2\int_{\mathbb{R}} |g(a)||a|^{2\alpha-2}da \\ &\quad + 3C_3(\alpha)C_5^2t^{(2\alpha-\varepsilon_0-2)/\alpha} + 3C_4(\alpha)C_5^2t.\end{aligned}$$

Thus, the integrals $\int_{\mathbb{R}} g(a)V_t^{a,n}da$ and $\int_{\mathbb{R}} g(a)L_t^a da$ are square integrable. Moreover, by $g * \rho_n \in C_c^\infty(\mathbb{R})$ and $g \in C_c(\mathbb{R})$, the integrals $\int_0^t (g * \rho_n)(X_s)ds$ and $\int_0^t g(X_s)ds$ are square integrable.

Now, we will show that the left-hand side of (3.18) converges in $L^2(\mathbb{P})$, i.e.,

$$\lim_{n\to\infty} \int_{\mathbb{R}} g(a)V_t^{a,n}da = \int_{\mathbb{R}} g(a)L_t^a da \quad \text{in } L^2(\mathbb{P}), \tag{3.19}$$

and the right-hand side of (3.18) converges in $L^2(\mathbb{P})$, i.e.,

$$\lim_{n\to\infty} \int_0^t (g * \rho_n)(X_s)ds = \int_0^t g(X_s)ds \quad \text{in } L^2(\mathbb{P}). \tag{3.20}$$

It follows from the Cauchy–Schwarz inequality, Fubini's theorem and (3.4) that for $g \in C_c(\mathbb{R})$,

$$\begin{aligned} \mathbb{E}\big[\big|\int_{\mathbb{R}} g(a)V_t^{a,n}da - \int_{\mathbb{R}} g(a)L_t^a da\big|^2\big] &\leqslant \mathbb{E}\big[C_5 \int_{\mathbb{R}} |g(a)||V_t^{a,n} - L_t^a|^2 da\big] \\ &\leqslant 3C_5 \int_{\mathbb{R}} |g(a)|\mathbb{E}[|F_n(X_t - a) - F(X_t - a)|^2]da \\ &\quad + 3C_5 \int_{\mathbb{R}} |g(a)|\mathbb{E}[|F_n(X_0 - a) - F(X_0 - a)|^2]da \\ &\quad + 3C_5 \int_{\mathbb{R}} |g(a)|\mathbb{E}[|H_t^{a,n} + K_t^{a,n} - M_t^a|^2]da. \end{aligned} \tag{3.21}$$

From (3.6), (3.7) and (3.14)–(3.16) we recall that

$$\begin{aligned} \mathbb{E}[|F_n(X_t - a)|^2] &\leqslant 12D(\alpha)^2(\mathbb{E}[|X_t|^{2\alpha-2}] + |a|^{2\alpha-2} + 1), \\ \mathbb{E}[|F(X_t - a)|^2] &\leqslant 8D(\alpha)^2(\mathbb{E}[|X_t|^{2\alpha-2}] + |a|^{2\alpha-2}), \\ \mathbb{E}[|H_t^{a,n}|^2] &\leqslant C_3(\alpha)t^{(2\alpha-\varepsilon_0-2)/\alpha}, \\ \mathbb{E}[|K_t^{a,n}|^2] &\leqslant C_4(\alpha)t, \\ \mathbb{E}[|M_t^a|^2] &\leqslant C_3(\alpha)t^{(2\alpha-\varepsilon_0-2)/\alpha} + C_4(\alpha)t. \end{aligned}$$

Thus, it follows from (3.8), (3.17) and the dominated convergence theorem that the right-hand side of (3.21) converges to zero as $n \to \infty$, and hence (3.19) follows.

Now we prove (3.20). It follows from the Cauchy–Schwarz inequality, Jensen's inequality and Fubini's theorem that for $g \in C_c(\mathbb{R})$,

$$
\begin{aligned}
(3.22)\quad & \mathbb{E}\big[\big|\int_0^t (g * \rho_n)(X_s)ds - \int_0^t g(X_s)ds\big|^2\big] \\
&\leqslant \mathbb{E}\big[t\int_0^t \big|\int_{\mathbb{R}} \rho_n(y)\{g(X_s - y) - g(X_s)\}dy\big|^2 ds\big] \\
&\leqslant \mathbb{E}\big[t\int_0^t \int_{\mathbb{R}} \rho_n(y)|g(X_s - y) - g(X_s)|^2 dyds\big] \\
&= t\int_0^t \int_{\mathbb{R}} n\rho(ny)\mathbb{E}[|g(X_s - y) - g(X_s)|^2]dyds \\
&= t\int_0^t \int_{\mathbb{R}} \rho(z)\mathbb{E}[|g(X_s - n^{-1}z) - g(X_s)|^2]dzds,
\end{aligned}
$$

by putting $z = ny$. Thus, it follows from the dominated convergence theorem that the right-hand side of (3.22) converges to zero as $n \to \infty$, and hence (3.20) follows. The proof is now complete. ∎

REMARK 3.1. *From the proof of Theorem* 3.1 *we obtain the existence of the local time* $L = \{L_t^a : a \in \mathbb{R}, t \geqslant 0\}$ *of* X.

REMARK 3.2. *The local time* $L = \{L_t^a : a \in \mathbb{R}, t \geqslant 0\}$ *of* X *can be also represented by*

$$L_t^a = \lim_{n\to\infty} \int_0^t \rho_n(X_s - a)ds \quad \text{in } L^2(\mathbb{P}),$$

where $(\rho_n)_{n\in\mathbb{N}}$ *is given by* $\rho_n(x) = n\rho(nx)$ *for all* $n \in \mathbb{N}$ *with a mollifier* ρ.

4. APPENDIX

Let $\alpha \in (1, 2)$. We establish the following inequalities: for all $x \in \mathbb{R}$,

$$\big||x+1|^{\alpha-1} - 1\big| \leqslant |x|^{(\alpha+\varepsilon)/2} \wedge |x|^{\alpha-1},$$
$$\big||x+1|^{\alpha-1}\mathrm{sgn}(x+1) - 1\big| \leqslant 3(|x|^{(\alpha+\varepsilon)/2} \wedge |x|^{\alpha-1})$$

if $\alpha - 2 \leqslant \varepsilon \leqslant 2 - \alpha$.

Since $0 < \alpha - 1 \leqslant (\alpha+\varepsilon)/2 \leqslant 1$, we have

$$|x|^{(\alpha+\varepsilon)/2} \wedge |x|^{\alpha-1} = |x|^{(\alpha+\varepsilon)/2}\mathbf{1}_{\{|x|<1\}} + |x|^{\alpha-1}\mathbf{1}_{\{|x|\geqslant 1\}}.$$

Thus, it is sufficient to prove the result in the case of $\varepsilon = 2 - \alpha$. By using the inequality (3.5), we have for $x \in \mathbb{R}$,

$$\big||x+1|^{\alpha-1} - 1\big| \leqslant |x|^{\alpha-1},$$

and for $x \leqslant -1$,

$$\big||x+1|^{\alpha-1}\mathrm{sgn}(x+1)-1\big| = |x+1|^{\alpha-1}+1 < 3|x|^{\alpha-1}.$$

Then, we have for $x \in \mathbb{R}$,

$$\big||x+1|^{\alpha-1}\mathrm{sgn}(x+1)-1\big| \leqslant 3|x|^{\alpha-1}.$$

By $0 < \alpha - 1 < 1$, we have for $x \geqslant 0$,

$$\big||x+1|^{\alpha-1}-1\big| = (x+1)^{\alpha-1}-1 \leqslant (\alpha-1)|x|,$$

and for $-1 < x < 0$,

$$\big||x+1|^{\alpha-1}-1\big| = 1-(x+1)^{\alpha-1} < |x|.$$

Therefore, the required inequalities hold.

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