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Modal ontologic

Abstract

The paper deals with several problems concerning ontological notions of existence, possibility, well-foundation and fusion, used in reference to objects, in relation to contemporary semantic analysis of modal terms. The name *ontologic* was suggested by Polish logician Jerzy Perzanowski for theoretical or formal part of ontology. Term *modal ontologic* refers to the formal logical study of ontological concepts within the framework of propositional modal logic, especially a study of logical interconnections between modal concepts as applied to propositions or some proposition-like entities, on the one hand, and ontological concepts of existence, possibility, well-foundation and fusion used in reference to objects, on the other hand. It is shown, that a slight modification to contemporary semantic analysis of modal terms can capture some intuitions of Aristotle and his scholastic followers, especially about so-called modalities *de re*.

0. The history of modal logic begins with Aristotle who studied the logical interconnections between the necessary, the impossible, the possible and the permitted. However, in *On Interpretation*, he argues, that every single assertion, such as premise or conclusion in a syllogism, is either the affirmation or the denial of a single predicate of a single subject. Thus, for Aristotle, modal terms in fact modify this assertion or denial, therefore modalities are well-rooted in things. Hence, modal terms are closely related to ontological notions. The Megarians and Stoics also developed various theories concerning modality but in connection with propositional logic. So, for them, modal terms modify propositions or some proposition-like entities, situations or states of affairs. Contemporary attention paid to the formal properties of modal terms begins with the work of C.I. Lewis *Survey of Symbolic Logic*.[1] Contemporary semantic analysis of modal terms, known as possible worlds semantics, initiated by S. Kripke, follows Leibniz suggestion that a necessary proposition is one which holds not merely in the actual world but in every other possible worlds as well.[2]

[1] See C.I. Lewis, *A Survey of Symbolic Logic*, Berkeley 1918.
[2] See S.A. Kripke, 'A Completeness Theorem in Modal Logic', *The Journal of Symbolic Logic* 24 (1959), pp. 1–14.

1. The name *ontologic* was suggested by Polish logician Jerzy Perzanowski for theoretical or formal part of ontology. [3] By *modal ontologic* I understand the formal logical study of ontological concepts within the framework of propositional modal logic, especially a study of logical interconnections between modal concepts, the necessary and the possible, on the one hand, and ontological concepts of existence, possibility, well-foundation and fusion on the other hand. There is one key difference in comparison to the standard approach. I assume, the concept of necessity as applied to propositions or proposition-like entities may be relativized to the objects in a fixed ontological universe. So, instead of contexts like *it is necessary that* **A**, where **A** stands for a proposition, I will study contexts like *for* **b** *it is necessary that* **A**, where **A** stands for a proposition and **b** stands for an object. Let me adopt an informal notation to express some basic insights. For object **a** and proposition **B** let **aB** mean that for **a** it is necessary that **B**. For object **a** let Ex**a** mean that **a** exists and let Pos**a** mean that **a** is possible. For objects **a** and **b** let **a**/**b** mean that **a** is well-founded in **b** and let (**a***b**) stand for the fusion of **a** and **b**. The signs $\sim$, &, $\vee$ and $\rightarrow$ will then be used respectively as symbols for negation, conjunction, disjunction and material implication. It is quite clear that if an object exists, then every state of affair that is necessary for the object obtains:

(1) $\mathrm{Ex}\mathbf{a} \rightarrow (\mathbf{aB} \rightarrow \mathbf{B})$.

So, the denial of something that is necessary for object **a**, implies the denial of the existence of **a**:

(2) $\mathbf{aB} \rightarrow (\sim\mathbf{B} \rightarrow\sim\mathrm{Ex}\mathbf{a})$.

On the other hand, the existence of object **a** is something that is necessary for object **a**:

(3) $\mathbf{a}(\mathrm{Ex}\mathbf{a})$,

therefore, existence is, in a sense, something necessary. If there is a contradictory pair of situations, such that each of them is necessary for object **a**, then **a** is not a possible object:

(4) $(\mathbf{aB}\ \&\ \mathbf{a}(\sim\mathbf{B})) \rightarrow\sim\mathrm{Pos}\mathbf{a}$,

which implies, that

(5) $\mathrm{Pos}\mathbf{a} \rightarrow (\mathbf{aB} \rightarrow\sim\mathbf{a}(\sim\mathbf{B}))$.

Of course, each object that exists, is a possible object:

(6) $\mathrm{Ex}\mathbf{a} \rightarrow \mathrm{Pos}\mathbf{a}$.

If object **a** is well-founded in object **b**, then everything that is necessary for **b** is also necessary for **a**:

[3] See J. Perzanowski, ‘Ontologies and Ontologics’, [in:] E. Żarnecka-Biały (ed.), *Logic Counts*, Dordrecht 1990, pp. 23–42.

(7) $\mathbf{a}/\mathbf{b} \to (\mathbf{bC} \to \mathbf{aC})$.

On the other hand, if the existence of object **b** is necessary for object **a**, then **a** is well-founded in **b**:

(8) $\mathbf{a}(\mathrm{Ex}\mathbf{b}) \to \mathbf{a}/\mathbf{b}$,

and conversely, if object **a** is well-founded in object **b**, then the existence of **b** is necessary for **a**:

(9) $\mathbf{a}/\mathbf{b} \to \mathbf{a}(\mathrm{Ex}\mathbf{b})$.

Well-founding is a transitive and reflexive relation, thus:

(10) $(\mathbf{a}/\mathbf{b}\ \&\ \mathbf{b}/\mathbf{c}) \to \mathbf{a}/\mathbf{b}$

and

(11) $(\mathbf{a}/\mathbf{a})$.

By the fusion of objects **a** and **b**, I understand a complex object composed of **a** and **b** as its direct parts. Therefore, if a situation **C** is necessary for object **a** or is necessary for object **b**, then it is necessary for the fusion of these objects:

(12) $(\mathbf{aC} \vee \mathbf{bC}) \to (\mathbf{a}{*}\mathbf{b})\mathbf{C}$,

but the converse implication is not generally valid. There could be a situation **C** that is neither necessary for **a** nor for **b**, but is necessary for the fusion of **a** and **b**. If the fusion of **a** and **b** exists, then **a** and **b** also exists:

(13) $\mathrm{Ex}(\mathbf{a}{*}\mathbf{b}) \to (\mathrm{Ex}\mathbf{a}\ \&\ \mathrm{Ex}\mathbf{b})$,

but not conversely. Finally, each complex object is well-founded in its direct parts:

(14) $(\mathbf{a}{*}\mathbf{b})/\mathbf{a}$

and

(15) $(\mathbf{a}{*}\mathbf{b})/\mathbf{b}$.

As was said, Leibniz suggested that necessity was an equivalent to the truth at all possible worlds, although perhaps he never stated it explicitly. So, a proposition is necessarily true in this world (or a situation, or a state of affairs necessarily obtains in this world) if and only if that proposition is true in all worlds alternative to this world (or that situation, or that state of affairs obtains in all worlds alternative to this world). Possible worlds are also referred to by the term 'stand points' or 'possibilities' or 'state descriptions'.

Assuming that, the concept of necessity as applied to propositions or proposition-like entities may be relativized to the objects in a fixed ontological universe, I will take into account not one, but many different alternativeness-relations, one relation for one object. So, instead of contexts like ***v*** *is an alternative to* ***w***, where ***v*** and ***w*** stand for possible worlds, I will deal with contexts like *for* **a** *possible world* ***v*** *is accessible from world* ***w***, or shortly, ***v*** *is* **a**-*accessible from* ***w***, where **a** stands for an object. It could be said, that if ***v*** *is* **a**-*accessible from* ***w***, then at a possible world $\boldsymbol{w}$, $\boldsymbol{v}$ is a possible world which is supportive for object **a**, or the transition from the possible world $\boldsymbol{w}$ to the possible world $\boldsymbol{v}$ is sustainable for object **a**. The idea is roughly as follows. An object can be faced with many different possibilities. Usually, some of these possibilities are sustainable for the object and some of them are not. Moreover, the class of possibilities which are sustainable for an object can vary from different standpoints. Thus, each object could be determined (as to its existential aspect) by the relation which correlates each standpoint with the possibilities, which are sustainable for the object in this standpoint. The possible worlds are simply formal counterparts of possibilities and standpoints. I assume that, **aB** is true at a possible world $\boldsymbol{w}$ if and only if **B** is true at all possible worlds which are **a**-*accessible from* $\boldsymbol{w}$, that is, a situation is necessary for an object at a possible world $\boldsymbol{w}$ if and only if it obtains in all possible worlds which are accessible for this object from world $\boldsymbol{w}$. I also assume that, Ex**a** is true at a possible world $\boldsymbol{w}$ if and only if the possible world $\boldsymbol{w}$ is **a**-*accessible* from itself, and that Pos**a** is true at a possible world $\boldsymbol{w}$ if and only if there is a possible world $\boldsymbol{v}$ which is **a**-*accessible* from the possible world $\boldsymbol{w}$.Furthermore, I assume, that **a**/**b** is true at a possible world $\boldsymbol{w}$ if and only if any possible world $\boldsymbol{v}$, which is **a**-*accessible* from the possible world $\boldsymbol{w}$, is also **b**-*accessible* from the possible world $\boldsymbol{w}$. Finally, I assume that if a possible world $\boldsymbol{v}$ is (**a*****b**)-*accessible* from a possible world $\boldsymbol{w}$, then the possible world $\boldsymbol{v}$ is **a**-*accessible* from the possible world $\boldsymbol{w}$ and **b**-*accessible* from the possible world $\boldsymbol{w}$. I presume that if a possible world $\boldsymbol{v}$ is **a**-*accessible* from a possible world $\boldsymbol{w}$, then the possible world $\boldsymbol{v}$ is **a**-*accessible* from itself. Thus, if a possibility $\boldsymbol{v}$ is sustainable for object **a** in a standpoint $\boldsymbol{w}$, then the transition from $\boldsymbol{w}$ to $\boldsymbol{v}$ doesn't change the quality of possibility $\boldsymbol{v}$ as a possibility sustainable for **a**. This assumption could be referred to as *the first ontological consistency principle*. I also presume that if any possible world $\boldsymbol{v}$, which is **a**-*accessible* from a possible world $\boldsymbol{w}$, is a possible world which is **b**-*accessible* from itself, then, any possible world $\boldsymbol{v}$, which is **a**-*accessible* from the possible world $\boldsymbol{w}$, is also **b**-*accessible* from the possible world $\boldsymbol{w}$. Thus, due to a possible world $\boldsymbol{v}$ that is **b**-accessible from itself is a possible world at which **b** exists, if that a possibility $\boldsymbol{v}$ is sustainable for object **a** in the standpoint $\boldsymbol{w}$ implies that $\boldsymbol{v}$ is a point at which **b** exist, then that a possibility $\boldsymbol{v}$ is sustainable for object **a** in the standpoint $\boldsymbol{w}$ implies that $\boldsymbol{v}$ is also a possibility sustainable for object **b** in the standpoint $\boldsymbol{w}$. This assumption could be referred to as *the second ontological consistency principle*.

2. A language for modal ontologic consists of the following:

The alphabet is given by

(a) a denumerable set P of propositional letters. I refer to these as p_1, p_2, p_3, ...etc.,

(b) the symbols of logical connectives $\sim$ and & for negation and conjunction respectively,

(c) a denumerable set O of object letters. I refer to these as a_1, a_2, a_3, ...etc.,

(d) the ontological symbols of fusion *, existence Ex, possibility Poss and well-founding /,

(e) the auxiliary symbols (and).

The set Ob of one-place object operators is the smallest set X satisfying the following conditions:

(Ob 1) Each object letter belongs to X.
(Ob 2) If x belongs to X and y belongs to X, then (x*y) belongs to X.

The letters **a**, **b**, **c**... are used as metalogical variables, to range over object operators.

The set For of well-formed sentential formulae is the smallest set X satisfying the following conditions:

(For 1) Each propositional letter belongs to X.
(For 2) If x belongs to X and y belongs to X, then (x & y), ∼x and ∼y belongs to X.
(For 3) If **a** is an object operator, then Ex**a** and Poss**a** belongs to X.
(For 4) If **a** and **b** are object operators, then **a**/**b** belongs to X.
(For 5) If x belongs to X and **a** is an object operator, then **a**x belongs to X.

The letters **A**, **B**, **C**... are used as metalogical variables, to range over well-formed sentential formulae.

Various other symbols are introduced by following definitions:

(D1) $(\mathbf{A} \vee \mathbf{B}) = \sim(\sim\mathbf{A} \;\&\; \sim\mathbf{B})$
(D2) $(\mathbf{A} \rightarrow \mathbf{B}) = \sim(\mathbf{A} \;\&\; \sim\mathbf{B})$
(D3) $(\mathbf{A} \equiv \mathbf{B}) = \sim(\mathbf{A} \;\&\; \sim\mathbf{B}) \;\&\; \sim(\sim\mathbf{A} \;\&\; \mathbf{B})$
(D4) $\cancel{\mathbf{a}}\mathbf{B} = \sim(\mathbf{a}\sim\mathbf{B})$.

The phrase $\cancel{\mathbf{a}}\mathbf{B}$ could be read: **B** is possible for **a**.

Within the frame of the formal language a simple *modal ontological calculus* can be constructed. A calculus is given by a set of sentential formulae, called axioms, and a finite set of inference rules. Let me construct the modal ontological calculus on the following axiomatic basis.

The first group of axioms consists of all tautologies of classical propositional logic, with well-formed formulae of ontological language substituted for the propositional letters.

The second group of axioms is determined by the following schemata:

(A1) $\mathbf{a}(\mathbf{B} \rightarrow \mathbf{C}) \rightarrow (\mathbf{aB} \rightarrow \mathbf{aC})$,
(A2) $\mathrm{Ex}\mathbf{a} \rightarrow (\mathbf{aB} \rightarrow \mathbf{B})$,
(A3) $\mathbf{a}\mathrm{Ex}\mathbf{a}$,
(A4) $\mathrm{Pos}\mathbf{a} \rightarrow (\mathbf{aB} \rightarrow \sim\mathbf{a}\sim\mathbf{B})$,
(A5) $\sim\mathbf{aB} \rightarrow \mathrm{Pos}\mathbf{a}$,
(A6) $\mathbf{a}/\mathbf{b} \rightarrow (\mathbf{bC} \rightarrow \mathbf{aC})$,
(A7) $\mathbf{a}\mathrm{Ex}\mathbf{b} \rightarrow \mathbf{a}/\mathbf{b}$,
(A8) $(\mathbf{aC} \vee \mathbf{bC}) \rightarrow (\mathbf{a}*\mathbf{b})\mathbf{C}$.

The rules of inference of the calculus are:

(R1) Rule of Detachment (*Modus Ponens*).

(R2) A Rule of Necessitation to the effect that if $\mathbf{B}$ is a provable formula, then $\mathbf{aB}$ is also a provable formula.

The axiom schema (A1) and the rule (R2) are respectively the multimodal versions of the axiom of regularity and the rule of necessitation applied in standard logical calculi for normal propositional modal logics. The axiom schemata (A2)–(A8) capture a proper ontological content discussed in the previous paragraph.

A proof is then defined as a finite sequence of formulas such that each member either belongs to axioms or is derived from earlier members of the sequence by *Modus Ponens* or the Rule of Necessitation. A proof is said to be a proof of the last member in its sequence, and a thesis is a formula of which there is a proof. I write $\vdash$ to mean that the formula $\mathbf{A}$ is a thesis. From the axiomatic basis of the modal ontological calculus we can prove a number of theses. Among them are all formal counterparts of (1)–(15). In fact, formal counterparts of (1), (3), (5), (7), (8) and (12) are axioms. As to the remaining of them, and some other theses, let me state the following theorem.

Theorem 1. The following expressions are thesis schemata of modal ontological calculus (the proofs of theorems are enclosed in the appendix, where proofs are easy, they are omitted):

(Th1) $\mathbf{aB} \to (\sim\mathbf{B} \to \sim\mathrm{Ex}\mathbf{a})$,

(Th2) $(\mathbf{aB} \;\&\; \mathbf{a}{\sim}\mathbf{B}) \to \sim\mathrm{Pos}\mathbf{a}$,

(Th3) $\mathrm{Ex}\mathbf{a} \to \mathrm{Pos}\mathbf{a}$,

(Th4) $\mathbf{a}/\mathbf{b} \to \mathbf{a}(\mathrm{Ex}\mathbf{b})$,

(Th5) $(\mathbf{a}/\mathbf{b} \;\&\; \mathbf{b}/\mathbf{c}) \to \mathbf{a}/\mathbf{c}$,

(Th6) $(\mathbf{a}/\mathbf{a})$,

(Th7) $\mathrm{Ex}(\mathbf{a}{*}\mathbf{b}) \to (\mathrm{Ex}\mathbf{a} \;\&\; \mathrm{Ex}\mathbf{b})$,

(Th8) $(\mathbf{a}{*}\mathbf{b})\mathrm{Ex}\mathbf{a}$,

(Th9) $(\mathbf{a}{*}\mathbf{b})\mathrm{Ex}\mathbf{b}$,

(Th10) $(\mathbf{a}{*}\mathbf{b})/\mathbf{a}$,

(Th11) $(\mathbf{a}{*}\mathbf{b})/\mathbf{b}$,

(Th12) $\mathrm{Pos}(\mathbf{a}{*}\mathbf{b}) \to (\mathrm{Pos}\mathbf{a} \;\&\; \mathrm{Pos}\mathbf{b})$,

(Th13) $(\mathbf{a}/\mathbf{b} \;\&\; \mathbf{b}/\mathbf{a}) \to (\mathrm{Ex}\mathbf{a} \equiv \mathrm{Ex}\mathbf{b})$,

(Th14) $\mathbf{a}/\mathbf{b} \to (\mathbf{a}{*}\mathbf{c})/\mathbf{b}$,

(Th15) $\mathbf{a}/(\mathbf{b}{*}\mathbf{c}) \to (\mathbf{a}/\mathbf{b} \;\&\; \mathbf{a}/\mathbf{c})$.

Let me draw your attention to (Th8), (Th9) and to the last four schemata (Th12)–(Th15) which capture some ontological intuition about objects. Thesis schemata (Th8) and (Th9) say that existence of $\mathbf{a}$ as well as existence of $\mathbf{b}$ is necessary for the fusion of $\mathbf{a}$ and $\mathbf{b}$. According to (Th12) if the fusion of $\mathbf{a}$ and $\mathbf{b}$

is possible, then **a** is possible and **b** is possible. Due to (Th13), if **a** is well-founded in **b** and **b** is well-founded in **a**, then **a** exist if and only if **b** exist. Scheme (Th14) says that if **a** is well-founded in **b**, then a fusion of **a** and **c** is also well-founded in **b** and scheme (Th15) says that if **a** is well-founded in the fusion of **b** and **c**, then **a** is well-founded in **b** and **a** is well founded in **c**.

By the *modal ontological theory*, I mean the class of all theses. Thus, the modal ontological theory is the smallest set containing all axioms and closed with respect to *Modus Ponens* and the Rule of Necessitation.

For any set of formulas X, I shall say that **A** is *deducible* from X if and only if there are formulas $\mathbf{B}_1$, $\mathbf{B}_2$,....$\mathbf{B}_n$ belonging to X such that the formula $(\mathbf{B}_1 \,\&\, \mathbf{B}_2 \,\&\ldots.\&\, \mathbf{B}_n) \rightarrow \mathbf{A}$ is a thesis. Note that each formula which is deducible from the modal ontological theory belongs to it. A set of formulas X is called *consistent* if and only if there is no formula **A**, such that **A** and ∼**A** are both deducible from X. Otherwise X is called *inconsistent*. The definition implies that if any set of formulae X is inconsistent, then some finite subset of X is also inconsistent, and that if any set of formulas X is inconsistent and a formula **A** is not deducible form X, then the set $X \cup \{\sim\mathbf{A}\}$is also inconsistent. Let me assume, that sets of formulae represent situations and consistent sets of formulae represent consistent, or ontologically possible situations. Note that if the modal ontological theory is inconsistent, then all sets of formulae are inconsistent. Fortunately the following theorem holds.

Theorem 2. The modal ontological theory is consistent.

A set of formulae is *complete* if and only if for any formula **A** either **A** belongs to X or ∼**A** belongs to X. A set of formulae which is both consistent and complete is called a *maximal consistent* set of formulae. Note that if X is maximal consistent set of formulae and **A** is deducible from X, then **A** belongs to X. It is easy to show, that for every maximal consistent set of formulae X and for every formulae **A** and **B**, (*i*) ∼**A** belongs to X if and only if **A** doesn't belong to X, (*ii*) (**A** & **B**) belongs to X if and only if **A** belongs to X and **B** belongs to X, (*iii*) (**A** ∨ **B**) belongs to X if and only if **A** belongs to X or **B** belongs to X, (*iv*) (**A** → **B**) belongs to X if and only if if **A** belongs to X, then **B** belongs to X, (*v*) (**A** ≡ **B**) belongs to X if and only if **A** belongs to X if and only if **B** belongs to X.

Maximal consistent sets of formulae might be regarded as complete state descriptions expressed in the language of modal ontologic. Thus, they might be regarded as counterparts of possible worlds. For the modal ontological theory holds the theorem known as the Lindenbaum's Lemma to the effect that any consistent set of formulae is a subset of a maximal consistent set of formulae. It follows that a set of formulas is consistent if and only if it is included in a maximal consistent set of formulas. It reflects the conviction that any ontologically possible situation obtains in some possible world and, in fact, that a situation is ontologically possible if and only if it obtains in some possible world.

3. The semantics of modal ontologic is a slight modification to the standard semantics of normal modal logics known as possible worlds semantics. The object

operators are to be interpreted as binary relations on the set of possible worlds. The intuition behind this modeling is that each object **a** is determined by the binary relation $\mathbf{R_a}$, which correlates a possible world $\boldsymbol{w}$ with possible worlds, which are supportive for object **a** at a possible world $\boldsymbol{w}$. The symbol of fusion is to be interpreted as a binary operation + defined on the set of binary relations assigned to modal operators. I assume that for any relations **R** and **S**, and for any possible worlds $\boldsymbol{w}$ and $\boldsymbol{v}$, if $\boldsymbol{v}\mathbf{R}+\mathbf{S}\mathbf{w}$ then $\boldsymbol{v}\mathbf{R}\boldsymbol{w}$ and $\boldsymbol{v}\mathbf{S}\boldsymbol{w}$. It reflects the conviction that a possible world which is supportive for a fusion is also supportive for the direct parts of the fusion.

Formally, I shall introduce the notion of an ontological model. An ontological model M is to consist of a non-empty set of possible worlds W, an infinite sequence P_1, P_2, P_3,... of subsets of W, let me abbreviate it as P_i, a set of binary relations on W, let me abbreviate it as R, a binary operation + defined on R, and an infinite sequence R_1, R_2, R_3, ... of binary relations from R, let me abbreviate it as R_i. Thus, I define an ontological model as a structure $M = < W, P_i, R, +, R_i >$ satisfying the following additional conditions:

(C1) for any relations **R** and **S** belonging to R, and for any $\boldsymbol{w}$ and $\boldsymbol{v}$ belonging to W, if $\boldsymbol{v}\mathbf{R}+\mathbf{S}\boldsymbol{w}$ then $\boldsymbol{v}\mathbf{R}\boldsymbol{w}$ and $\boldsymbol{v}\mathbf{S}\boldsymbol{w}$;

(C2) for any relation **R** belonging to R, and for any $\boldsymbol{w}$ and $\boldsymbol{v}$ belonging to W, if $\boldsymbol{v}\mathbf{R}\boldsymbol{w}$ then $\boldsymbol{v}\mathbf{R}\boldsymbol{v}$;

(C3) for any relations **R** and **S** belonging to R, and for any $\boldsymbol{w}$ and $\boldsymbol{v}$ belonging to W, if $\boldsymbol{v}\mathbf{R}\boldsymbol{w}$ implies that $\boldsymbol{v}\mathbf{S}\boldsymbol{v}$, then $\boldsymbol{v}\mathbf{R}\boldsymbol{w}$ implies that $\boldsymbol{v}\boldsymbol{S}\boldsymbol{w}$.

Condition (C1) reflects the conviction that a possible world which is supportive for a fusion is also supportive for the direct parts of the fusion. Conditions (C2) and (C3) are formal counterparts of the first ontological consistency principle and the second ontological consistency principle stated at the end of the paragraph 1.

Given the definition of an ontological model, I shall state the following theorem.

Theorem 3. There are structures which are ontological models.

For any modal operator **a** there is a unique binary relation $\mathbf{R_a}$, which correspond to **a** in a model M. For each natural number k, R_k correspond to a_k and $\mathbf{R_a}+\mathbf{R_b}$ correspond to (**a*b**). In terms of possible world in a model I state the truth conditions for formulae according to their forms. I write $\boldsymbol{w}\models^M \mathbf{A}$ to mean that **A** is true at the possible world $\boldsymbol{w}$ in the model M. The truth conditions are as follows.

1. $\boldsymbol{w}\models^M \mathrm{p}_k$ if and only if $\boldsymbol{w}$ belongs to P_k, for $k = 1, 2, 3, \ldots$.
2. $\boldsymbol{w}\models^M \sim\mathbf{A}$ if and only if not $\boldsymbol{w} \models^M \mathbf{A}$.
3. $\boldsymbol{w}\models^M (\mathbf{A}\ \&\ \mathbf{B})$ if and only if both $\boldsymbol{w}\models^M \mathbf{A}$ and $\boldsymbol{w}\models^M \mathbf{B}$.
4. $\boldsymbol{w}\models^M \mathbf{aB}$ if and only if for any possible world $\boldsymbol{v}$ if $\boldsymbol{v}\mathbf{R_a}\boldsymbol{w}$ then $\boldsymbol{v}\models^M \mathbf{B}$.
5. $\boldsymbol{w}\models^M \mathrm{Ex}\mathbf{a}$ if and only if $\boldsymbol{w}\mathbf{R_a}\boldsymbol{w}$.
6. $\boldsymbol{w}\models^M \mathrm{Pos}\mathbf{a}$ if and only if there is a possible world $\boldsymbol{v}$ such that $\boldsymbol{v}\mathbf{R_a}\boldsymbol{w}$.
7. $\boldsymbol{w}\models^M \mathbf{a/b}$ if and only if for any possible world $\boldsymbol{v}$ if $\boldsymbol{v}\mathbf{R_a}\boldsymbol{w}$ then $\boldsymbol{v}\mathbf{R_b}\boldsymbol{w}$.

Clause (1) reflects the stipulation that in a model M, a propositional letter

p_k is true at a possible world $\boldsymbol{w}$ just in the case $\boldsymbol{w}$ is a member of the set P_k. Clauses (2) and (3) are simply repeats of the usual propositional truth clauses. Due to definitions (D1)–(D4) they yield the classical truth tables for standard propositional connectives. Clause (4) formulates the interpretation of object relative necessity: $\mathbf{aB}$ is true at a possible world $\boldsymbol{w}$ if and only if $\mathbf{B}$ is true at all possible worlds which are **a**-*accessible from* $\boldsymbol{w}$, or a situation is necessary for an object at a possible world $\boldsymbol{w}$ if and only if it obtains in all possible worlds which are supportive for this object at the world $\boldsymbol{w}$. Clauses (5), (6) and (7) reflect the remarks about ontological concepts of existence, possibility, well-foundation and fusion contained in the paragraph 1.

A formula true at every possible world in a model M is said to be *valid* in the model M, a formula valid in every model is said to be *ontologically valid*. I write $\models^M$ $\mathbf{A}$ to mean that the formula $\mathbf{A}$ is valid in the model M, and $\models$ $\mathbf{A}$ to mean that the formula $\mathbf{A}$ is ontologically valid.

For any possible world $\boldsymbol{w}$ let $[\boldsymbol{w}]^{\mathbf{Ra}}$ be the set of possible worlds which are supportive for object **a** at the possible world $\boldsymbol{w}$. Thus $[\boldsymbol{w}]^{\mathbf{Ra}} = \{\boldsymbol{v} : \boldsymbol{v}\mathbf{R_a}\boldsymbol{w}\}$. I shall call it the *range of object* **a** at the possible world $\boldsymbol{w}$. The range of object **a** at the possible world $\boldsymbol{w}$ contains the possibilities which are sustainable for object **a** according to the standpoint $\boldsymbol{w}$. Thus each object **a** can be depicted as a function which to any possible world assigns the range of object **a** at the possible world $\boldsymbol{w}$. Let me reformulate the clauses (5) – (7) in terms of range of an object. The clauses are as follows.

(5*) $\boldsymbol{w}\models^M \mathrm{Ex}\mathbf{a}$ if and only if $\boldsymbol{w}$ belongs to $[\boldsymbol{w}]^{\mathbf{Ra}}$.
(6*) $\boldsymbol{w}\models^M \mathrm{Pos}\mathbf{a}$ if and only if $[\boldsymbol{w}]^{\mathbf{Ra}} \neq \emptyset$.
(7*) $\boldsymbol{w}\models^M \mathbf{a}/\mathbf{b}$ if and only if $[\boldsymbol{w}]^{\mathbf{Ra}} \subseteq [\boldsymbol{w}]^{\mathbf{Rb}}$.

According to (5*) object **a** exists at point $\boldsymbol{w}$ if and only if from standpoint $\boldsymbol{w}$, $\boldsymbol{w}$ is itself a sustainable possibility for object **a**. Clause (6*) states that object **a** is possible at point $\boldsymbol{w}$ if and only if the class of possibilities which are sustainable for object **a** according to standpoint $\boldsymbol{w}$ is not empty. The content of (7*) is that object **a** is well-founded in object **b** at point $\boldsymbol{w}$ if and only if the class of possibilities which are sustainable for object **a** according to standpoint $\boldsymbol{w}$ is included in the class of possibilities which are sustainable for object **b** according to standpoint $\boldsymbol{w}$.

4. In the previous two paragraphs modal ontological formulae were studied in two quite different ways, a syntactical one in paragraph 2 and a semantic one in paragraph 3. The syntactical approach is concerned with a modal ontological calculus and a modal ontological theory as a set of all theses. On the other hand, the semantic approach is concerned with ontological models and with truth conditions in a model. By the help of these semantic notions a set of ontologically valid formulae has been singled out. A fruitful blend of the two approaches results in the following completeness theorem.

Theorem 4. A formula is a thesis of modal ontologic if and only if it is an ontologically valid formula.

Thus, the syntactical and the semantic approach depict the same class of ontologically true formulae.

Appendix

Proof of theorem 1. To prove the theorem it is sufficient to show that the schemata (Th1) – (Th15) are thesis schemata.

(Th1) $\mathbf{aB} \to (\sim\mathbf{B} \to \sim\mathrm{Ex}\mathbf{a})$: From (A2) by propositional logic.

(Th2) $(\mathbf{aB}\ \&\ \mathbf{a}\sim\mathbf{B}) \to \sim\mathrm{Pos}\mathbf{a}$: From (A4) by propositional logic.

(Th3) $\mathrm{Ex}\mathbf{a} \to \mathrm{Pos}\mathbf{a}$: By (A2), $\mathrm{Ex}\mathbf{a} \to (\mathbf{aB} \to \mathbf{B})$ is a thesis schema. Then, by propositional logic, $\mathrm{Ex}\mathbf{a} \to (\sim\mathbf{B} \to \sim\mathbf{aB})$ is a thesis schema and $\mathrm{Ex}\mathbf{a} \to (\sim\mathbf{0} \to \sim\mathbf{a0})$ is also a thesis schema ($\mathbf{0}$ stands for arbitrary chosen counter-tautology of propositional logic). Thus, by propositional logic, $\mathrm{Ex}\mathbf{a} \to \sim\mathbf{a0}$ is a thesis schema. But, by (A5), $\sim\mathbf{a0} \to \mathrm{Pos}\mathbf{a}$ is a thesis schema and, by propositional logic $\mathrm{Ex}\mathbf{a} \to \mathrm{Pos}\mathbf{a}$ is a thesis schema.

(Th4) $\mathbf{a}/\mathbf{b} \to \mathbf{a}\mathrm{Ex}\mathbf{b}$: By (A6), $\mathbf{a}/\mathbf{b} \to (\mathbf{b}\mathrm{Ex}\mathbf{b} \to \mathbf{a}\mathrm{Ex}\mathbf{b})$ is a thesis schema. But, by (A3), $\mathbf{b}\mathrm{Ex}\mathbf{b}$ is a thesis schema, then, by propositional logic, $\mathbf{a}/\mathbf{b} \to \mathbf{a}\mathrm{Ex}\mathbf{b}$ is a thesis schema.

(Th5) $(\mathbf{a}/\mathbf{b}\ \&\ \mathbf{b}/\mathbf{c}) \to \mathbf{a}/\mathbf{c}$: By (A6) and propositional logic, $(\mathbf{a}/\mathbf{b}\ \&\ \mathbf{b}/\mathbf{c}) \to (\mathbf{cD}\to\mathbf{aD})$ is a thesis schema and therefore $(\mathbf{a}/\mathbf{b}\ \&\ \mathbf{b}/\mathbf{c}) \to (\mathbf{c}\mathrm{Ex}\mathbf{c}\to\mathbf{a}\mathrm{Ex}\mathbf{c})$ is a thesis schema. But, by (A3), $\mathbf{c}\mathrm{Ex}\mathbf{c}$ is a thesis schema and $(\mathbf{a}/\mathbf{b}\ \&\ \mathbf{b}/\mathbf{c}) \to \mathbf{a}\mathrm{Ex}\mathbf{c}$ is a thesis schema. Hence, by (A7) and propositional logic, $(\mathbf{a}/\mathbf{b}\ \&\ \mathbf{b}/\mathbf{c}) \to \mathbf{a}/\mathbf{c}$ is a thesis schema.

(Th6) $\mathbf{a}/\mathbf{a}$: From (A3) and (A7) by propositional logic.

(Th7) $\mathrm{Ex}(\mathbf{a}*\mathbf{b}) \to (\mathrm{Ex}\mathbf{a}\ \&\ \mathrm{Ex}\mathbf{b})$: By(A2), $\mathrm{Ex}(\mathbf{a}*\mathbf{b}) \to ((\mathbf{a}*\mathbf{b})\mathrm{Ex}\mathbf{a} \to \mathrm{Ex}\mathbf{a})$ and $\mathrm{Ex}(\mathbf{a}*\mathbf{b}) \to ((\mathbf{a}*\mathbf{b})\mathrm{Ex}\mathbf{b} \to \mathrm{Ex}\mathbf{b})$ are thesis schemata. But, by (A8), $\mathbf{a}\mathrm{Ex}\mathbf{a} \to (\mathbf{a}*\mathbf{b})\mathrm{Ex}\mathbf{a}$ and $\mathbf{b}\mathrm{Ex}\mathbf{b} \to (\mathbf{a}*\mathbf{b})\mathrm{Ex}\mathbf{b}$ are thesis schemata. Hence, by propositional logic, $\mathrm{Ex}(\mathbf{a}*\mathbf{b}) \to (\mathbf{a}\mathrm{Ex}\mathbf{a} \to \mathrm{Ex}\mathbf{a})$ and $\mathrm{Ex}(\mathbf{a}*\mathbf{b}) \to (\mathbf{b}\mathrm{Ex}\mathbf{b} \to \mathrm{Ex}\mathbf{b})$ are thesis schemata and, by (A3) and propositional logic, $\mathrm{Ex}(\mathbf{a}*\mathbf{b}) \to (\mathrm{Ex}\mathbf{a}\ \&\ \mathrm{Ex}\mathbf{b})$ is a thesis schema.

(Th8) $(\mathbf{a}*\mathbf{b})\mathrm{Ex}\mathbf{a}$: By (Th7), $\mathrm{Ex}(\mathbf{a}*\mathbf{b}) \to \mathrm{Ex}\mathbf{a}$ is a thesis schema. Hence, by (A1) and (R2), $(\mathbf{a}*\mathbf{b})\mathrm{Ex}(\mathbf{a}*\mathbf{b}) \to (\mathbf{a}*\mathbf{b})\mathrm{Ex}\mathbf{a}$ is a thesis schema. But, by (A3), $(\mathbf{a}*\mathbf{b})\mathrm{Ex}(\mathbf{a}*\mathbf{b})$ is a thesis schema and, by (R1), $(\mathbf{a}*\mathbf{b})\mathrm{Ex}\mathbf{a}$ is a thesis schema.

(Th9) $(\mathbf{a}*\mathbf{b})\mathrm{Ex}\mathbf{b}$: In the same way as for (Th8).

(Th10) $(\mathbf{a}*\mathbf{b})/\mathbf{a}$: By (Th8), $(\mathbf{a}*\mathbf{b})\mathrm{Ex}\mathbf{a}$ is a thesis schema. Hence, by (A7), $(\mathbf{a}*\mathbf{b})/\mathbf{a}$ is a thesis schema.

(Th11) $(\mathbf{a}*\mathbf{b})/\mathbf{b}$: In the same way as for (Th10).

(Th12) $\mathrm{Pos}(\mathbf{a}*\mathbf{b}) \to (\mathrm{Pos}\mathbf{a}\ \&\ \mathrm{Pos}\mathbf{b})$: By (A4) $\mathrm{Pos}(\mathbf{a}*\mathbf{b}) \to ((\mathbf{a}*\mathbf{b})\mathbf{C} \to \sim(\mathbf{a}*\mathbf{b})\sim\mathbf{C})$ is a thesis schema. But, by (A8), $\mathbf{aC} \to (\mathbf{a}*\mathbf{b})\mathbf{C}$ and $\mathbf{bC} \to (\mathbf{a}*\mathbf{b})\mathbf{C}$ are thesis schemata and, by propositional logic, $\mathrm{Pos}(\mathbf{a}*\mathbf{b}) \to (\mathbf{aC} \to \sim(\mathbf{a}*\mathbf{b})\sim\mathbf{C})$ and $\mathrm{Pos}(\mathbf{a}*\mathbf{b}) \to (\mathbf{bC} \to \sim(\mathbf{a}*\mathbf{b})\sim\mathbf{C})$ are thesis schemata. In particular, $\mathrm{Pos}(\mathbf{a}*\mathbf{b}) \to (\mathbf{a0} \to \sim(\mathbf{a}*\mathbf{b})\sim\mathbf{0})$ and $\mathrm{Pos}(\mathbf{a}*\mathbf{b}) \to (\mathbf{b0} \to \sim(\mathbf{a}*\mathbf{b})\sim\mathbf{0})$ are thesis schemata and, by propositional logic, $\mathrm{Pos}(\mathbf{a}*\mathbf{b}) \to ((\mathbf{a}*\mathbf{b})\sim\mathbf{0} \to \sim\mathbf{a0})$ and $\mathrm{Pos}(\mathbf{a}*\mathbf{b}) \to ((\mathbf{a}*\mathbf{b})\sim\mathbf{0} \to \sim\mathbf{b0})$ are thesis schemata. ($\mathbf{0}$ stands for arbitrary chosen counter-tautology of propositional logic.) But, by (R2), (**a*b**)~**0** is a thesis schema and therefore Pos(**a*b**) → ~**a0** and Pos(**a*b**) → ~**b0** are thesis schemata. Hence, by (A5) and propositional logic, Pos(**a*b**) → (Pos**a** & Pos**b**) is a thesis schema.

(Th13) (**a**/**b** & **b**/**a**) → (Ex**a** ≡ Ex**b**): By (A6), **a**/**b** → (**b**Ex**b** → **a**Ex**b**) is a thesis schema and, by (A3) and propositional logic, **a**/**b** → **a**Ex**b** is a thesis schema. By (A2), Ex**a** → (**a**Ex**b** → Ex**b**) is a thesis schema. Hence, by propositional logic, **a**/**b** → (Ex**a** → Ex**b**) is a thesis schema. On the other hand, by (A6), **b**/**a** → (**a**Ex**a** → **b**Ex**a**) is a thesis schema and, by (A3) and propositional logic, **b**/**a** → **b**Ex**a** is a thesis schema. By (A2), Ex**b** → (**b**Ex**a** → Ex**a**) is a thesis schema. Hence, by propositional logic, **b**/**a** → (Ex**b** → Ex**a**) is a thesis schema. Thus, by propositional logic, (**a**/**b** & **b**/**a**) → (Ex**a** ≡ Ex**b**) is a thesis schema.

(Th14) **a**/**b** → (**a*c**)/**b**: From (Th10) and (Th5) by propositional logic.

(Th15) **a**/(**b*c**) → (**a**/**b** & **a**/**c**): By (A6), **a**/(**b*c**) → ((**b*c**)**C** → **aC**), is a thesis schema. Hence, **a**/(**b*c**) → ((**b*c**)Ex**b** → **a**Ex**b**) and **a**/(**b*c**) → ((**b*c**)Ex**c** → **a**Ex**c**) are thesis schemata. But, by (Th8) and (Th9), (**b*c**)Ex**b** and (**b*c**)Ex**c** are thesis schemata, and therefore **a**/(**b*c**) → **a**Ex**b** and **a**/(**b*c**) → **a**Ex**c** are thesis schemata. Thus, by (A7) and propositional logic, **a**/(**b*c**) → (**a**/**b** & **a**/**c**) is a thesis schema.

Proof of theorem 2. To prove that the modal ontological theory is consistent let me take advantage of a standard language of classical propositional logic. The alphabet is given by a denumerable set Q of propositional letters, I refer to these as q_1, q_2, q_3, ...etc., the symbols of logical connectives ~ and & for negation and conjunction respectively and parentheses (and). The set of well-formed sentential formulae is defined inductively in the standard way. Thus (i) every propositional letter is a well-formed sentential formula, (ii) if x and y are well-formed sentential formulae, then ~x, ~y and (x&y) are well-formed sentential formulae, (iii) nothing else is a well-formed sentential formula. The symbols ∨, → and ≡ are introduced by definitions in the standard way. Let me define the mapping T, from the language of the modal ontologic to the language of classical propositional calculus, as follows.

(T1) $\mathrm{T}(\mathrm{p}_n) = \mathrm{q}_{2n}$.
(T2) $\mathrm{T}(\mathrm{a}_n) = \mathrm{q}_{2n+1}$.
(T3) T(**a*b**) = T(**a**) & T(**b**).
(T4) T(~**A**) = ~T(**A**).
(T5) T(**A** & **B**) = T(**A**) & T(**B**).
(T5) T(Ex**a**) = T(**a**).
(T6) T(pos**a**) = T(**a**).
(T7) T(**a**/**b**) = T(**a**) →T(**b**).
(T8) T(**aB**) = T(**a**) → T(**B**).

Thus, T associates with each well-formed sentential formula **A** in modal ontologic language a unique formula T(**A**) in the language of classical propositional logic. Let me call it the PC-transform of **A**. It is easy to show, that the PC-transform of every thesis of modal ontological calculus is a tautology of classical propositional calculus. It follows that for every well-formed sentential formula **A**,

$\mathbf{A}$ and $\sim\mathbf{A}$ are not theses, for if they were, T($\mathbf{A}$) and $\sim$T($\mathbf{A}$) would both be tautologies of classical propositional calculus, which is impossible.

Proof of theorem 3. Let W^* be the set of maximal consistent sets of formulae. Due to *theorem 2*, W^* is a non-empty set. Let P_i^* be the infinite sequence of subsets of W^*, such that for each natural number k, P_k^* is the set of maximal consistent sets of formulae containing propositional letter p_k. For each object operator $\mathbf{a}$, let $\mathbf{R_a}$ be the binary relation onW^*, such that for any $\boldsymbol{v}$ and $\boldsymbol{w}$ belonging to W^*, $\boldsymbol{v}\ \mathbf{R_a}\boldsymbol{w}$ if and only if $\{\mathbf{C} : \mathbf{aC}\ \epsilon\ \boldsymbol{w}\}\subseteq \boldsymbol{v}$. Let R^* be the set of binary relations on W^* which contains for every object operator $\mathbf{a}$ the relation $\mathbf{R_a}$ and no other relations. Let $+^*$ be the binary operation on R^* such that for any $\mathbf{R_a}$ and $\mathbf{R_b}$ belonging to W^*, $\mathbf{R_a}+^*\ \mathbf{R_b} = \mathbf{R_{(a*b)}}$. Let R_i^* be the infinite sequence of binary relations on W^*, such that for each natural number k, for any $\boldsymbol{v}$ and $\boldsymbol{w}$ belonging to W^*, $\boldsymbol{v}\ R_k^*\boldsymbol{w}$ if and only if $\{\mathbf{C} : \mathrm{a}_k\mathbf{C}\ \epsilon\ \boldsymbol{w}\}\subseteq \boldsymbol{v}$. In order to show that the structure $M^* = <W^*, P_i^*, R^*, +^*, R_i^*>$is an ontological model, it is sufficient to prove that the structure satisfy the conditions (C1), (C2) and (C3).

To prove (C1) assume that $\boldsymbol{v}\mathbf{R_a}+^*\mathbf{S_b}\boldsymbol{w}$. Thus $\boldsymbol{v}\mathbf{R_{(a*b)}}\boldsymbol{w}$ and $\{\mathbf{C} : (\mathbf{a*b})\mathbf{C}\ \epsilon\ \boldsymbol{w}\}\subseteq \boldsymbol{v}$. Now suppose $\mathbf{aC}$ belongs to $\boldsymbol{w}$. Then, by (A8), $(\mathbf{a*b})\mathbf{C}$ also belongs to $\boldsymbol{w}$, and by the assumption, $\mathbf{C}$ belongs to $\boldsymbol{v}$. Hence $\{\mathbf{C} : \mathbf{aC}\ \epsilon\ \boldsymbol{w}\}\subseteq \boldsymbol{v}$. Next, suppose $\mathbf{bC}$ belongs to $\boldsymbol{w}$. Then, by (A8), $(\mathbf{a*b})\mathbf{C}$ also belongs to $\boldsymbol{w}$, and by the assumption, $\mathbf{C}$ belongs to $\boldsymbol{v}$. Hence $\{\mathbf{C} : \mathbf{bC}\ \epsilon\ \boldsymbol{w}\}\subseteq \boldsymbol{v}$. Thus, $\boldsymbol{v}\mathbf{R_a}\boldsymbol{w}$ and $\boldsymbol{v}\mathbf{R_b}\boldsymbol{w}$.

To prove (C2) assume that $\boldsymbol{v}\mathbf{R_a}\boldsymbol{w}$. Thus $\{\mathbf{C} : \mathbf{aC}\ \epsilon\ \boldsymbol{w}\}\subseteq \boldsymbol{v}$ and, by (A3), Ex$\mathbf{a}$ belongs to $\boldsymbol{v}$. Hence, by (A2) any formula depicted by schema ($\mathbf{aC} \to \mathbf{C}$), also belongs to $\boldsymbol{v}$. Thus for any formula $\mathbf{aC}$, if $\mathbf{aC}$ belongs to $\boldsymbol{v}$, then C belongs to $\boldsymbol{v}$. Hence $\{\mathbf{C} : \mathbf{aC}\ \epsilon\ \boldsymbol{v}\}\subseteq \boldsymbol{v}$ and $\boldsymbol{v}\mathbf{R_a}\boldsymbol{v}$.

To prove (C3) assume that for any $\boldsymbol{v}$, $\boldsymbol{v}\mathbf{R_a}\boldsymbol{w}$ implies that $\boldsymbol{v}\mathbf{R_b}\boldsymbol{v}$. Thus for any $\boldsymbol{v}$, if $\{\mathbf{C} : \mathbf{aC}\ \epsilon\ \boldsymbol{w}\}\subseteq \boldsymbol{v}$, then $\{\mathbf{C} : \mathbf{bC}\ \epsilon\ \boldsymbol{v}\}\subseteq \boldsymbol{v}$. But, due to (A3) for any $\boldsymbol{v}$, if $\{\mathbf{C} : \mathbf{bC}\ \epsilon\ \boldsymbol{v}\}\subseteq \boldsymbol{v}$, then Ex$\mathbf{b}$ belongs to $\boldsymbol{v}$. Thus, for any $\boldsymbol{v}$, if $\{\mathbf{C} : \mathbf{aC}\ \epsilon\ \boldsymbol{w}\}\subseteq \boldsymbol{v}$, then Ex$\mathbf{b}$ belongs to $\boldsymbol{v}$. Hence, no maximal consistent set of formulae includes the set $\{\mathbf{C} : \mathbf{aC}\ \epsilon\ \boldsymbol{w}\}\cup\{\sim\mathrm{Ex}\mathbf{b}\}$and therefore this set is inconsistent. Thus, there is a finite subset of this set $\{\mathbf{C}_1, \mathbf{C}_2, \mathbf{C}_3, \ldots, \mathbf{C}_k, \sim\mathrm{Ex}\mathbf{b}\}$which is inconsistent and therefore formula $\mathbf{C}_1 \to (\mathbf{C}_2 \to (\mathbf{C}_3 \to \ldots (\mathbf{C}_k \to \mathrm{Ex}\mathbf{b})\ldots))$ is a thesis. Hence $\mathbf{C}_1 \to (\mathbf{C}_2 \to (\mathbf{C}_3 \to \ldots (\mathbf{C}_k \to \mathrm{Ex}\mathbf{b})\ldots))$ belongs to $\boldsymbol{w}$ and, by (A1), (R1) and (R2), $\mathbf{aC}_1 \to (\mathbf{aC}_2 \to (\mathbf{aC}_3 \to \ldots (\mathbf{aC}_k \to \mathbf{a}\mathrm{Ex}\mathbf{b})\ldots))$ also belongs to $\boldsymbol{w}$. But formulae $\mathbf{aC}_1$, $\mathbf{aC}_2$, $\mathbf{aC}_3$, ..., and $\mathbf{aC}_k$ belong to $\boldsymbol{w}$, and by (R1), $\mathbf{a}\mathrm{Ex}\mathbf{b}$ also belongs to $\boldsymbol{w}$. Thus, by (A7), $\mathbf{a/b}$ belongs to $\boldsymbol{w}$ and, by (A6), any formula depicted by scheme ($\mathbf{bC} \to \mathbf{aC}$) also belongs to $\boldsymbol{w}$. Hence $\{\mathbf{C} : \mathbf{bC}\ \epsilon\ \boldsymbol{w}\}\subseteq \{\mathbf{C} : \mathbf{aC}\ \epsilon\ \boldsymbol{w}\}$and therefore for any $\boldsymbol{v}$, if $\{\mathbf{C} : \mathbf{aC}\ \epsilon\ \boldsymbol{w}\}\subseteq \boldsymbol{v}$, then $\{\mathbf{C} : \mathbf{bC}\ \epsilon\ \boldsymbol{w}\}\subseteq \boldsymbol{v}$. Thus, $\boldsymbol{v}\mathbf{R_a}\boldsymbol{w}$ implies that $\boldsymbol{v}\mathbf{R_b}\boldsymbol{w}$.

It completes the proof that the structure $M^* = < W^*, P_i^*, R^*, +^*, R_i^* >$ is an ontological model. I shall call it *canonical ontological model*. For the canonical ontological model holds the lemma to the effect that for any formula $\mathbf{A}$ and any $\boldsymbol{w}\ \epsilon W^*$, $\boldsymbol{w}\models^{M*}\ \mathbf{A}$ if and only if $\mathbf{A}\ \epsilon\ \boldsymbol{w}$. I shall call it *the fundamental lemma*.

The proof of the lemma is of course by induction on the construction of formulae. The definition of the canonical ontological model assures that for any propositional letter p_k, and for any $\boldsymbol{w}\ \epsilon W^*$, $\boldsymbol{w}\models^{M*}\ \mathrm{p}_k$ if and only if $\mathrm{p}_k\epsilon\ \boldsymbol{w}$. In case of propositional connectives $\sim$ and & you rely on the maximal consistency of each $\boldsymbol{w}$, to assure you that $\mathbf{A} \,\epsilon\, \boldsymbol{w}$ if and only if it is not the case that $\sim\mathbf{A} \,\epsilon\, \boldsymbol{w}$ and that A & B $\epsilon\, \boldsymbol{w}$ if and only if $\mathbf{A} \,\epsilon\, \boldsymbol{w}$ and $\mathbf{B} \,\epsilon\, \boldsymbol{w}$.

The case of one-place object operator $\mathbf{a}$ is a standard one. The induction hypothesis is that for any $\boldsymbol{w} \,\epsilon\, W^*$, $\boldsymbol{w} \models^{M*} \mathbf{A}$ if and only if $\mathbf{A} \,\epsilon\, \boldsymbol{w}$. Now, suppose $\mathbf{aA}$ belongs to $\boldsymbol{w}$. Hence if $\boldsymbol{v}\mathbf{R_a}\boldsymbol{w}$, then $\{\mathbf{B} : \mathbf{aB} \,\epsilon\, \boldsymbol{w}\} \subseteq \boldsymbol{v}$ and therefore $\mathbf{A} \,\epsilon\, \boldsymbol{v}$. Thus, by the induction hypothesis, if $\boldsymbol{v}\mathbf{R_a}\boldsymbol{w}$, then $\boldsymbol{v} \models^{M*} \mathbf{A}$, hence $\boldsymbol{w} \models^{M*} \mathbf{aA}$. Next, suppose that $\boldsymbol{w} \models^{M*} \mathbf{aA}$. Thus, if $\boldsymbol{v}\mathbf{R_a}\boldsymbol{w}$, then $\boldsymbol{v} \models^{M*} \mathbf{A}$ and, by the definition of the canonical ontological model and by the induction hypothesis, if $\{\mathbf{C} : \mathbf{aC} \,\epsilon\, \boldsymbol{w}\} \subseteq \boldsymbol{v}$, then $\mathbf{A} \,\epsilon\, \boldsymbol{v}$. Hence, no maximal consistent set of formulae includes the set $\{\mathbf{C} : \mathbf{aC} \,\epsilon\, \boldsymbol{w}\} \cup \{\sim\mathbf{A}\}$and therefore this set is inconsistent. Thus, there is a finite subset of this set $\{\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3, \ldots, \mathbf{B}_k, \mathbf{A}\}$which is inconsistent and therefore formula $\mathbf{B}_1 \to (\mathbf{B}_2 \to (\mathbf{B}_3 \to \ldots (\mathbf{B}_k \to \mathbf{A}) \ldots))$ is a thesis. Hence $\mathbf{B}_1 \to (\mathbf{B}_2 \to (\mathbf{B}_3 \to \ldots (\mathbf{B}_k \to \mathbf{A}) \ldots))$ belongs to $\boldsymbol{w}$ and, by (A1), (R1) and (R2), $\mathbf{aB}_1 \to (\mathbf{aB}_2 \to (\mathbf{aB}_3 \to \ldots (\mathbf{aB}_k \to \mathbf{aA}) \ldots))$ also belongs to $\boldsymbol{w}$. But formulae $\mathbf{aB}_1$, $\mathbf{aB}_2$, $\mathbf{aB}_3$, …, and $\mathbf{aB}_k$ belong to $\boldsymbol{w}$, and by (R1), $\mathbf{aA}$ also belongs to $\boldsymbol{w}$.

The only essentially new cases are the symbols of existence (Ex), possibility (Pos) and well-foundation (/).

To prove that $\boldsymbol{w} \models^{M*}$ Ex$\mathbf{a}$ if and only if Ex$\mathbf{a} \,\epsilon\, \boldsymbol{w}$, at first suppose that $\boldsymbol{w} \models^{M*}$ Ex$\mathbf{a}$. Hence, $\boldsymbol{w}\ \mathbf{R_a}\boldsymbol{w}$ and, by the definition of M^*, $\{\mathbf{C} : \mathbf{aC} \,\epsilon\, \boldsymbol{w}\} \subseteq \boldsymbol{w}$. Thus, any formula depicted by scheme $\mathbf{aB} \to \mathbf{B}$ belongs to $\boldsymbol{w}$. But, by (A3), $\mathbf{a}$Ex$\mathbf{a}$ belongs to $\boldsymbol{w}$, and therefore Ex$\mathbf{a}$ also belongs to $\boldsymbol{w}$. Next, suppose Ex$\mathbf{a}$ belongs to $\boldsymbol{w}$. Thus, by (A2), any formula depicted by scheme $\mathbf{aC} \to \mathbf{C}$ belongs to $\boldsymbol{w}$. Hence $\{\mathbf{C} : \mathbf{aC} \,\epsilon\, \boldsymbol{w}\} \subseteq \boldsymbol{w}$ and, by the definition of M^*, $\boldsymbol{w}\ \mathbf{R_a}\boldsymbol{w}$. Therefore $\boldsymbol{w} \models^{M*}$ Ex$\mathbf{a}$.

To prove that $\boldsymbol{w} \models^{M*}$ Pos$\mathbf{a}$ if and only if Pos$\mathbf{a} \,\epsilon\, \boldsymbol{w}$, at first suppose $\boldsymbol{w} \models^{M*}$ Pos$\mathbf{a}$. Hence, $[\boldsymbol{w}]^{\mathbf{Ra}} \neq \emptyset$ and therefore there is a maximal consistent set of formulae $\boldsymbol{v}$, such that $\{\mathbf{C} : \mathbf{aC} \,\epsilon\, \boldsymbol{w}\} \subseteq \boldsymbol{v}$. Thus, $\{\mathbf{C} : \mathbf{aC} \,\epsilon\, \boldsymbol{w}\}$is a consistent set and therefore any formula depicted by scheme $\mathbf{aC} \to \sim\mathbf{a}\sim\mathbf{C}$ belongs to $\boldsymbol{w}$. Let $\mathbf{1}$ abbreviates the arbitrary chosen tautology of propositional logic. Of course, by (R2), $\mathbf{a1}$ belongs to $\boldsymbol{w}$. Hence, $\sim\mathbf{a}\sim\mathbf{1}$ belongs to $\boldsymbol{w}$ and, by (A5), Pos$\mathbf{a}$ belongs to $\boldsymbol{w}$. Next, suppose Pos$\mathbf{a}$ belongs to $\boldsymbol{w}$. Hence, by (A4), any formula depicted by scheme $\mathbf{aC} \to \sim\mathbf{a}\sim\mathbf{C}$ belongs to $\boldsymbol{w}$ and therefore $\{\mathbf{C} : \mathbf{aC} \,\epsilon\, \boldsymbol{w}\}$is a consistent set. Thus, there is a maximal consistent set of formulae $\boldsymbol{v}$, such that $\{\mathbf{C} : \mathbf{aC} \,\epsilon\, \boldsymbol{w}\} \subseteq \boldsymbol{v}$ and, by the definition of M^*, $[\boldsymbol{w}]^{\mathbf{Ra}} \neq \emptyset$. Therefore $\boldsymbol{w} \models^{M*}$ Pos$\mathbf{a}$.

To prove that $\boldsymbol{w} \models^{M*} \mathbf{a}/\mathbf{b}$ if and only if $\mathbf{a}/\mathbf{b} \,\epsilon\, \boldsymbol{w}$, at first suppose $\boldsymbol{w} \models^{M*} \mathbf{a}/\mathbf{b}$. Hence, $[\boldsymbol{w}]^{\mathbf{Ra}} \subseteq [\boldsymbol{w}]^{\mathbf{Rb}}$ and, by the definition of M^*, for any $\boldsymbol{v}$, if $\{\mathbf{C} : \mathbf{aC} \,\epsilon\, \boldsymbol{w}\} \subseteq \boldsymbol{v}$ then $\{\mathbf{C} : \mathbf{bC} \,\epsilon\, \boldsymbol{w}\} \subseteq \boldsymbol{v}$. But, due to (A3), $\mathbf{b}$Ex$\mathbf{b}$ belongs to $\boldsymbol{w}$, and therefore for any $\boldsymbol{v}$, if $\{\mathbf{C} : \mathbf{aC} \,\epsilon\, \boldsymbol{w}\} \subseteq \boldsymbol{v}$ then Ex$\mathbf{b}$ belongs to $\boldsymbol{v}$. Hence, no maximal consistent set of formulae includes the set $\{\mathbf{C} : \mathbf{aC} \,\epsilon\, \boldsymbol{w}\} \cup \{\sim$Ex$\mathbf{b}\}$and therefore this set is inconsistent. Thus, there is a finite subset of this set $\{\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3, \ldots, \mathbf{B}_k,$ $\sim$Ex$\mathbf{b}\}$which is inconsistent and therefore formula $\mathbf{B}_1 \to (\mathbf{B}_2 \to (\mathbf{B}_3 \to \ldots (\mathbf{B}_k \to$ Ex$\mathbf{b}$ $) \ldots))$ is a thesis. Hence $\mathbf{B}_1 \to (\mathbf{B}_2 \to (\mathbf{B}_3 \to \ldots (\mathbf{B}_k \to$ Ex$\mathbf{b}) \ldots))$ belongs to $\boldsymbol{w}$ and, by (A1), (R1) and (R2), $\mathbf{aB}_1 \to (\mathbf{aB}_2 \to (\mathbf{aB}_3 \to \ldots (\mathbf{aB}_k \to \mathbf{a}$ Ex$\mathbf{b}) \ldots))$ also belongs to $\boldsymbol{w}$. But formulae $\mathbf{aB}_1$, $\mathbf{aB}_2$, $\mathbf{aB}_3$, …, and $\mathbf{aB}_k$ belong to $\boldsymbol{w}$, and by (R1), $\mathbf{a}$Ex$\mathbf{b}$ also belongs to $\boldsymbol{w}$. But, by (A7), any formula depicted by schema

$\mathbf{a}\mathrm{Ex}\mathbf{b} \rightarrow \mathbf{a}/\mathbf{b}$ belongs to $\boldsymbol{w}$, and therefore $\mathbf{a}/\mathbf{b}$ also belongs to $\boldsymbol{w}$.

Next, suppose $\mathbf{a}/\mathbf{b}$ belongs to $\boldsymbol{w}$. Hence, by (A6), any formula depicted by schema $\mathbf{bC} \rightarrow \mathbf{aC}$ belongs to $\boldsymbol{w}$, and therefore $\{\mathbf{C} : \mathbf{bC} \epsilon\ \boldsymbol{w}\} \subseteq \{\mathbf{C} : \mathbf{aC} \epsilon\ \boldsymbol{w}\}$. Thus, for any $\boldsymbol{v}$, if $\{\mathbf{C} : \mathbf{aC} \epsilon\ \boldsymbol{w}\} \subseteq \boldsymbol{v}$ then $\{\mathbf{C} : \mathbf{bC} \epsilon\ \boldsymbol{w}\} \subseteq \boldsymbol{v}$. Therefore, $[\boldsymbol{w}]^{\mathbf{Ra}} \subseteq [\boldsymbol{w}]^{\mathbf{Rb}}$ and $\boldsymbol{w} \models^{M*} \mathbf{a}/\mathbf{b}$.

Proof of theorem 4. In order to show that, a formula is a thesis of modal ontologic if and only if it is an ontologically valid formula it is sufficient to prove that any thesis is an ontologically valid formula and that any formula, which is not a thesis is not ontologically valid. The proof of the first implication requires the establishment of the ontological validity of all axioms and the demonstration that the rules of inference (R1) and (R2) preserve ontological validity. It could be easily done. To prove the converse implication, suppose a formula $\mathbf{A}$ is not a thesis. Then $\{\sim\mathbf{A}\}$is a consistent set. Thus, there is a maximal consistent set of formulae $\boldsymbol{v}$, such that $\{\sim\mathbf{A}\} \subseteq \boldsymbol{v}$. Hence, by the definition of M^*, for some $\boldsymbol{w}\ \epsilon W^*$, $\sim\mathbf{A}$ belongs to $\boldsymbol{w}$, and $\mathbf{A}$ doesn't belong to $\boldsymbol{w}$. Therefore, by the fundamental lemma, for some $\boldsymbol{w}\ \epsilon W^*$, $\boldsymbol{w} \not\models^{M*} \mathbf{A}$, and so, $\mathbf{A}$ is not ontologically valid.